THE CHAIN PROPERTIES AND LI-YORKE SENSITIVITY OF ZADEH'S EXTENSION ON THE SPACE OF UPPER SEMI-CONTINUOUS FUZZY SETS

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Abstract. Some characterizations on the chain recurrence, chain transitivity, chain mixing property, shadowing and h-shadowing for Zadeh's extension are obtained. Besides, it is proved that a dynamical system is spatiotemporally chaotic provided that the Zadeh's extension is Li-Yorke sensitive.

1. Introduction

A *dynamical system* is a pair (X, T), where X is a nontrivial compact metric space with a metric d and $T : X \longrightarrow X$ is a continuous surjection. A nonempty invariant closed subset $Y \subset X$ (i.e., $T(Y) \subset Y$) defines naturally a *subsystem* $(Y, T|_Y)$ of (X, T). Throughout this paper, let $\mathbb{N} = \{1, 2, 3, \ldots\}$ and $\mathbb{Z}^+ = \{0, 1, 2, \ldots\}$. For any $n \in \mathbb{N}$, let $(X^n, T^{(n)})$ be the *n-fold product system* $(\underbrace{X \times \cdots \times X}_{n \text{ times}}, \underbrace{T \times \cdots \times T}_{n \text{ times}})$.

Given a dynamical system (X, T), one can naturally obtain some associated systems induced by (X, T). The first one is $(M(X), T_M)$ on the space $M(X)$ consisting of all Borel probability measures with the Prohorov metric, which can be viewed as statistical states, representing imperfect knowledge of the system. Its topological parallelism is $(K(X), T_K)$ on the hyperspace $K(X)$ consisting of all nonempty closed subsets of X with the Hausdorff metric. And the third one is the Zadeh's extension $(\mathbb{F}_0(X), T_F)$ (more generally g-fuzzification $(\mathbb{F}_0(X), T_F^g)$ which was introduced by Kupka [16] to generalize Zadeh's extension) on the space $\mathbb{F}_0(X)$ consisting of all nonempty upper semi-continuous fuzzy sets with the level-wise metric induced by the extended Hausdorff metric. A systematic study on the connections between dynamical properties of (X, T) and its two induced systems $(K(X), T_K)$ and $(M(X), T_M)$ was initiated by Bauer and Sigmund in [5], and later has been widely developed by several authors. For more results on this topic, one is referred

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to [3, 9, 10, 11, 12, 13, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 28, 29, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40] and references therein. For $n \in \mathbb{N}$, denote

$$K_n(X) = \{A \in K(X) : |A| \leq n\} \text{ and } K_\infty(X) = \cup_{n\in\mathbb{N}} K_n(X).$$

Definition 1.1. [8, 14] A dynamical system (X, T) is

(1) *exact* if for any nonempty open subset $U \subset X$, there exists $n \in \mathbb{Z}^+$ such that $T^n(U) = X$;
(2) *transitive* if for any pair of nonempty open subsets $U, V \subset X$, there exists $n \in \mathbb{Z}^+$ such that $T^n(U) \cap V \neq \varnothing$;
(3) *totally transitive* if T^n is transitive for any $n \in \mathbb{N}$;
(4) *weakly mixing* if $(X^2, T^{(2)})$ is transitive;
(5) *mildly mixing* if for every transitive system (Y, S), $(X \times Y, T \times S)$ is transitive;
(6) *topologically mixing* if for any pair of nonempty open subsets $U, V \subset X$, there exists $m \in \mathbb{Z}^+$ such that for any $n \geq m$, $T^n(U) \cap V \neq \varnothing$.

Banks [3] proved that (X, T) is weakly mixing if and only if $(K(X), T_K)$ is transitive, which is equivalent to the weakly mixing property of $(K(X), T_K)$ (also see [23]). Román-Flores and Chalco-Cano [26, 27] studied some chaotic properties (for example, transitivity, turbulence, sensitive dependence, periodic density) for the Zadeh's extension. Wang and Wei [30] studied the dynamics of Zadeh's extension in the hit-or-miss topology. Then, Kupka [15] investigated the relations between Devaney chaos in the original system and in the Zadeh's extension and proved the following:

Lemma 1.2. [15, Lemma 1, Remark 1, Theorem 1] *Let (X, T) be a dynamical system and $\lambda \in (0, 1]$. Then,*

(1) the set of piecewise constants is dense in $\mathbb{F}(X)$, $\mathbb{F}^{\geq\lambda}(X)$ and $\mathbb{F}^{=\lambda}(X)$.
(2) $(\mathbb{F}^{=\lambda}(X), T_F|_{\mathbb{F}^{=\lambda}(X)})$ is periodically dense in $\mathbb{F}^{=\lambda}(X)$ if and only if $(K(X), T_K)$ is periodically dense in $K(X)$.

Kupka [17] continued in studying chaotic properties (for example, Li-Yorke chaos, distributional chaos, ω-chaos, transitivity, total transitivity, exactness, sensitive dependence, weakly mixing, mildly mixing, topologically mixing) of g-fuzzification and showed that if the g-fuzzification $(\mathbb{F}^{=1}(X), T_F^g)$ has the property P, then (X, T) also has the property P, where P denotes the following properties: exactness, sensitive dependence, weakly mixing, mildly mixing, or topologically mixing. Meanwhile, he asked that does the P-property of (X, T) imply the P-property of $(\mathbb{F}^{=1}(X), T_F^g)$? To negatively answer this question, we [31] obtained a sufficient condition on $g \in D_m(I)$[1] to ensure that for every dynamical system (X, T), its g-fuzzification $(\mathbb{F}^{=1}(X), T_F^g)$ is not transitive (thus, not weakly mixing) and constructed a sensitive dynamical system whose g-fuzzification is not sensitive for any $g \in D_m(I)$.

