SOME SIMILARITY MEASURES FOR PICTURE FUZZY SETS AND THEIR APPLICATIONS

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Abstract. In this work, we shall present some novel process to measure the similarity between picture fuzzy sets. Firstly, we adopt the concept of intuitionistic fuzzy sets, interval-valued intuitionistic fuzzy sets and picture fuzzy sets. Secondly, we develop some similarity measures between picture fuzzy sets, such as, cosine similarity measure, weighted cosine similarity measure, set-theoretic similarity measure, weighted set-theoretic cosine similarity measure, grey similarity measure and weighted grey similarity measure. Then, we apply these similarity measures between picture fuzzy sets to building material recognition and minerals field recognition. Finally, two illustrative examples are given to demonstrate the efficiency of the similarity measures for building material recognition and minerals field recognition.

1. Introduction

Fuzzy set theory, introduced by Zadeh [52], has been widely used to model uncertainty present in real-world applications. Many researchers have paid their attention to the generalization of fuzzy set theory and its applications. Out of several generalizations of fuzzy sets, the concept of intuitionistic fuzzy sets (IFSs), introduced by Atanassov [1-2], has been found to be highly useful to deal with vagueness. By adding the degree of non-membership to fuzzy set, IFS [1-2] was introduced, which reflects the fact that the degree of non-membership is not always equal to one minus degree of membership. Atanassov and Gargov [3] and Atanassov[4] proposed the concept of interval-valued intuitionistic fuzzy sets, which are characterized by a membership function, a non-membership function, and a hesitancy function whose values are intervals. Thus, there are some situations where intuitionistic fuzzy sets and interval-valued intuitionistic fuzzy sets theory provides a strong and suitable framework to deal with incomplete information present in real-world decision making problems [6-9, 13-15, 19, 23, 26-33, 41, 44, 45, 46, 48, 55].

Recently, Cuong [10] proposed picture fuzzy set (PFS) and investigated the some basic operations and properties of PFS. The picture fuzzy set is characterized by three functions expressing the degree of membership, the degree of neutral membership and the degree of non-membership. The only constraint is that the sum of the three degrees must not exceed 1. Basically, PFS based models can be applied to

Received: April 2016; Revised: December 2016; Accepted: May 2017

Key words and phrases: Picture fuzzy set, Cosine similarity measure, Set-theoretic similarity measure, Grey similarity measure, Building material recognition, Minerals field recognition.

situations requiring human opinions involving more answers of types: yes, abstain, no, refusal, which cant be accurately expressed in the traditional FS and IFS. Until now, some progress has been made in the research of the PFS theory. Singh [21] investigated the correlation coefficients for picture fuzzy set and apply the correlation coefficient to clustering analysis with picture fuzzy information. Son[22] introduce several novel fuzzy clustering algorithms on the basis of picture fuzzy sets and applications to time series forecasting and weather forecasting. Thong & Son[24] and Thong [25] developed a novel hybrid model between picture fuzzy clustering and intuitionistic fuzzy recommender systems for medical diagnosis and application to health care support systems.

Although, Atanassovs intuitionistic fuzzy set theory has been successfully applied in different areas, but there are situations in real life which cant be represented by Atanassovs intuitionistic fuzzy sets. Voting can be a good example of such situation as the human voters may be divided into four groups of those who: vote for, abstain, refusal of voting. Basically, picture fuzzy sets[10] based models may be adequate in situations when we face human opinions involving more answers of the type: yes, abstain, no, refusal. Therefore in order to deal with these types of situations, in this paper we introduce the concept of similarity measures for picture fuzzy sets, which is a new extension of the similarity measure of Atanassovs intuitionistic fuzzy set. In order to do so, the remainder of this paper is set out as follows. In the next section, we introduce some basic concepts related to intuitionistic fuzzy set, interval-valued intuitionistic fuzzy sets and picture fuzzy sets. In Section 3, we shall propose some similarity measure and some weighted similarity measure between PFSs based on the concept of the similarity measure for fuzzy sets. In Section 4, the similarity measures for PFSs are applied to building material recognition and minerals field recognition. Section 5 concludes the paper with some remarks.

2. **Preliminaries**

In the following, we introduce some basic concepts related to intuitionistic fuzzy sets and interval-valued intuitionistic fuzzy sets.

Definition 2.1. [1-3] An IFS A in X is given by

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in X.\}, \tag{1}$$

where $\mu_A:\ X \to [0,1]$ and $\nu_A:\ X \to [0,1]$, with the condition $0 \leq \mu_A(x)+\nu_A(x) \leq 1$, $\forall\ x \in X$. The numbers $\mu_A(x)$ and $\nu_A(x)$ represent, respectively, the membership degree and non- membership degree of the element to the set A.

