L-FUZZY BILINEAR OPERATOR AND ITS CONTINUITY

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Abstract. The purpose of this paper is to introduce the concept of L-fuzzy bilinear operators. We obtain a decomposition theorem for L-fuzzy bilinear operators and then prove that a L-fuzzy bilinear operator is the same as a powerset operator for the variable-basis introduced by S.E.Rodabaugh (1991). Finally we discuss the continuity of L-fuzzy bilinear operators.

1. **Introduction**

The concept of L-topological vector spaces was introduced by Fang and Yan [3]. Since then many properties of this kind of spaces have been discussed. In particular, Fang Jin-xuan first introduced the concept of fuzzy linear operators in [1]. For detailed discussions of the theory of fuzzy linear operator and its applications in L-topological vector spaces, we refer to [2, 11, 12, 13, 15, 16]. As we know, the theory of duality of classical topological vector spaces is a significant part of topological vector spaces and bilinear operators play an important role in the theory of duality. Hence it is natural to consider introducing the concept of L-fuzzy bilinear operator in the theory of L-topological vector spaces.

How do we introduce the concept of L-fuzzy bilinear operators? In other words, how do we define a powerset operator from $L_1^{X\times Y}$ to L_2^{Z} ? In his pioneering paper [17], Zadeh introduced the Extension Principle for fixed-basis fuzzy sets with $L = [0, 1]$. S. E. Rodabaugh first introduced the concept of powerset operators for variable-basis fuzzy sets in [5, 7, 8]. He also discussed for generalized operators (numbered 4.5) in [8]. In [1], Fang gave the notion of pointwise L-fuzzy linear operators. In fact, we may prove that the definition of Fang is a special case of that of Rodabaugh. To sum up, we easily find there are two ways of introducing powerset operators for variable-basis fuzzy sets. The pointwise method is a very powerful tool in studying the theory of L-topological vector spaces. In this paper we intend to introduce the concept of pointwise L-fuzzy bilinear operators and study some of their properties. A decomposition theorem for L-fuzzy bilinear operators is also obtained: An L-fuzzy bilinear operator can be determined uniquely by the corresponding bilinear operator and a finitely meet-preserving order-homomorphism.

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Then we prove that our L-fuzzy bilinear operator is a special case of powerset operators in the sense of S.E. Rodabaugh. Finally we discuss the continuity of L-fuzzy bilinear operators.

2. **Preliminaries**

Throughout this paper $L,\ L_1,\ L_2$ denote the regular *Hutton algebras* [7], i.e. the *Hutton algebras* with $1 \in M(L)$, where $M(L)$ is the set of all non-zero union-irreducible elements in L (L_1 or L_2), and 0 and 1 are respectively their smallest and greatest elements. L^X denotes the collection of all L-fuzzy sets on X and $\underline{\alpha}$ denotes an L-fuzzy set which takes the constant value $\alpha \in L$ on X. An L-fuzzy subset of X is called an L-fuzzy point [1, 3] iff it takes the value 0 for all $y \in X$ except one, say, $x \in X$. If the value at x is $\alpha \in L \setminus \{0\}$, we call *alpha* the height of the L-fuzzy point, and denote it by x_α. A crisp point x in X can be regarded as a fuzzy point x_1 with height 1. Henceforth $\mathrm{Pt}(L^X)$ will denote the set of all L-fuzzy points in L^X. Also, for $A \in L^X$, the set $\{x \in X \mid A(x) \neq 0\}$, called the support of A will be denoted by $\mathrm{supp}\,A$, and the value $\bigvee\{A(x) \mid x \in X\}$, called the height of A , will be denoted by $\mathrm{hgt}\,A$.

Let X, Y and Z be the vector spaces over $\mathbf{K}$ (real or complex number field). For convenience, the zero element in X, Y and Z will be denoted by the same letter θ. For $A,\ B \in L^X$ and $k \in \mathbf{K}$, $A + B$ and kA are defined by the ZEP [1]. The product $A \times B$ of L-fuzzy sets A and B is defined by

$$(A \times B)(x, y) = A(x) \wedge B(y) \quad \text{for all } (x, y) \in X \times X.$$

In particular, $x_\alpha \times y_\beta = (x, y)_{\alpha\wedge\beta}$ for all $(x_\alpha, y_\beta) \in \mathrm{Pt}(L^X) \times \mathrm{Pt}(L^X)$.