Very recently, Fernández and Good [10] proved the following result:

[1] $D_m(I)$ is the set of all nondecreasing right-continuous functions $g : I \longrightarrow I$ with $g(0) = 0$ and $g(1) = 1$.

Lemma 1.3. [10, Theorem 3.3, Theorem 3.4] *Let (X, T) be a dynamical system. Then, the following statements are equivalent:*

(1) (X, T) has shadowing;
(2) $(K_\infty(X), T_K|_{K_\infty(X)})$ has finite shadowing;
(3) $(K(X), T_K)$ has shadowing.

Lemma 1.4. [10, Theorem 4.5, Theorem 4.6] *Let (X, T) be a dynamical system. Then, the following statements are equivalent:*

(1) (X, T) has h-shadowing;
(2) $(K_\infty(X), T_K|_{K_\infty(X)})$ has h-shadowing;
(3) $(K(X), T_K)$ has h-shadowing.

Remark 1.5. Gómez-Rueda et al. [11] proved that if $(K_n(X), T_K|_{K_n(X)})$ has shadowing, then (X, T) has shadowing and obtained that if (X, T) has shadowing, then $(K_2(X), T_K|_{K_2(X)})$ has shadowing. They also constructed a dynamical system (X, T) for which T has shadowing but $(K_n(X), T_K|_{K_n(X)})$ does not have shadowing for any $n \geq 3$.

In this paper, we further investigate chain recurrence, chain transitivity, shadowing, h-shadowing, and Li-Yorke sensitivity of the Zadeh's extension through further developing the results in [9, 10, 13, 28, 42]. In particular, we prove that a dynamical system has chain recurrence (resp., chain mixing property, shadowing, h-shadowing) if and only if the Zadeh's extension has chain recurrence (resp., chain mixing property, finite shadowing, h-shadowing) and obtain that if the Zadeh's extension is Li-Yorke sensitive, then the dynamical system is spatiotemporally chaotic.

2. Basic definitions and notations

2.1. **Hyperspace $K(X)$.** For any $\delta > 0$ and any nonempty closed subset $A \subset X$, the *δ-neighborhood of* A is given by $B(A, \delta) = \{y \in X : \inf\{d(x, y) : x \in A\} < \delta\}$. In particular, when $A = \{x\}$, denote this by $B(x, \delta)$ instead of $B(\{x\}, \delta)$. Let $K(X)$ be the hyperspace on X, i.e., the space of all nonempty closed subsets of X with the Hausdorff metric d_H defined by

$$\begin{aligned} d_H(A_1, A_2) &= \max\left\{\max_{x \in A_1} \min_{y \in A_2} d(x, y), \max_{y \in A_2} \min_{x \in A_1} d(x, y)\right\} \\ &= \inf\left\{\varepsilon : A_1 \subset B(A_2, \varepsilon) \text{ and } A_2 \subset B(A_1, \varepsilon)\right\}, \quad \forall A_1, A_2 \in K(X). \end{aligned}$$

It is known that $(K(X), d_H)$ is also a compact metric space (see [12]).

A dynamical system (X, T) induces naturally a set-valued dynamical system $(K(X), T_K)$, where $T_K : K(X) \longrightarrow K(X)$ is defined as $T_K(A) = T(A)$ for any $A \in K(X)$. For any finite collection $A_1, \ldots, A_n$ of nonempty subsets of X, take

$$\langle A_1, \ldots, A_n \rangle = \left\{A \in K(X) : A \subset \bigcup_{i=1}^{n} A_i, A \cap A_i \neq \varnothing \text{ for all } i = 1, \ldots, n\right\}.$$

It follows from [12] that the topology on $K(X)$ given by the metric d_H is same as the Vietoris or finite topology, which is generated by a basis consisting of all sets of the following form,

$$\langle U_1, \dots, U_n \rangle, \qquad \text{where } U_1, \dots, U_n \text{ are an arbitrary finite collection of nonempty open subsets of } X.$$

Clearly, under this topology $K_\infty(X)$ is a dense invariant subset of $K(X)$.

2.2. **Zadeh's extension.** Let $I = [0, 1]$. A *fuzzy set* A on space X is a function $A : X \longrightarrow I$. Given a fuzzy set A, its *α-cuts* (or *α-level sets*) $[A]_\alpha$ and *support* $\mathrm{supp}(A)$ are defined respectively by

$$[A]_\alpha = \{x \in X : A(x) \geq \alpha\}, \quad \forall \alpha \in I,$$

and

$$\mathrm{supp}(A) = \overline{\{x \in X : A(x) > 0\}}.$$

Let $\mathbb{F}(X)$ denote the set of all upper semicontinuous fuzzy sets defined on X and set

$$\mathbb{F}^{\geq \lambda}(X) = \{A \in \mathbb{F}(X) : A(x) \geq \lambda \text{ for some } x \in X\},$$
$$\mathbb{F}^{= \lambda}(X) = \{A \in \mathbb{F}(X) : \max A := \max\{A(x) : x \in X\} = \lambda\}.$$

For any $A \in \mathbb{F}(X)$, let $\xi = \sup\{A(x) : x \in X\}$. It is easy to see that there exists a sequence $\{x_n\}_{n=1}^\infty \subset X$ such that $\xi - \frac{1}{n} < A(x_n) \leq \xi$. Assume that $\lim_{n \to \infty} x_n = z$. The upper semicontinuity of A implies that $\xi \geq A(z) \geq \limsup_{n \to \infty} A(x_n) = \xi$, i.e., $\max A = \sup\{A(x) : x \in X\}$. Especially, let $\mathbb{F}^{=1}(X)$ denote the system of all normal fuzzy sets on X.