Atanassov and Gargov[3] further introduced the interval-valued intuitionistic fuzzy set (IVIFS) based on the intuitionistic fuzzy sets.

Definition 2.2. [3] Let X be a universe of discourse, An interval-valued intuitionistic fuzzy set (IVIFS) $\tilde{A}$ over X is an object having the form:

$$\tilde{A} = \{\langle x, \tilde{\mu}_A(x), \tilde{\nu}_A(x)\rangle | x \in X\}, \tag{2}$$

where $\tilde{\mu}_A(x) \subset [0,1]$ and $\tilde{\nu}_A(x) \subset [0,1]$ are interval numbers, and $0 \leq \sup(\tilde{\mu}_A(x)) + \sup(\tilde{\nu}_A(x)) \leq 1$, $\forall\, x \in X$. For convenience, let $\tilde{\mu}_A(x) = [a,b]$, $\tilde{\nu}_A(x) = [c,d]$, so $\tilde{A} = ([a,b],[c,d])$.

Picture fuzzy set[10] based models may be adequate in situations when we face human opinions involving more answers of types: yes, abstain, no, refusal. It can be considered as a powerful tool represent the uncertain information in the process of patterns recognition and cluster analysis.

Definition 2.3. [10] A picture fuzzy set (PFS) A on the universe X is an object of the form

$$A = \{\langle x, \mu_A(x), \eta_A(x), \nu_A(x)\rangle \,|x \in X\} \tag{3}$$

where $\mu_A(x) \in [0,1]$ is called the "degree of positive membership of A", $\eta_A(x) \in [0,1]$ is called the "degree of neutral membership of A" and $\nu_A(x) \in [0,1]$ is called the "degree of negative membership ", and $\mu_A(x)$, $\eta_A(x)$, $\nu_A(x)$ satisfy the following condition: $0 \leq \mu_A(x) + \eta_A(x) + \nu_A(x) \leq 1$, $\forall\, x \in X$. Then for $x \in X$, $\rho_A(x) = 1 - (\mu_A(x) + \eta_A(x) + \nu_A(x))$ could be called the degree of refusal membership of x in A.

Cuong et al.[10] also defined some operations as follows.

Definition 2.4. [10] Given two PFEs represented by and A on B universe X, the inclusion, union, intersection and complement operations are defined as follows:

(1) $A \subseteq B$, if $\mu_A(x) \leq \mu_B(x)$, $\eta_A(x) \leq \eta_B(x)$ and $\nu_A(x) \geq \nu_B(x)$, $\forall x \in X$;

(2) $A \cup B = \{(x, \max(\mu_A(x), \mu_B(x)), \min(\eta_A(x), \eta_B(x)), \min(\nu_A(x), \nu_B(x)))\,|x \in X\}$

(3) $A \cap B = \{(x, \min(\mu_A(x), \mu_B(x)), \max(\eta_A(x), \eta_B(x)), \max(\nu_A(x), \nu_B(x)))\,|x \in X\}$

(4) $\bar{A} = \{(x, \nu_A(x), \eta_A(x), \mu_A(x))\,|x \in X\}$

3. Cosine Similarity Measure for Picture Fuzzy Sets

In this section, we shall propose some similarity measure and some weighted similarity measure between PFSs based on the concept of the similarity measure for fuzzy sets[20].

3.1. Linguistic Term Set.

Let A be PFS in universe of discourse $X = \{x\}$, the PFS is characterized by the degree of positive membership $\mu_A(x)$, the degree of neutral membership $\eta_A(x)$ and the degree of negative membership $\nu_A(x)$ which can be considered as a vector representation with the three elements. Therefore, a cosine similarity measure and a weighted cosine similarity measure for PFSs are proposed in an analogous manner to the cosine similarity measure based on Bhattacharyas distance [5, 20,51] and cosine similarity measure for intuitionistic fuzzy set[51].