Definition 2.1. (Wang Guo-jun [9]). A mapping $\phi : L_1 \to L_2$ is called an order-homomorphism, if the following conditions hold:

(OH-1) ϕ preserves arbitrary $\vee$;

(OH-2) ϕ^* preserves all involutions, i.e., for each $b \in L_2$, $\phi^*(b') = [\phi^*(b)]'$, where the mapping $\phi^* : L_2 \to L_1$ is given by

$$\phi^*(b) = \bigvee\{a \in L_1 :\ \phi(a) \le b\},$$

Definition 2.2. (Ming He [4]). Let $f : X \to Y$ and $\phi : L \to L_1$ be two mappings. From f and ϕ we can induce a mapping $F : L^X \to L_1^Y$ as follows:

$$F(A)(y) = \bigvee\{\ \phi(A(x)) \mid f(x) = y\}, \quad \text{for all } A \in L^X \text{ and } y \in Y.$$

The mapping F is called a bi-induced mapping of f and ϕ.

Definition 2.3. (Fang Jin-xuan [1]). A mapping $\tilde{f} : \mathrm{Pt}(L_1^X) \to \mathrm{Pt}(L_2^Y)$ is called an L-fuzzy linear operator if the following conditions are satisfied:

(1) $\tilde{f}(sx_\lambda + ty_\mu) = s\tilde{f}(x_\lambda) + t\tilde{f}(y_\mu)$, for all $x_\lambda, y_\mu \in \mathrm{Pt}(L_1^X)$ and $s, t \in \mathbf{K}$;

(2) $\tilde{f}(\theta_{\vee\lambda_d}) = \bigvee \tilde{f}(\theta_{\lambda_d})$;

(3) $\mathrm{hgt}\,\tilde{f}^*(\theta_{\lambda'}) = [\mathrm{hgt}\,\tilde{f}^*(\theta_\lambda)]'$, for all $\lambda \in L_2 \setminus \{0, 1\}$, where

$$\tilde{f}^*(\theta_{\lambda'}) = \bigvee\{x_\alpha \mid \tilde{f}(x_\alpha) \le \theta_{\lambda'}\}.$$

Lemma 2.4. *(Fang Jin-xuan [1]). Suppose that $\tilde{f} : \mathrm{Pt}(L_1^X) \to \mathrm{Pt}(L_2^Y)$ is a fuzzy linear operator. Then*

(1) $\mathrm{supp}\,\tilde{f}(x_\lambda) = \mathrm{supp}\,\tilde{f}(x_\mu)$, *for all* $\lambda,\ \mu \in L_1 \setminus \{0\}$;

(2) $\mathrm{supp}\,\tilde{f}(\theta_\lambda) = \theta$ *and* $\mathrm{hgt}\,\tilde{f}(\theta_\lambda) = \mathrm{hgt}\,\tilde{f}(x_\lambda)$, *for all* $x \in X$.

Lemma 2.5. *(Fang Jin-xuan [1]). The mapping $\tilde{f} : \mathrm{Pt}(L_1^X) \to \mathrm{Pt}(L_2^Y)$ is an L-fuzzy linear operator iff there exist an ordinary linear mapping $f : X \to Y$ and an order-homomorphism preserving finite meets $\phi : L_1 \to L_2$ such that*

$$\tilde{f}(x_\lambda) = [f(x)]_{\phi(\lambda)}, \quad \text{for all } x_\lambda \in \mathrm{Pt}(L_1^X).$$

3. *L*-fuzzy Bilinear Operator and Its Decomposition Theorem

Definition 3.1. Suppose that X, Y and Z are vector spaces over $\mathbf{K}$. A mapping $\tilde{f} : \mathrm{Pt}(L^{X\times Y})$ to $\mathrm{Pt}(L_1^Z)$ is called an L-fuzzy bilinear order-homomorphism, if it satisfies the following conditions:

(1) For each $x \in X$, the mapping $\tilde{f}_{x_1} : y_\alpha \to \tilde{f}(x_1, y_\alpha)$ is a fuzzy linear operator;

(2) For each $y \in Y$, the mapping $\tilde{f}_{y_1} : x_\alpha \to \tilde{f}(x_\alpha, y_1)$ is a fuzzy linear operator.