Define $\emptyset_X$ as the empty fuzzy set ($\emptyset_X \equiv 0$) in X, and $\mathbb{F}_0(X)$ as the set of all nonempty upper semicontinuous fuzzy sets. Since the Hausdorff metric d_H is measured only between two nonempty closed subsets in X, one can consider the following extension of the Hausdorff metric:

$$d_H(\varnothing, \varnothing) = 0 \text{ and } d_H(\varnothing, A) = d_H(A, \varnothing) = \mathrm{diam}(X), \quad \forall A \in K(X).$$

Under this Hausdorff metric, one can define a *levelwise metric* d_∞ on $\mathbb{F}(X)$ by

$$d_\infty(A, B) = \sup\{d_H([A]_\alpha, [B]_\alpha) : \alpha \in (0, 1]\}, \quad \forall A, B \in \mathbb{F}(X).$$

It is well known that the spaces $(\mathbb{F}(X), d_\infty)$ and $(\mathbb{F}^{=1}(X), d_\infty)$ are complete, but not compact and not separable (see [16] and references therein).

A fuzzy set $A \in \mathbb{F}(X)$ is *piecewise constant* if there exists a finite number of sets $D_i \subset X$ such that $\bigcup \overline{D_i} = X$ and $A|_{\mathrm{int}\overline{D_i}}$ is constant. In this case, a piecewise constant A can be represented by a strictly decreasing sequence of closed subsets $\{A_1, A_2, \dots, A_k\} \subset K(X)$ and a strictly increasing sequence of reals $\{\alpha_1, \alpha_2, \dots, \alpha_k = \max\{A(x) : x \in X\}\} \subset (0, 1]$ if

$$[A]_\alpha = A_{i+1}, \text{ whenever } \alpha \in (\alpha_i, \alpha_{i+1}].$$

Remark 2.1. Fix any two piecewise constants $A, B \in \mathbb{F}(X)$ which are represented by strictly decreasing sequences of closed subsets $\{A_1, A_2, \dots, A_k\}$, $\{B_1, B_2, \dots, B_s\} \subset K(X)$ and strictly increasing sequences of reals $\{\alpha_1, \alpha_2, \dots, \alpha_k\}$, $\{\beta_1, \beta_2, \dots, \beta_s\} \subset (0, 1]$ with

$$[A]_\alpha = A_{i+1}, \ \forall \alpha \in (\alpha_i, \alpha_{i+1}] \text{ and } [B]_\alpha = B_{i+1}, \ \forall \beta \in (\beta_i, \beta_{i+1}],$$

respectively. Arrange all reals $\alpha_1, \alpha_2, \ldots, \alpha_k, \beta_1, \beta_2, \ldots, \beta_s$ by the natural order '$<$' and denote them by $\gamma_1, \gamma_2, \ldots, \gamma_n$ $(n \leq k+s)$. Then, it can be verified that for any $1 \leq t < n$, there exist $1 \leq i \leq k$ and $1 \leq j \leq s$ such that for any $\gamma \in (\gamma_t, \gamma_{t+1}]$,

$$[A]_\gamma = A_i \text{ and } [B]_\gamma = B_j.$$

This implies that there exist (not necessarily strictly) decreasing sequences of closed subsets $\{C_1, C_2, \ldots, C_n\}$, $\{D_1, D_2, \ldots, D_n\} \subset K(X)$ and a strictly increasing sequence of reals $\gamma_1, \gamma_2, \ldots, \gamma_n \subset (0,1]$ such that

$$[A]_\gamma = C_{i+1} \text{ and } B_\gamma = D_{i+1}, \text{ whenever } \gamma \in (\gamma_i, \gamma_{i+1}].$$

Zadeh's extension (also called *usual fuzzification*) of a dynamical system (X, T) is a map $T_F : \mathbb{F}(X) \longrightarrow \mathbb{F}(X)$ defined by

$$T_F(A)(x) = \sup\left\{A(y) : y \in T^{-1}(x)\right\}, \ \forall A \in \mathbb{F}(X), \forall x \in X.$$

Remark 2.2. For any $A, B \in K(X)$, any $n \in \mathbb{N}$ and any $\lambda \in (0,1]$,

$$d_H(T_K^n(A), T_K^n(B)) = d_\infty(T_F^n(\lambda \cdot \chi_A), T_F^n(\lambda \cdot \chi_B)). \tag{1}$$

This shows that for any fixed $\lambda \in (0,1]$, the subsystem $(\mathbb{F}_{\lambda\cdot\chi} := \{\lambda \cdot \chi_A \in \mathbb{F}_0(X) : A \in K(X)\}, T_F|_{\mathbb{F}_{\lambda\cdot\chi}})$ is topologically conjugated to $(K(X), T_K)$. Meanwhile, it is not difficult to check that $(\mathbb{F}^{=\lambda}(X), T_F|_{\mathbb{F}^{=\lambda}(X)})$ is topologically conjugated to $(\mathbb{F}^{=1}(X), T_F|_{\mathbb{F}^{=1}(X)})$ for any $\lambda \in (0,1]$.