Assume that there are two PFSs A and B in $X = \{x_1, x_2, \cdots, x_n\}$, a cosine similarity measure between PIFSs A and B is proposed as follows:

$$C^1_{PFS}(A,B) = \frac{1}{n}\sum_{i=1}^{n} \frac{\mu_A(x_i)\,\mu_B(x_i) + \eta_A(x_i)\,\eta_B(x_i) + \nu_A(x_i)\,\nu_B(x_i)}{\sqrt{\mu_A^2(x_i) + \eta_A^2(x_i) + \nu_A^2(x_i)}\sqrt{\mu_B^2(x_i) + \eta_B^2(x_i) + \nu_B^2(x_i)}} \tag{4}$$

If we take $n = 1$, then the cosine similarity measure between PFSs A and B becomes the correlation coefficient between PFSs A and B, i.e. $C_{PFS}(A, B) = K_{PFS}(A, B)$. Therefore, the cosine similarity measure between PFSs A and B also satisfies the following properties:

(1) $0 \leq C^1_{_{PFS}}(A, B) \leq 1$;
(2) $C^1_{_{PFS}}(A, B) = C^1_{_{PFS}}(B, A)$;
(3) $C^1_{_{PFS}}(A, B) = 1$, if $A = B$,$i = 1, 2, \cdots, n$
(4) if $A \subseteq B \subseteq C$, then $C^1_{_{PFS}}(A, C) \leq C^1_{_{PFS}}(A, B)$, $C^1_{_{PFS}}(A, C) \leq C^1_{_{PFS}}(B, C)$.

Proof. (1) It is obvious that the proposition is true according to the cosine value.

(2) It is obvious that the proposition is true.

(3)When $A = B$, there are $\mu_A(x_i) = \mu_B(x_i)$, $\eta_A(x_i) = \eta_B(x_i)$ and $\nu_A(x_i) = \nu_B(x_i)$ for $i = 1, 2, \cdots, n$. So, there is $C^1_{_{PFS}}(A, B) = 1$.

(4) if $A \subseteq B \subseteq C$, geometrically the angle between A and C should be larger than the angle between A and B and the angle between B and C for any element $i\,(i = 1, 2, \cdots, n)$. Obviously the relations for $C^1_{_{PFS}}(A, C) \leqslant C^1_{_{PFS}}(A, B)$ and $C^1_{_{PFS}}(A, C) \leqslant C^1_{_{PFS}}(B, C)$ can be obtained from equation (4).

Therefore, we have finished the proofs. □

In the following, we shall investigate the distance measure of the angle as

$$d(A, B) = \arccos\left(C^1_{_{PFS}}(A, B)\right).$$

It satisfies the following properties:

(1) $d(A, B) \geqslant 0$, if $0 \leqslant C_{PFS}(A, B) \leqslant 1$;
(2) $d(A, B) = \arccos(1) = 0$, if $C_{PFS}(A, A) = 1$;
(3) $d(A, B) = d(B, A)$, if $C_{PFS}(A, B) = C_{PFS}(B, A)$,
(4) $d(A, C) \leqslant d(A, B) + d(B, C)$, if $A \subseteq B \subseteq C$for any PFS C.

Proof. Obviously, $d(A, B)$ satisfies the property (1)-(3). In the following, $d(A, B)$ will be proved to satisfy the property (4).

For any $C = \{\langle x_i, (\mu_C(x_i), \eta_C(x_i), \nu_C(x_i))\rangle\, |x_i \in x\}$, $A \subseteq B \subseteq C$, Since equation(4) is the sum of terms, let us investigate the distance measures of the angle between the vectors:

$d_i(A(x_i), B(x_i)) = \arccos(C_{PFS_i}(A(x_i), B(x_i)))$,

$d_i(B(x_i), C(x_i)) = \arccos(C_{PFS_i}(B(x_i), C(x_i)))$,

$d_i(A(x_i), C(x_i)) = \arccos(C_{PFS_i}(A(x_i), C(x_i)))\, i = 1, 2, \cdots, n$, where

$C_{PFS_i}(A(x_i), B(x_i)) = \frac{\mu_A(x_i)\mu_B(x_i)+\eta_A(x_i)\eta_B(x_i)+\nu_A(x_i)\nu_B(x_i)}{\sqrt{\mu_A^2(x_i)+\eta_A^2(x_i)+\nu_A^2(x_i)}\sqrt{\mu_B^2(x_i)+\eta_B^2(x_i)+\nu_A^2(x_B)}}$

$C_{PFS_i}(B(x_i), C(x_i)) = \frac{\mu_B(x_i)\mu_C(x_i)+\eta_B(x_i)\eta_C(x_i)+\nu_B(x_i)\nu_C(x_i)}{\sqrt{\mu_B^2(x_i)+\eta_B^2(x_i)+\nu_B^2(x_i)}\sqrt{\mu_C^2(x_i)+\eta_C^2(x_i)+\nu_C^2(x_i)}}$