Theorem 3.2. *Let $\tilde{f}$ be an L-fuzzy bilinear operator, then the following relations hold:*

(1) supp $\tilde{f}(\theta_\alpha, \theta^1_\alpha) = \theta^2$, and hgt $\tilde{f}(\theta_\alpha, \theta^1_\alpha) =$ hgt $\tilde{f}(x_\alpha, y_\alpha)$; here $\theta,\ \theta^1,\ \theta^2$ are separately zero elements in X, Y, Z.

(2) supp $\tilde{f}(x_\alpha, y_\alpha) =$ supp$\tilde{f}(x_\lambda, y_\lambda)$.

Proof. (1) Suppose that $\tilde{f}(x_\alpha, y_\alpha) = z_\mu$. Then

$$\begin{aligned}&\tilde{f}(\theta_\alpha, \theta^1_\alpha) = \tilde{f}(\theta_\alpha, \theta^1_1) = \tilde{f}_{\theta^1_1}(\theta_\alpha) = \tilde{f}_{\theta^1_1}(x_\alpha - x_\alpha) = \tilde{f}_{\theta^1_1}(x_\alpha) - \tilde{f}_{\theta^1_1}(x_\alpha)\\ &= \tilde{f}(x_\alpha, \theta^1_1) - \tilde{f}(x_\alpha, \theta^1_1) = \tilde{f}_{x_1}(\theta^1_\alpha) - \tilde{f}_{x_1}(\theta^1_\alpha) = \tilde{f}_{x_1}(y_\alpha - y_\alpha) - \tilde{f}_{x_1}(y_\alpha - y_\alpha)\\ &= \tilde{f}_{x_1}(y_\alpha) - \tilde{f}_{x_1}(y_\alpha) - \tilde{f}_{x_1}(y_\alpha) + \tilde{f}_{x_1}(y_\alpha) = \tilde{f}(x_\alpha, y_\alpha) - \tilde{f}(x_\alpha, y_\alpha)\\ &- \tilde{f}(x_\alpha, y_\alpha) + \tilde{f}(x_\alpha, y_\alpha) = z_\mu - z_\mu - z_\mu + z_\mu = \theta^2_\mu.\end{aligned}$$

Hence supp $\tilde{f}(\theta_\alpha, \theta^1_\alpha) = \theta^2$ and hgt $\tilde{f}(\theta_\alpha, \theta^1_\alpha) =$ hgt $\tilde{f}(x_\alpha, y_\alpha)$, here $\theta,\ \theta^1,\ \theta^2$ are separately zero elements in X, Y, Z.

(2). Suppose $\tilde{f}(x_\alpha, y_\alpha) = z_\mu,\ \tilde{f}(x_\lambda, y_\lambda) = z^1_\nu$. By the proof of (1) we may assume that $\tilde{f}(\theta_{\alpha\wedge\lambda}, \theta^1_{\alpha\wedge\lambda}) = \theta^2_\varepsilon$. Then $(z - z^1)_{\mu\wedge\nu} = z_\mu - z^1_\nu = \tilde{f}(x_\alpha, y_\alpha) - \tilde{f}(x_\lambda, y_\lambda) = \tilde{f}_{x_1}(y_\alpha) - \tilde{f}_{x_1}(y_\lambda) = \tilde{f}_{x_1}(y_\alpha - y_\lambda) = \tilde{f}_{x_1}(\theta_{\alpha\wedge\lambda}) = \theta^2_\varepsilon$. Hence $z = z^1$, i.e. supp $\tilde{f}(x_\alpha, y_\alpha) =$ supp$\tilde{f}(x_\lambda, y_\lambda)$. □

Theorem 3.3. *A mapping $\tilde{f} : \mathrm{Pt}(L^{X\times Y}) \to \mathrm{Pt}(L_1^Z)$ is an L-fuzzy bilinear operator iff there exist an ordinary bilinear operator $f : X \to Y$ and a finitely meet-preserving order-homomorphism $\phi : L \to L_1$ such that*

$$\tilde{f}(x_\alpha, y_\alpha) = [f(x, y)]_{\phi(\alpha)},\ \forall (x_\alpha, y_\alpha) \in Pt(L^{X\times Y})$$