3. Chain properties and shadowing for Zadeh's extension

3.1. **Chain properties for Zadeh's extension.** For any $\delta > 0$, a *δ-pseudo-orbit* is a finite or infinite sequence $\{x_0, x_1, \ldots\}$ such that $d(T(x_i), x_{i+1}) < \delta$ for any $i \geq 0$. A finite δ-pseudo-orbit $\Gamma = \{x_0, x_1, \ldots, x_n\}$ is also called a *δ-chain* from x_0 to x_n of length $|\Gamma| = n$. A map is *chain transitive* if for any $\delta > 0$ and any two points $x, y \in X$, there exists a δ-chain from x to y. Chain transitivity is a natural generalization of transitivity. There is a surprising result [25] which shows that chains do not distinguish between total transitivity and mixing. Precisely speaking, if (X, T^n) is chain transitive for all $n \in \mathbb{N}$ then it is *chain mixing*, that is, for any $x, y \in X$ and any $\delta > 0$, there exists $N \in \mathbb{N}$ such that there exists a δ-chain from x to y of length n for every $n \geq N$. Recently, we [41] also proved that this also holds for iterated function systems.

A point $x \in X$ is *chain recurrent* if for every $\delta > 0$ there exists a δ-chain from x to x (see [7]). The set of all chain recurrent points is denoted $\mathrm{CR}(T)$. By compactness, it is not difficult to check that $\mathrm{CR}(T)$ is a closed set and $T(\mathrm{CR}(T)) = \mathrm{CR}(T)$. If $\mathrm{CR}(T) = X$, then (X, T) is *chain recurrent.* The chain recurrence which was introduced by Conley [7] as a generalization of non-wandering property is a way of describing the recurrent phenomena of dynamical systems.

If $\Gamma_1 = \{x_0, x_1, \ldots, x_n\}$ and $\Gamma_2 = \{y_0, y_1, \ldots, y_m\}$ are two δ-chains such that $x_n = y_0$, then $\Gamma_1 + \Gamma_2$ denotes the *concatenation* of Γ_1 with Γ_2,

$$\Gamma_1 + \Gamma_2 = \{x_0, x_1, \ldots, x_n = y_0, y_1, \ldots, y_m\}.$$

Clearly, $\Gamma_1 + \Gamma_2$ is an δ-chain and $|\Gamma_1 + \Gamma_2| = |\Gamma_1| + |\Gamma_2|$. In particular, if Γ is an δ-chain from a point to itself and $m \in \mathbb{N}$, $m\Gamma$ denotes $\underbrace{\Gamma + \Gamma + \cdots + \Gamma}_{m \text{ times}}$.

Lemma 3.1. [10, Lemma 3.1] *Let (X, T) be a dynamical system and Y be a dense invariant subset of X. Then, (X, T) has finite shadowing if and only if $(Y, T|_Y)$ has finite shadowing.*

Lemma 3.2. *Let (X, T) be a dynamical system and Y be a dense invariant subset of X. Then, (X, T) is chain recurrent (resp., chain transitive, chain mixing) if and only if $(Y, T|_Y)$ is chain recurrent (resp., chain transitive, chain mixing).*

Proof. The proof of this theorem is Similarly to Lemma 3.1 in [10]. For completeness, we give the proof for chain recurrence.

($\Longrightarrow$). For any $\delta > 0$ and any $x \in Y$, as (X, T) is chain recurrent, there exists a $\frac{\delta}{3}$-chain $\{x, x_1, \ldots, x_{n-1}, x_n = x\}$. The uniform continuity of (X, T) implies that there exists $0 < \varepsilon < \frac{\delta}{3}$ such that for any $y, z \in X$ with $d(y, z) < \varepsilon$, $d(T(y), T(z)) < \frac{\delta}{3}$. As Y is dense in X, then for any $1 \leq i \leq n-1$, there exists $x_i' \in Y$ such that $d(x_i', x_i) < \varepsilon$. This implies that

$$\begin{aligned} d(T(x_i'), x_{i+1}') &\leq d(T(x_i'), T(x_i)) + d(T(x_i), x_{i+1}) + d(x_{i+1}, x_{i+1}') \\ &< \frac{\delta}{3} + \frac{\delta}{3} + \varepsilon < \delta. \end{aligned}$$

Therefore, $\{x, x_1', \ldots, x_{n-1}', x\}$ is a δ-chain in Y, i.e., $(Y, T|_Y)$ is chain recurrent.

($\Longleftarrow$). For any $\delta > 0$ and any $x \in X$, there exists $0 < \varepsilon < \frac{\delta}{2}$ such that for any $y, z \in X$ with $d(y, z) < \varepsilon$, $d(T(y), T(z)) < \frac{\delta}{2}$. As Y is dense in X, there exists $y \in Y$ such that $d(x, y) < \varepsilon$. The chain recurrence of $(Y, T|_Y)$ implies that there a $\frac{\delta}{2}$-chain $\{y, y_1, \ldots, y_{y-1}, y\}$ in Y. Meanwhile, $d(T(x), y_1) \leq d(T(x), T(y)) + d(T(y), y_1) < \delta$ and $d(T(y_{n-1}), x) \leq d(T(y_{n-1}), y) + d(y, x) < \delta$. This implies that $\{x, y_1, \ldots, y_{n-1}, x\}$ is a δ-chain from x to x. Therefore, (X, T) is chain recurrent. □

Theorem 3.3. *Let (X, T) be a dynamical system. Then, the following statements are equivalent:*

(1) (X, T) is chain recurrent;
(2) $(K(X), T_K)$ is chain recurrent;
(3) $(\mathbb{F}_0(X), T_F)$ is chain recurrent;
(4) $(\mathbb{F}^{=1}(X), T_F|_{\mathbb{F}^{=1}(X)})$ is chain recurrent.