$C_{PFS_i}(A(x_i), C(x_i)) = \frac{\mu_A(x_i)\mu_C(x_i)+\eta_A(x_i)\eta_C(x_i)+\nu_A(x_i)\nu_C(x_i)}{\sqrt{\mu_A^2(x_i)+\eta_A^2(x_i)+\nu_A^2(x_i)}\sqrt{\mu_C^2(x_i)+\eta_C^2(x_i)+\nu_C^2(x_i)}}$

For three vectors $A(x_i) = \langle\mu_A(x_i), \eta_A(x_i), \nu_A(x_i)\rangle$, $B(x_i) = \langle\mu_B(x_i), \eta_B(x_i),$ $\nu_B(x_i)\rangle$, $C(x_i) = \langle\mu_C(x_i), \eta_C(x_i), \nu_C(x_i)\rangle$ in one plane, if $A(x_i) \subseteq B(x_i) \subseteq$

$C(x_i)$, $i = 1, 2, \cdots, n$. Then, it is obvious that

$$d_i(A(x_i), C(x_i)) \leqslant d_i(A(x_i), B(x_i)) + d_i(B(x_i), C(x_i))$$

according to the triangle inequality. Combining the inequality with equation(4), we can obtain $d(A, C) \leqslant d(A, B) + d(B, C)$. Thus $d(A, B)$ satisfies the property (4). So we finished the proof. □

If we consider the weights of x_i, a weighted cosine similarity measure between PFSs A and B is proposed as follows:

$$W^1_{PFS}(A, B) = \sum_{i=1}^{n} w_i \frac{\mu_A(x_i)\mu_B(x_i) + \eta_A(x_i)\eta_B(x_i) + \nu_A(x_i)\nu_B(x_i)}{\sqrt{\mu^2_A(x_i) + \eta^2_A(x_i) + \nu^2_A(x_i)}\sqrt{\mu^2_B(x_i) + \eta^2_B(x_i) + \nu^2_B(x_i)}} \quad (5)$$

where $w = (w_1, w_2, \cdots, w_n)^T$ is the weight vector of $x_i\,(i = 1, 2, \cdots, n)$,with $w_i \in [0, 1]$,$i = 1, 2, \cdots, n$, $\sum\limits_{i=1}^{n} w_i = 1$. In particular, if $w = (1/n, 1/n, \cdots, 1/n)^T$, then the weighted cosine similarity measure reduces to cosine similarity measure. Thatis to say, if we take $w_i = \frac{1}{n}, i = 1, 2\cdots, n$, then there is $W^1_{PFS}(A, B) = C^1_{PFS}(A, B)$.

Obviously, the weighted cosine similarity measure of two PFSs A and B also satisfies the following properties:

(1)$0 \leqslant W^1_{PFS}(A, B) \leqslant 1$

(2)$W^1_{PFS}(A, B) = W^1_{PFS}(B, A)$

(3)$W^1_{PFS}(A, B) = 1,\ if\ A = B, i = 1, 2, \cdots, n.$

Similar to the previous proof method, we can prove the above three properties.

3.2. **Set-theoretic Similarity Measure for Picture Fuzzy Sets.**

Assume that there are two PFSs A and B in $X = \{x_1, x_2, \cdots, x_n\}$. Based on the set-theoretic viewpoint[50], we shall propose another similarity measure between PFSs A and B as follows:

$$C^2_{PFS}(A, B) = \frac{1}{n}\sum_{i=1}^{n} \frac{\mu_A(x_i)\mu_B(x_i) + \eta_A(x_i)\eta_B(x_i) + \nu_A(x_i)\nu_B(x_i)}{\max\left(\mu^2_A(x_i) + \eta^2_A(x_i) + \nu^2_A(x_i), \mu^2_B(x_i) + \eta^2_B(x_i) + \nu^2_B(x_i)\right)} \quad (6)$$

Obviously, equation(6) satisfies the three properties of the similarity measures as follows:

(1) $0 \leqslant C^2_{PFS}(A, B) \leqslant 1$;

(2) $C^2_{PFS}(A, B) = C^2_{PFS}(B, A)$;

(3) $C^2_{PFS}(A, B) = 1,\ if\ A = B, i = 1, 2, \cdots, n.$

(4) if $A \subseteq B \subseteq C$,then $C^2_{PFS}(A, C) \leqslant C^2_{PFS}(A, B)$, $C^2_{PFS}(A, C) \leqslant C^2_{PFS}(B, C)$.