Proof. Necessity. Define a mapping $f: X \to Y$ and a mapping $\phi: L \to L_1$ as follows:

$$f(x,y) = \text{supp}\tilde{f}(x_1,y_1), \quad \text{and } \phi(\alpha) = \text{hgt}\tilde{f}(\theta_\alpha,\theta_\alpha^1).$$

From Theorem 3.2, it follows that supp $\tilde{f}(x_\alpha,y_\alpha) =$ supp $\tilde{f}(x_1,y_1) = f(x,y)$, and hgt $\tilde{f}(x_\alpha,y_\alpha) =$ hgt $\tilde{f}(\theta_\alpha,\theta_\alpha^1) = \phi(\alpha)$. Hence $\tilde{f}(x_\alpha,y_\alpha) = [f(x,y)]_{\phi(\alpha)}$.

Since $\tilde{f}$ is an L-fuzzy bilinear operator for each $s,t \in \mathbf{K}$ and $x, x^1 \in X$, we have for each $y \in Y$,

$$\begin{aligned} f(sx+tx^1,y) &= \text{supp}\tilde{f}(sx_1+tx_1^1,y_1) = \text{supp}[s\tilde{f}(x_1,y_1)+t\tilde{f}(X_1^1,y_1)] \\ &= \text{supp}[sf(x,y)+tf(x^1,y)]_{\phi(1)} = sf(x,y)+tf(x^1,y). \end{aligned}$$

So f is linear operator on the second variable. By the same method we may obtain that f is linear operator on the first variable. Hence the mapping $f: X \times Y \to Z$ is a bilinear operator. Moreover,

$$\begin{aligned} \phi(\vee\lambda_d) &= \text{hgt}\tilde{f}(\theta_{\vee\lambda_d},\theta^1_{\vee\lambda_d}) = \text{hgt}\tilde{f}_{\theta_1^1}(\theta_{\vee\lambda_d}) = \text{hgt}\Big(\vee\tilde{f}_{\theta_1^1}(\theta_{\lambda_d})\Big) \\ &= \text{hgt}\Big(\vee\tilde{f}(\theta_{\lambda_d},\theta^1_{\lambda_d})\Big) = \text{hgt}(\vee\theta^2_{\phi(\lambda_d)}) = \text{hgt}(\theta^2_{\vee\phi(\lambda_d)}) = \vee\phi(\lambda_d). \end{aligned}$$

$$\begin{aligned} \phi(\alpha\wedge\lambda) &= \text{hgt}\tilde{f}(\theta_{\alpha\wedge\lambda},\theta^1_{\alpha\wedge\lambda}) = \text{hgt}\tilde{f}_{\theta_1^1}(\theta_{\alpha\wedge\lambda}) = \text{hgt}\tilde{f}_{\theta_1^1}(\theta_\alpha+\theta_\lambda) \\ &= \text{hgt}[\tilde{f}(\theta_\alpha,\theta^1_\alpha)+\tilde{f}(\theta_\lambda,\theta^1_\lambda) = \text{hgt}[\theta^2_{\phi(\alpha)}+\theta^2_{\phi(\lambda)}] = \phi(\alpha)\wedge\phi(\lambda). \end{aligned}$$

Now, since

$$\begin{aligned} \tilde{f}^{-1}_{\theta_1}(\theta^2_\mu)(y) &= \vee\{\ \alpha \mid \tilde{f}_{\theta_1}(y_\alpha) \le \theta^2_\mu\} = \vee\{\ \alpha \ \mid\ \tilde{f}(\theta_\alpha,y_\alpha) \ \le \theta^2_\mu\} \\ &= \vee\{\ \alpha \mid\ [f(\theta,y)]_{\phi(\alpha)} \le \theta^2_\mu\} = \phi^*(\mu). \end{aligned}$$

$$\Big(\phi^*(\alpha)\Big)^{'} = \Big(\text{hgt}\tilde{f}^{-1}_{\theta_1}(\theta^2_\alpha)\Big)^{'} = \text{hgt}\tilde{f}^{-1}_{\theta_1}(\theta^2_{\alpha^{'}}) = \phi^*(\alpha^{'}).$$

Therefore ϕ is a finitely meet-preserving order-homomorphism.