Proof. Applying [13, Proposition 4.1, Proposition 4.2, Proposition 4.6] yields that (3) $\Longrightarrow$ (2) $\Longrightarrow$ (1). It follows immediately from the definition of chain recurrence that (4) $\Longleftrightarrow$ (3).

(1) $\Longrightarrow$ (2). Applying Lemma 3.2, it suffices to prove that $(K_\infty(X), T_K|_{K_\infty(X)})$ is chain recurrent. Given any $A \in K_\infty(X)$ and any $\delta > 0$, without loss of generality, assume that $A = \{x_1, x_2, \ldots, x_n\}$. As (X, T) is chain recurrent, then for any $1 \leq i \leq n$, there exists a δ-chain Γ_i from x_i to x_i. Take a sequence

$\Gamma(i) = |\Gamma_1|\cdots|\Gamma_{i-1}||\Gamma_{i+1}|\cdots|\Gamma_n|\Gamma_i$ which is denoted by $\{z_0^{(i)}, z_1^{(i)}, \ldots, z_{|\Gamma_1|\cdots|\Gamma_n|}^{(i)}\}$ and choose $\{A_0, A_1, \ldots, A_{|\Gamma_1|\cdots|\Gamma_n|}\} \subset K_\infty(X)$ with $A_k = \{z_k^{(1)}, \ldots, z_k^{(n)}\}$ for any $0 \leq k \leq |\Gamma_1|\cdots|\Gamma_n|$. It is not difficult to verify that $\{A_0, A_1, \ldots, A_{|\Gamma_1|\cdots|\Gamma_n|}\}$ is a δ-chain from A to A. Therefore, $(K_\infty(X), T_K)$ is chain recurrent.

(2) $\Longrightarrow$ (3). Given any piecewise constant $A \in \mathbb{F}_0(X)$ which is represented by strictly decreasing sequences of closed subsets $\{A_1, A_2, \ldots, A_k\} \subset K(X)$ and strictly increasing sequences of reals $\{\alpha_1, \alpha_2, \ldots, \alpha_k\} \subset (0, 1]$ with $[A]_\alpha = A_{i+1}$, $\forall \alpha \in (\alpha_i, \alpha_{i+1}]$ and any $\delta > 0$, as $(K(X), T_K)$ is chain recurrent, it follows that for any $1 \leq i \leq k$, there exists a δ-chain from A_i to A_i, denoted it by $\{A_0^{(i)}, A_1^{(i)}, \ldots, A_{n_i}^{(i)}\}$. Take $B_0 = A$ and $B_l \in \mathbb{F}_0(X)$ with $[B_l]_\alpha = A^{(i+1)}_{l - \left[\frac{l}{n_{i+1}}\right] n_{i+1}}$, $\forall \alpha \in (\alpha_i, \alpha_{i+1}]$ for any $l \in \mathbb{N}$. It can be verified that $\{B_0, B_1, \ldots, B_{n_1 \cdots n_k}\}$ is a δ-chain from A to A. This, together with Lemma 3.2, implies that $(\mathbb{F}_0(X), T_F)$ is chain recurrent. □

Remark 3.4. It follows from Theorem 3.3 that the converses of [13, Proposition 4.1, Proposition 4.2, Theorem 4.7] hold trivially. Theorem 3.3 also implies that the assumption of connectedness of [13, Corollary 4.9] can be removed.

Theorem 3.5. *Let (X, T) be a dynamical system and Y be an invariant closed subset of $\mathbb{F}_0(X)$. If $(Y, T_F|_Y)$ is chain transitive, then for any $A, B \in Y$, $\max A = \max B$, i.e., there exists $\lambda \in [0, 1]$ such that $Y \subset \mathbb{F}^{=\lambda}(X)$.*

Proof. For any $A, B \in Y$ and any $0 < \delta < \frac{\mathrm{diam}(X)}{2}$, as $(Y, T_F|_Y)$ is chain transitive, there exists a δ-chain $\{A_0, A_1, \ldots, A_n\}$ from A to B. Then, for any $0 \leq i < n$ and any $\alpha \in (0, 1]$,

(2)
$$d_H(T([A_i]_\alpha), [A_{i+1}]_\alpha) = d_H([T_F(A_i)]_\alpha, [A_{i+1}]_\alpha) \leq d_\infty(T_F(A_i), A_{i+1}) < \delta < \frac{\mathrm{diam}(X)}{2}.$$

This implies that $\max A_i = \max A_{i+1}$.

In fact, suppose that $\xi := \max A_i \neq \eta := \max A_{i+1}$, then

$$d_H(T([A_i]_{\max\{\xi,\eta\}}), [A_{i+1}]_{\max\{\xi,\eta\}}) = \mathrm{diam}(X),$$

which contradicts (2).

Thus, $\max A = \max A_0 = \max A_n = \max B$. □

Corollary 3.6. *$(\mathbb{F}_0(X), T_F)$ is not chain transitive.*

Remark 3.7. Corollary 3.6 implies that the assumptions of [13, Proposition 4.3, Lemma 4.4] do not hold.