If we consider the weights of x_i, a weighted set-theoretic similarity measure between PFSs A and B is proposed as follows:

$$W^2_{PFS}(A, B) = \sum_{i=1}^{n} w_i \frac{\mu_A(x_i)\mu_B(x_i) + \eta_A(x_i)\eta_B(x_i) + \nu_A(x_i)\nu_B(x_i)}{\max\left(\mu^2_A(x_i) + \eta^2_A(x_i) + \nu^2_A(x_i), \mu^2_B(x_i) + \eta^2_B(x_i) + \nu^2_B(x_i)\right)} \quad (7)$$

where$w = (w_1, w_2, \cdots, w_n)^T$ is the weight vector of $x_i\,(i = 1, 2, \cdots, n)$, with $w_i \in [0, 1]$, $i = 1, 2, \cdots, n$, $\sum\limits_{i=1}^{n} w_i = 1$. In particular, if $w = (1/n, 1/n, \cdots, 1/n)^T$, then equation(7) reduces to equation(6).

Obviously, the weighted set-theoretic similarity measure of two PFSs A and B also satisfies the following properties:

(1) $0 \leqslant W^2_{PFS}(A,B) \leqslant 1$,

(2) $W^2_{PFS}(A,B) = W^2_{PFS}(B,A)$,

(3) $W^2_{PFS}(A,B) = 1,\ if\ A = B, i = 1,2,\cdots,n$.

3.3. **Grey Similarity Measure for Picture Fuzzy Sets.**

Assume that there are two PFSs A and B in $X = \{x_1, x_2, \cdots, x_n\}$. In what follows, we shall propose grey similarity measure and a weighted grey similarity measure between PFSs based on the concept of the grey relational analysis[47].

Assume that there are two PFSs A and B in $X = \{x_1, x_2, \cdots, x_n\}$. Based on the extension of the grey relational analysis, a grey similarity measure between PIFSs A and B is proposed as follows:

$$C^3_{PFS}(A,B) = \frac{1}{3n}\sum_{i=1}^{n}\left(\frac{\Delta\mu_{\min}+\Delta\mu_{\max}}{\Delta\mu_i+\Delta\mu_{\max}} + \frac{\Delta\eta_{\min}+\Delta\eta_{\max}}{\Delta\eta_i+\Delta\eta_{\max}} + \frac{\Delta\nu_{\min}+\Delta\nu_{\max}}{\Delta\nu_i+\Delta\nu_{\max}}\right) \quad (8)$$

where $\Delta\mu_i = |\mu_A(x_i) - \mu_B(x_i)|$, $\Delta\mu_{\min} = \min_i\{|\mu_A(x_i) - \mu_B(x_i)|\}$,
$\Delta\mu_{\max} = \max_i\{|\mu_A(x_i) - \mu_B(x_i)|\}$, $\Delta\eta_i = |\eta_A(x_i) - \eta_B(x_i)|$,
$\Delta\eta_{\min} = \min_i\{|\eta_A(x_i) - \eta_B(x_i)|\}$, $\Delta\eta_{\max} = \max_i\{|\eta_A(x_i) - \eta_B(x_i)|\}$,
$\Delta\nu_i = |\nu_A(x_i) - \nu_B(x_i)|$, $\Delta\nu_{\min} = \min_i\{|\nu_A(x_i) - \nu_B(x_i)|\}$,
$\Delta\nu_{\max} = \max_i\{|\nu_A(x_i) - \nu_B(x_i)|\}$.

Obviously, the greater the value of $C_{PFS}(A,B)$, the closer A to B. By equation(8), the grey similarity measure $C_{PFS}(A,B)$ satisfies the following properties:

(1) $0 \leqslant C^3_{PFS}(A,B) \leqslant 1$;

(2) $C^3_{PFS}(A,B) = C^3_{PFS}(B,A)$;

(3) $C^3_{PFS}(A,B) = 1,\ if\ A = B$,$i = 1,2,\cdots,n$.

(4) if $A \subseteq B \subseteq C$,then $C^3_{PFS}(A,C) \leqslant C^3_{PFS}(A,B)$, $C^3_{PFS}(A,C) \leqslant C^3_{PFS}(B,C)$.

In many situations, the weight of the elements $x_i \in X$ should be taken into account. For example, in multiple attribute decision making, the considered attributes usually have different importance, and thus need to be assigned different weights. As a result, a weighted cosine similarity measure between PFSs A and B is proposed as follows:

$$W^3_{PFS}(A,B) = \frac{1}{3}\sum_{i=1}^{n} w_i\left(\frac{\Delta\mu_{\min}+\Delta\mu_{\max}}{\Delta\mu_i+\Delta\mu_{\max}} + \frac{\Delta\eta_{\min}+\Delta\eta_{\max}}{\Delta\eta_i+\Delta\eta_{\max}} + \frac{\Delta\nu_{\min}+\Delta\nu_{\max}}{\Delta\nu_i+\Delta\nu_{\max}}\right) \quad (9)$$