Sufficiency. Because of symmetry, we only need to verify that $\tilde{f}$ satisfies condition (1) of Definition 3.1, i.e. For each $x \in X$, we verify that $\tilde{f}$ satisfies (1)-(3) of Definition 2.3. By the assumption of the theorem and Lemma 2.5, we have:

(1) For each $y_\alpha,\ y^1_\lambda \in \hat{Y}(L)$, and $s,\ t \in \mathbf{K}$,

$\tilde{f}_{x_1}(sy_\alpha + ty^1_\lambda) = \tilde{f}(x_1, sy_\alpha + ty^1_\lambda) = \tilde{f}\Big((x_{\alpha\wedge\lambda},(sy+ty^1)_{\alpha\wedge\lambda}\Big) = [f(x,sy+ty^1)]_{\phi(\alpha\wedge\lambda)}$

$$= [sf(x,y)+tf(x,y^1)]_{\phi(\alpha)\wedge\phi(\lambda)} = s\tilde{f}_{x_1}(y_\alpha)+t\tilde{f}(y^1_\lambda).$$

(2) $\tilde{f}_{x_1}(\theta^1_{\vee\lambda_d}) = [f(x,\theta^1)]_{\phi(\vee\lambda_d)} = \theta^2_{\vee\phi(\lambda_d)} = \vee\tilde{f}_{x_1}(\theta^1_{\lambda_d})$.

(3) $\text{hgt}\tilde{f}^{-1}_{x_1}(\theta^1_{\lambda^{'}}) = \phi^*(\lambda^{'}) = (\phi^*(\lambda))^{'} = [\text{hgt}f^{-1}_{x_1}(\theta^1_\lambda)]^{'}$.

This completes the proof. □

Theorem 3.4. *Let $\tilde{f}$ be an L-fuzzy bilinear operator from* $\text{Pt}(L^{X\times Y})$ *to* $\text{Pt}(L_1^Z)$, *i.e. there exist a bilinear operator $f: X\times Y \to Z$ and a finitely meet-preserving order-homomorphism $\phi: L \to L_1$ such that*

$$\tilde{f}(x,y)_\alpha = [f(x,y)]_{\phi(\alpha)}, \quad \forall (x,y)_\alpha \in \text{Pt}(\text{L}^{\text{X}\times\text{Y}}).$$

Then the bi-induced mapping of f and ϕ, $F: L^{X\times Y} \to L_1^Z$, is an order-homomorphism.

Proof. Since F is a bi-induced mapping of f and ϕ, then

$$\begin{aligned} F(\vee A_t)(z) &= \vee\{\ \phi((\vee A_t)(x,y))\ |\ \ f(x,y)=z\ \} = \vee\{\ \phi(\vee A_t(x))\ |\ f(x,y)=z\} \\ &= \bigvee_{f(x,y)=z} \vee\phi(A_t(x)) = \vee \bigvee_{f(x,y)=z} \phi(A_t(x)) \\ &= \vee F(A_t)(z) = [\vee F(A_t)](z). \end{aligned}$$

This means F preserves arbitrary $\vee$.

On the other hand, by [Theorem 2.3A, 8], there exists an unique right adjoint functor F^* from L_1^Z to $L^{X\times Y}$ given by

$$F^*(B) = \vee\{\ A \in L^{X\times Y}\ \mid F(A) \le B)\ \}.$$

Thus $F^*(B) = \vee\{\ (x,y)_\lambda \mid F((x,y)_\lambda) \le B\}$. So

$F^*(B)(x,y) = \vee\{\ \lambda \mid \phi(\lambda) \le B(f(x,y))\} = \ \phi^*(B(f(x,y)))$. Hence

$F^*(B^{'})(x,y) = \phi^*\Big((B(f(x,y)))^{'}\Big) = (\phi^*(B(f(x,y))))^{'} = (F^*(B))^{'}(x,y)$.