Lemma 3.8. [13, Theorem 4.8] *A dynamical system (X, T) is chain mixing if and only if $(K(X), T_K)$ is chain mixing.*

Theorem 3.9. *Let (X, T) be a dynamical system. Then, the following statements are equivalent:*

(1) (X, T) is chain mixing;
(2) $(K(X), T_K)$ is chain transitive;
(3) $(K(X), T_K)$ is chain mixing;

(4) $(\mathbb{F}^{=1}(X), T_F|_{\mathbb{F}^{=1}(X)})$ *is chain transitive;*
(5) $(\mathbb{F}^{=1}(X), T_F|_{\mathbb{F}^{=1}(X)})$ *is chain mixing.*

Proof. It follows from [9, Theorem 3, A1, A2 and F2] that T is chain mixing if and only if $(K_\infty(X), T_K|_{K_\infty(X)})$ is chain transitive if and only if $(K(X), T_K)$ is chain transitive. This, together with Lemma 3.2 and [13, Theorem 4.8], implies that (1) $\iff$ (2) $\iff$ (3). (5) $\implies$ (4) holds trivially.

(4) $\implies$ (2). For any $A, B \in K(X)$ and any $0 < \delta < \frac{\mathrm{diam}(X)}{2}$, noting that $\chi_A, \chi_B \in \mathbb{F}^{=1}(X)$, as $(\mathbb{F}^{=1}(X), T_F|_{\mathbb{F}^{=1}(X)})$ is chain transitive, then there exists a δ-chain $\{F_0, F_1, \ldots, F_n\}$ from χ_A to χ_B. Take a sequence $\Gamma = \{[F_0]_1, [F_1]_1, \ldots, [F_n]_1\}$. According to the proof of Theorem 3.5, it is easy to see that $\{[F_0]_1, [F_1]_1, \ldots, [F_n]_1\} \subset K(X)$. Meanwhile, it can be verified that for any $0 \le i < n$,

$$d_H(T_K([F_i]_1), [F_{i+1}]_1) = d_H([T_F(F_i)]_1, [F_{i+1}]_1) \le d_\infty(T_F(F_i), F_{i+1}) < \delta,$$

and

$$[F_0]_1 = A, \quad [F_n]_1 = B,$$

i.e., Γ is a δ-chain from A to B. Thus, $(K(X), T_K)$ is chain transitive.

(3) $\implies$ (5). For any two piecewise constants $A, B \in \mathbb{F}^{=1}(X)$ which are represented by decreasing sequences of closed subsets $\{A_1, A_2, \ldots, A_k\}$, $\{B_1, B_2, \ldots, B_k\} \subset K(X)$ and a strictly increasing sequence of reals $\{\alpha_1, \alpha_2, \ldots, \alpha_k = 1\} \subset [0,1]$ such that

$$[A]_\alpha = A_{i+1},\ [B]_\alpha = B_{i+1},\ \text{whenever } \alpha \in (\alpha_i, \alpha_{i+1}],$$

and any $\delta > 0$, as $(K(X), T_K)$ is chain mixing, there exists $N \in \mathbb{N}$ such that for any $1 \le i \le k$ and any $n \ge N$, there exists a δ-chain $\{C_0^{(i)}, C_1^{(i)}, \ldots, C_n^{(i)}\} \subset K(X)$ from A_i to B_i of length n. For any $0 \le j \le n$, take a fuzzy set $F_j \in \mathbb{F}^{=1}(X)$ with

$$[F_j]_\alpha = \cup_{l=i+1}^{k} C_j^{(l)},\ \text{whenever } \alpha \in (\alpha_i, \alpha_{i+1}].$$

It is not difficult to verify the following:

(i) each F_j is a piecewise constant;
(ii) $F_0 = A$, $F_n = B$;
(iii) $\{F_0, F_1, \ldots, F_n\}$ is a δ-chain of T_F.

This, together with Lemma 3.2, implying that $(\mathbb{F}^{=1}(X), T_F|_{\mathbb{F}^{=1}(X)})$ is chain mixing, because the set of piecewise constants is dense in $\mathbb{F}^{=1}(X)$ (see Lemma 1.2). □

3.2. **Shadowing for Zadeh's extension.** We say that a dynamical system (X, T) has *shadowing* if for any $\varepsilon > 0$, there exists $\delta > 0$ such that for every δ-pseudo-orbit $\{x_n\}_{n=0}^\infty$ of T, there exists $z \in X$ such that $\{x_n\}_{n=0}^\infty$ is ε-shadowed by z, i.e., $d(T^n(z), x_n) < \varepsilon$ for all $n \in \mathbb{Z}^+$ (see [2]). In the case that only finite pseudo-orbits are shadowed, then we say that (X, T) has *finite shadowing.* By compactness of X, it can be verified that (X, T) has shadowing if and only if it has finite shadowing.

Theorem 3.10. *A dynamical system* (X, T) *has shadowing if and only if* $(\mathbb{F}^{=1}(X), T_F|_{\mathbb{F}^{=1}(X)})$ *has finite shadowing.*

Proof. ($\Longleftarrow$). As $(\mathbb{F}^{=1}(X), T_F|_{\mathbb{F}^{=1}(X)})$ has finite shadowing, noting that $\{\chi_{\{x\}} : x \in X\} \subset \mathbb{F}^{=1}(X)$, it can be verified that (X, T) has finite shadowing. This implies that (X, T) has shadowing by the compactness of X.