where $\Delta\mu_i = |\mu_A(x_i) - \mu_B(x_i)|$, $\Delta\mu_{\min} = \min_i\{|\mu_A(x_i) - \mu_B(x_i)|\}$,
$\Delta\mu_{\max} = \max_i\{|\mu_A(x_i) - \mu_B(x_i)|\}$, $\Delta\eta_i = |\eta_A(x_i) - \eta_B(x_i)|$,
$\Delta\eta_{\min} = \min_i\{|\eta_A(x_i) - \eta_B(x_i)|\}$, $\Delta\eta_{\max} = \max_i\{|\eta_A(x_i) - \eta_B(x_i)|\}$
$\Delta\nu_i = |\nu_A(x_i) - \nu_B(x_i)|$, $\Delta\nu_{\min} = \min_i\{|\nu_A(x_i) - \nu_B(x_i)|\}$,
$\Delta\nu_{\max} = \max_i\{|\nu_A(x_i) - \nu_B(x_i)|\}$ and $w = (w_1, w_2, \cdots, w_n)^T$ is the weight vector of $x_i\,(i = 1,2,\cdots,n)$,with $w_i \in [0,1]$,$i = 1,2,\cdots,n$, $\sum_{i=1}^{n} w_i = 1$.

	A_1	A_2	A_3	A_4	A
x_1	$(0.17, 0.53, 0.13)$	$(0.51, 0.24, 0.21)$	$(0.31, 0.39, 0.25)$	$(1.00, 0.00, 0.00)$	$(0.91, 0.03, 0.05)$
x_2	$(0.10, 0.81, 0.05)$	$(0.62, 0.12, 0.07)$	$(0.60, 0.26, 0.11)$	$(1.00, 0.00, 0.00)$	$(0.78, 0.12, 0.07)$
x_3	$(0.53, 0.33, 0.09)$	$(1.00, 0.00, 0.00)$	$(0.91, 0.03, 0.02)$	$(0.85, 0.09, 0.05)$	$(0.90, 0.05, 0.02)$
x_4	$(0.89, 0.08, 0.03)$	$(0.13, 0.64, 0.21)$	$(0.07, 0.09, 0.05)$	$(0.74, 0.16, 0.10)$	$(0.68, 0.08, 0.21)$
x_5	$(0.42, 0.35, 0.18)$	$(0.03, 0.82, 0.13)$	$(0.04, 0.85, 0.10)$	$(0.02, 0.89, 0.05)$	$(0.05, 0.87, 0.06)$
X_6	$(0.08, 0.89, 0.02)$	$(0.73, 0.15, 0.08)$	$(0.68, 0.26, 0.06)$	$(0.08, 0.84, 0.06)$	$(0.13, 0.75, 0.09)$
X_7	$(0.33, 0.51, 0.12)$	$(0.52, 0.31, 0.16)$	$(0.15, 0.76, 0.07)$	$(0.16, 0.71, 0.05)$	$(0.15, 0.73, 0.08)$

Table 1. The Data on Building Materials

similarity measures	(A_1, A)	(A_2, A)	(A_3, A)	(A_4, A)
$W^1_{PFS}(A_i, A)$	0.716	0.763	0.858	0.994
$W^2_{PFS}(A_i, A)$	0.556	0.657	0.693	0.920
$W^3_{PFS}(A_i, A)$	0.660	0.762	0.830	0.901

Table 2. The Similarity Measures Between $A_i\,(i = 1, 2, 3, 4)$ and A

In particular, if $w = (1/n, 1/n, \cdots, 1/n)^T$, then the weighted grey similarity measure reduces to grey similarity measure. That is to say, if we take $w_i = \frac{1}{n}, i = 1, 2 \cdots, n$, then there is $W^3_{PFS}(A, B) = C^3_{PFS}(A, B)$.

Obviously, the weighted grey similarity measure of two PFSs A and B also satisfies the following properties:

(1) $0 \leqslant W^3_{PFS}(A, B) \leqslant 1$,

(2) $W^3_{PFS}(A, B) = W^3_{PFS}(B, A)$,

(3) $W^3_{PFS}(A, B) = 1,\ if\ A = B, i = 1, 2, \cdots, n$.

4. **Applications**

In this section, the similarity measures for PFSs are applied to building material recognition and minerals field recognition(adapted from[49]).