($\Longrightarrow$). For any $\varepsilon > 0$, as (X, T) has shadowing, then there exists $0 < \delta < \frac{\mathrm{diam}(X)}{2}$ such that every δ-pseudo-orbit of T is $\frac{\varepsilon}{2}$-shadowed by some point in X. For every finite δ-pseudo-orbit $\{A_0, A_1, \ldots, A_n\} \subset \mathbb{F}^{=1}(X)$ of piecewise constants, applying Remark 2.1 implies that there exist decreasing sequences of closed subsets $\{A_1^{(j)}, A_2^{(j)}, \ldots, A_m^{(j)}\} \subset K(X)$ $(0 \leq j \leq n)$, and a strictly increasing sequence of reals $\{\alpha_1, \alpha_2, \ldots, \alpha_m = 1\} \subset (0, 1]$ such that for any $0 \leq j \leq n$,

$$[A_j]_\alpha = A_{i+1}^{(j)}, \text{ whenever } \alpha \in (\alpha_i, \alpha_{i+1}].$$

It is easy to verify that every $\{A_i^0, A_i^{(1)}, \ldots, A_i^{(n)}\}$ $(1 \leq i \leq m)$ is a δ-chain. Then there exists $F_i \in K(X)$ such that for any $0 \leq k \leq n$, $d_H(T_K^k(F_i), A_i^{(k)}) < \frac{\varepsilon}{2}$. Take a fuzzy set $F \in \mathbb{F}^{=1}(X)$ with

$$[F]_\alpha = \cup_{l=i+1}^k F_l, \text{ whenever } \alpha \in (\alpha_i, \alpha_{i+1}].$$

Then it can be verified that $\{A_0, A_1, \ldots, A_n\} \subset \mathbb{F}^{=1}(X)$ can be ε-shadowed by F. Therefore, $(\mathbb{F}^{=1}(X), T_F|_{\mathbb{F}^{=1}(X)})$ has finite shadowing by applying Lemma 1.2 and Lemma 3.1. □

Remark 3.11. As $\mathbb{F}^{=1}(X)$ is not compact, we do not know whether the shadowing of (X, T) implies that $(\mathbb{F}^{=1}(X), T_F|_{\mathbb{F}^{=1}(X)})$ has shadowing.

Definition 3.12. [4] A dynamical system (X, T) has *h-shadowing* if for any $\varepsilon > 0$, there exists $\delta > 0$ such that for every finite δ-pseudo-orbit $\Gamma = \{x_0, x_1, \ldots, x_n\}$, there exists $z \in X$ such that $d(T^i(z), x_i) < \varepsilon$ for any $0 \leq i \leq n$ and $T^n(x) = x_n$.

Lemma 3.13. [10, Lemma 4.4] *Let (X, T) be a dynamical system and Y be a dense invariant subset of X. If $(Y, T|_Y)$ has h-shadowing, then (X, T) has h-shadowing.*

Theorem 3.14. *Let (X, T) be a dynamical system. Then, the following statements are equivalent:*
(1) (X, T) has h-shadowing;
(2) $(K(X), T_K)$ has h-shadowing;
(3) $(\mathbb{F}^{=1}(X), T_F|_{\mathbb{F}^{=1}(X)})$ has h-shadowing.

Proof. Similarly to the proof of Theorem 3.10, applying Lemma 3.13 and Lemma 1.4, it is not difficult to verify that this is true. □

4. Li-Yorke sensitivity for Zadeh's extension

In [6], Blanchard et al. introduced the concept of spatiotemporal chaos. A dynamical system (X, T) is called *spatiotemporally chaotic* (also called *chaotic dependence on initial conditions* in [28]) if any neighbourhood of any $x \in X$ contains a point y such that (x, y) is a Li-Yorke pair, i.e., $\liminf_{n\to\infty} d(T^n(x), T^n(y)) = 0$ and $\limsup_{n\to\infty} d(T^n(x), T^n(y)) > 0$. Later, Akin and Kolyada [1] introduced the concept of Li-Yorke sensitivity which links the Li-Yorke chaos with the notion of sensitivity and proved that the weakly mixing property implies Li-Yorke sensitivity. Recall [1] that a dynamical system (X,T) is *Li-Yorke sensitive* if there exists $\varepsilon > 0$ (Li-Yorke sensitive constant) such that any neighbourhood of any $x \in X$ contains a point y satisfying

$$\liminf_{n\to\infty} d(T^n(x), T^n(y)) = 0 \text{ and } \limsup_{n\to\infty} d(T^n(x), T^n(y)) > \varepsilon.$$

Recently, We [38] studied the sensitivity of $(K(X), T_K)$ in Furstenberg families and proved that $\mathscr{F}$-sensitivity of $(K(X), T_K)$ implies that of (X,T), and the converse is also true if the Furstenberg family $\mathscr{F}$ is a filter. In [28, Proposition 2.6], Sharma and Nagar proved that (X,T) is spatiotemporally chaotic provided that $(K(X), T_K)$ is Li-Yorke sensitive. Similarly to this, here we shall show that (X,T) is spatiotemporally chaotic if its Zadeh's extension $(\mathbb{F}_0(X), T_F)$ is Li-Yorke sensitive (see Theorem 4.1).

Theorem 4.1. *Let (X,T) be a dynamical system. If $(\mathbb{F}_0(X), T_F)$ is Li-Yorke sensitive, then (X,T) is spatiotemporally chaotic.*

Proof. Let $\varepsilon > 0$ be a Li-Yorke sensitive constant of T_F. For any $x \in X$ and any $\delta > 0$, it suffices to prove that there exists $z \in B(x,\delta)$ such that

$$\liminf_{n\to\infty} d(T^n(x), T^n(z)) = 0 \text{ and } \limsup_{n\to\infty} d(T^n(x), T^n(z)) > 0.$$

Noting that $\chi_{\{x\}} \in \mathbb{F}_0(X)$, as T_F is Li-Yorke sensitive, there exists $A \in B(\chi_{\{x\}}, \delta)$ such that

$$\liminf_{n\to\infty} d_\infty(T_F^n(\chi_{\{x\}}), T_F^n(A)) = 0, \tag{3}$$

and

$$\limsup_{n\to\infty} d_\infty(T_F^n(\chi_{\{x\}}), T_F^n(A)) > \varepsilon. \tag{4}$$