4.1. **Example1- building Materials Recognition.**

Let us consider four building materials: sealant, floor varnish, wall paint and polyvinyl chloride flooring, which are represented by the PFSs $A_i\,(i = 1, 2, 3, 4)$ in the feature space $X = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$. The weight vector of $x_i(i = 1, 2, \cdots, 7)$ is: $w = (0.12, 0.15, 0.09, 0.16, 0.20, 0.10, 0.18)^{\mathrm{T}}$

Now, we consider another kind of unknown building material A, with data as listed in Table 1. Based on the weight vector w and the data in Table 1, we can use the above similarity measures to identify to which type the unknown material A belongs. According to the recognition principle of maximum degree of similarity between IFSs proposed by Li and Cheng[12], the process of assigning A to A_k is described by

$$k = \arg \underset{1 \leqslant i \leqslant 4}{Max} \{W_{PFS}(A_i, A)\}$$

	A_1	A_2	A_3	A_4	
x_1	$(0.53, 0.33, 0.09)$	$(1.00, 0.00, 0.00)$	$(0.91, 0.03, 0.02)$	$(0.85, 0.09, 0.05)$	$(0.90, 0.05, 0.02)$
x_2	$(0.89, 0.08, 0.03)$	$(0.13, 0.64, 0.21)$	$(0.07, 0.09, 0.05)$	$(0.74, 0.16, 0.10)$	$(0.68, 0.08, 0.21)$
x_3	$(0.42, 0.35, 0.18)$	$(0.03, 0.82, 0.13)$	$(0.04, 0.85, 0.10)$	$(0.02, 0.89, 0.05)$	$(0.05, 0.87, 0.06)$
x_4	$(0.08, 0.89, 0.02)$	$(0.73, 0.15, 0.08)$	$(0.68, 0.26, 0.06)$	$(0.08, 0.84, 0.06)$	$(0.13, 0.75, 0.09)$
x_5	$(0.33, 0.51, 0.12)$	$(0.52, 0.31, 0.16)$	$(0.15, 0.76, 0.07)$	$(0.16, 0.71, 0.05)$	$(0.15, 0.73, 0.08)$
X_6	$(0.17, 0.53, 0.13)$	$(0.51, 0.24, 0.21)$	$(0.31, 0.39, 0.25)$	$(1.00, 0.00, 0.00)$	$(0.91, 0.03, 0.05)$

TABLE 3. The Data on Minerals

similarity measures	(A_1, A)	(A_2, A)	(A_3, A)	(A_4, A)
$W^1_{PFS}(A_i, A)$	0.813	0.656	0.787	0.994
$W^2_{PFS}(A_i, A)$	0.634	0.559	0.576	0.935
$W^3_{PFS}(A_i, A)$	0.696	0.700	0.793	0.913

TABLE 4. The Similarity Measures Between $A_i\,(i = 1, 2, 3, 4)$ and A

In the above numerical results in Table 2, all the similarity measures derive the same ranking, in which the degree of similarity between A_4 and A is the largest one, the degree of similarity between A_3 and A ranks the second, the degree of similarity between A_2 and A ranks the third, the degree of similarity between A_1 and A is the smallest one. Therefore, the building material A should belong to the class of building material A_4 according to the principle of the maximum degree of similarity between PFSs.

4.2. **Example 2-mineral Fields Recognition.**

Let us consider four kinds of mineral fields, which are represented by PFSs $A_i\,(i = 1, 2, 3, 4)$. Each of which is featured by the content of six minerals in the feature space $X = \{x_1, x_2, x_3, x_4, x_5, x_6\}$. The weight vector of $x_i\,(i = 1, 2, \cdots, 6)$ is: $w = (0.12, 0.25, 0.09, 0.16, 0.20, 0.18)^{\mathrm{T}}$.

Now, we consider another kind of unknown mineral A, with data as listed in Table 3. Based on the weight vector w and the data in Table 3, we can use the above similarity measures to identify to which type the unknown material A should belong. According to the recognition principle of maximum degree of similarity between IFSs proposed by Li and Cheng[12], the process of assigning A to A_k is described by

$$k = \arg \underset{1 \leqslant i \leqslant 4}{Max} \{W_{PFS}(A_i, A)\}$$

From the above numerical results in Table 4, we know that the degree of similarity between A_4 and A is the largest one as derived by three similarity measures. That is, all the three similarity measures assign the unknown mineral A to the class of mineral field A_4 according to the principle of the maximum degree of similarity between PFSs. Yet, there exist two slightly different ranking results: the cosine similarity measure and set-theoretic similarity measures derive the same ranking of the mineral fields, in which the degree of similarity between A_1 and A ranks the second, the degree of similarity between A_3 and A ranks the third, the degree of similarity between A_2 and A is the smallest one. While for the grey similarity measure, the degree of similarity between A_2 and A ranks the second, the degree of similarity between A_2 and A ranks the third, the degree of similarity between A_1 and A is the smallest one.

4.3. **Comparison Studies.**

The cross entropy of picture fuzzy sets, called picture fuzzy cross entropy[35], is proposed as an extension of the cross entropy of fuzzy sets. In order to show my proposed model effectively, in the following, we shall compare the proposed method with picture fuzzy cross entropy method which was proposed by Wei[35].

For Example 1, by using the picture fuzzy cross entropy method, we can calculate the cross-entropy $C_\omega (A_i, A)$ between $A_i\,(i = 1, 2, 3, 4)$ and A by using equation(18) in Ref.[50]: $C_\omega (A_1, A) = 0.219, C_\omega (A_2, A) = 0.150, C_\omega (A_3, A) = 0.117, C_\omega (A_4, A) = 0.021$

The smaller the value of $C (A_i, A)$ is, the alternative is closer A_i to A. The picture fuzzy cross entropy between A_1 and A is the largest one, the picture fuzzy cross entropy between A_2 and A ranks the second, the picture fuzzy cross entropy between A_3 and A ranks the third, the picture fuzzy cross entropy between A_4 and A is the smallest one. Therefore, the building material should belong to the class of building material A_4 according to the principle of the minimum picture fuzzy cross entropy between PFSs.

For Example 2, by using the picture fuzzy cross entropy method, we can calculate the cross-entropy $C_\omega (A_i, A)$ between $A_i\,(i = 1, 2, 3, 4)$ and the A by using equation(18) in Ref.[35]: $C_\omega (A_1, A) = 0.155, C_\omega (A_2, A) = 0.214, C_\omega (A_3, A) = 0.173, C_\omega (A_4, A) = 0.014$ The smaller the value of $C (A_i, A)$ is, the alternative is closer $A_i$to A. The picture fuzzy cross entropy between A_2 and A is the largest one, the picture fuzzy cross entropy between A_3 and A ranks the second, the picture fuzzy cross entropy between A_1 and A ranks the third, the picture fuzzy cross entropy between A_4 and A is the smallest one. Therefore, the unknown mineral A should belong to the class of mineral field A_4according to the principle of the minimum picture fuzzy cross entropy between PFSs.

From the above analysis, it can be seen that the proposed model is effective.

4.4. **Advantages of the Proposed Method.**

(1) As mentioned above, the existing similarity measures for intuitionistic fuzzy set have some limitations and are not able to represent the full information about the situation. Picture fuzzy set is a further generalization of the intuitionistic fuzzy set. So the PFS contains more information (degree of positive membership, degree of neutral membership, degrees of negative membership and degrees of refusal membership) than intuitionistic fuzzy set (both membership degree and nonmembership degree). Thus, the proposed similarity measures for picture fuzzy set can be considered as a further generalization of the similarity measures of intuitionistic fuzzy set [50]. Also the proposed similarity measures reflect the amount of information expressed by the degree of positive membership, neutral membership and negative membership and the reliability of the information expressed by refusal membership.

(2) The similarity measures for intuitionistic fuzzy set are special cases of the similarity measures of picture fuzzy set. Therefore, similarity measures proposed in this paper can be used to find not only the similarity measures for the problems with picture fuzzy set but also the similarity measures of the problems with intuitionistic fuzzy set, whereas the method in [50] is only suitable to find the similarity measures for intuitionistic fuzzy set.

5. **Conclusion**

In this paper, we presented some novel process to measure the similarity between PFSs. Firstly, we adopt the concept of intuitionistic fuzzy sets, interval-valued intuitionistic fuzzy sets and picture fuzzy sets. Secondly, we develop some similarity measures between picture fuzzy sets, such as, cosine similarity measure, weighted cosine similarity measure, set-theoretic similarity measure, weighted set-theoretic cosine similarity measure, grey similarity measure and weighted grey similarity measure. Then, we applied these similarity measures between PFSs to building material recognition and minerals field recognition. Finally, two illustrative examples are given to demonstrate the efficiency of the similarity measures for building material recognition and minerals field recognition. In the future, the pattern recognition application of the proposed similarity measure of PFSs needs to be explored on the basis of the similarity measures[11, 16-18, 34, 36-43, 53-57].

Acknowledgements. The work is supported by National Natural Science Foundation of China under Grants No. 61174149 and 71571128 and Humanities and Social Sciences Foundation of Ministry of Education of Peoples Republic of China (No.16XJA630005, 15YJCZH138) and the construction plan of scientific research innovation team for colleges and universities in Sichuan Province (15TD0004).

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