Applying (3) implies that there exists $n_1 \in \mathbb{N}$ such that for any $\lambda \in (0,1]$,

$$d_H(T^{n_1}(\{x\}), T^{n_1}([A]_\lambda)) = \sup\{d(T^{n_1}(x), T^{n_1}(y)) : y \in [A]_\lambda\} < \frac{1}{2}.$$

Meanwhile, applying (4) implies that there exist $m_1 \in \mathbb{N}$, $\lambda \in (0,1]$ and $x_1 \in [A]_\lambda$ such that

$$d(T^{m_1}(x), T^{m_1}(x_1)) > \varepsilon.$$

Clearly, $d(x_1, x) < \delta$. It follows from (3) that $\liminf_{n\to\infty} d(T^n(x), T^n(x_1)) = 0$. If (x, x_1) is a Li-Yorke pair, we are done. Otherwise, $\lim_{n\to\infty} d(T^n(x), T^n(x_1)) = 0$. Then, take $0 < \delta_1 < \frac{1}{2}$ satisfying the following conditions:

(a_1) $\overline{B(x_1, \delta_1)} \subset B(x, \delta)$;
(b_1) for any $y \in B(x_1, \delta_1)$, $d(T^{m_1}(x), T^{m_1}(y)) > \varepsilon$;
(c_1) for any $y \in B(x_1, \delta_1)$, $d(T^{n_1}(x), T^{n_1}(y)) < \frac{1}{2}$.

As T_F is Li-Yorke sensitive, there exists $A_1 \in B(\chi_{\{x_1\}}, \delta_1)$ such that

$$\liminf_{n\to\infty} d_\infty(T_F^n(\chi_{\{x_1\}}), T_F^n(A_1)) = 0,$$

and

$$\limsup_{n\to\infty} d_\infty(T_F^n(\chi_{\{x_1\}}), T_F^n(A_1)) > \varepsilon.$$

This, together with $\lim_{n\to\infty} d(T^n(x), T^n(x_1)) = 0$, implies that

$$\liminf_{n\to\infty} d_\infty(T_F^n(\chi_{\{x\}}), T_F^n(A_1)) = 0, \tag{5}$$

and

$$\limsup_{n\to\infty} d_\infty(T_F^n(\chi_{\{x\}}), T_F^n(A_1)) > \varepsilon. \tag{6}$$

Similarly, there exist $x_2 \in B(x_1, \delta_1)$, $n_2 > n_1$ and $m_2 > m_1$ such that

$$d(T^{n_2}(x), T^{n_2}(x_2)) < \frac{1}{4},$$

$$d(T^{m_2}(x), T^{m_2}(x_2)) > \varepsilon,$$

and

$$\liminf_{n\to\infty} d(T^n(x), T^n(x_2)) = 0.$$

If (x, x_2) is a Li-Yorke pair, we are done. Otherwise, $\lim_{n\to\infty} d(T^n(x), T^n(x_2)) = 0$. Then, take $0 < \delta_2 < \frac{1}{2^2}$ satisfying the following conditions:

(a$_2$) $\overline{B(x_2, \delta_2)} \subset B(x_1, \delta_1)$;
(b$_2$) for any $y \in B(x_2, \delta_2)$, $d(T^{m_2}(x), T^{m_2}(y)) > \varepsilon$;
(c$_2$) for any $y \in B(x_2, \delta_2)$, $d(T^{n_2}(x), T^{n_2}(y)) < \frac{1}{2^2}$.

Proceeding inductively, we either get a point $x_n \in B(x, \delta)$ such that (x, x_n) is a Li-Yorke pair or we get $\{x_k\}_{k=1}^\infty \subset X$, $n_1 < n_2 < \cdots < n_k < n_{k+1} < \cdots$, $m_1 < m_2 < \cdots < m_k < m_{k+1} < \cdots$ and $0 < \delta_k < \frac{1}{k}$ such that for any $k \in \mathbb{N}$,

(a$_k$) $\overline{B(x_k, \delta_k)} \subset B(x_{k-1}, \delta_{k-1})$, where $x_0 = x$, $\delta_0 = \delta$;
(b$_k$) for any $y \in B(x_k, \delta_k)$, $d(T^{m_k}(x), T^{m_k}(y)) > \varepsilon$;
(c$_k$) for any $y \in B(x_k, \delta_k)$, $d(T^{n_k}(x), T^{n_k}(y)) < \frac{1}{2^k}$.

Choose $z \in \bigcap_{k=1}^\infty B(x_k, \delta_k)$. It is no difficult to verify that $\liminf_{n\to\infty} d(T^n(x), T^n(z)) = 0$ and $\limsup_{n\to\infty} d(T^n(x), T^n(z)) \geq \varepsilon$. Thus, T is spatiotemporally chaotic. □

Similarly, it can be verified that the following theorem holds.

Theorem 4.2. *Let (X, T) be a dynamical system. If $(\mathbb{F}_0(X), T_F)$ is Li-Yorke sensitive, then $(K(X), T_K)$ is spatiotemporally chaotic.*

Remark 4.3. We believe that the converses of Theorem 4.1 and 4.2 do not hold. However, we can not obtain some proper examples. Meanwhile, the converse of [28, Proposition 2.6] which claims that the Li-Yorke sensitivity of $(K(X), T_K)$ implies that (X, T) is spatiotemporally chaotic is also open.

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