Therefore F is an order-homomorphism. □

By Theorem 3.3 and 3.4, we may obtain enough nontrivial examples of L-fuzzy bilinear operators. The following is an example.

Example 3.5. Let $L_1 = L_2 = \{0, c, a, b, d, 1\}$, here $a \bigwedge b = c,\ a \bigvee b = d$ and $0^{'} = 1, 1^{'} = 0, c^{'} = d, d^{'} = c, a^{'} = b, b^{'} = a$. Then L_1, L_2 are regular Hutton Algebras. A mapping $\phi : L_1 \to L_2$ is defined as follows: $\phi(0) = 0,\ \phi(c) = c,\ \phi(d) = d,\ \phi(1) = 1,\ \phi(a) = b,\ \phi(b) = a$. It is easy to verify that the mapping ϕ is a finitely meet-preserving order-homomorphism. Suppose that vector spaces $X = Y = Z = \mathbb{R}$ and classical bilinear operator f from $X \times Y$ to Z is defined by $f(x,y) = 2x+3y+c$, (c is constant). Then the mapping $\widetilde{f} : \mathrm{Pt}(L_1^{X\times Y}) \to \mathrm{Pt}(L_2^Z),\ \widetilde{f}\Big((x,y)_\alpha\Big) = (2x+3y+c)_{\phi(\alpha)}$ is an L-fuzzy bilinear operator.

Theorem 3.6. *Let $\tilde{f}$ be an L-fuzzy bilinear operator from $\mathrm{Pt}(L^{X\times Y})$ to $\mathrm{Pt}(L_1^Z)$, i.e. there exist a bilinear operator $f : X \times Y \to Z$ and a finitely meet-preserving order-homomorphism $\phi : L \to L_1$ such that*

$$\tilde{f}(x,y)_\alpha = [f(x,y)]_{\phi(\alpha)},\ \ \forall (x,y)_\alpha \in \mathrm{Pt}(\mathrm{L^{X\times Y}}).$$

The mappings $(f,\phi)^{\rightarrow} : L^{X\times Y} \to L_1^Z$ and $(f,\phi)^{\leftarrow} : L_1^Z \to L^{X\times Y}$ are defined as in [7], i.e.

$$(f,\phi)^{\rightarrow} \equiv < [\phi] > \circ f_L^{\rightarrow} : L^{X\times Y} \to L^Z \to L_1^Z$$

and

$$(f,\phi)^{\leftarrow} \equiv < \phi^{op} > \circ f_{L_1}^{\leftarrow} : L^{X\times Y} \leftarrow L_1^{X\times Y} \leftarrow L_1^Z.$$

Then $(f,\phi)^{\rightarrow}$ is equal to the bi-induced mapping F of f and ϕ, or, equivalently, $(f,\phi)^{\rightarrow}$ is exactly the mapping F of Theorem 3.4. In addition, $(f,\phi)^{\leftarrow} = F^*$.

Proof. For fixed order-homomorphism $\phi : L \to L_1$, from the properties of order-homomorphisms [9, 10], the right adjoint $\phi^* : L_1 \to L$ preserves arbitrary $\wedge$ and arbitrary $\vee$. Let $\phi^{op} = \phi^*$, by the Fundamental Theorem [Theorem 7.10, 8], both definitions of $(f,\phi)^{\rightarrow}$ coincide universally.

Thus

$[\phi](a) = \wedge\{b: \ a \leq \phi^{op}(b)\} = \wedge\{b : a \leq \phi^*(b)\} = \wedge\{b : \phi(a) \leq b\} = \phi(a).$

So for each $A \in L^{X\times Y}$ and every $z \in Z$, we have

$$(f,\phi)^{\rightarrow}(A)(z) =< [\phi] > \big(f_L^{\rightarrow}(A)(z)\big) = [\phi](\vee\{A(x,y): \ f(x,y) = z\})$$
$$= \bigvee\{\phi(A(x,y)): \ f(x,y) = z\} = F(A)(z).$$

Moreover, for each $B \in L_1^Z$ and for all $(x,y) \in X \times Y$, $(f,\phi)^{\leftarrow}(B)(x,y) =< \phi^{op} > \circ f_{L_1}^{\leftarrow}(B)(x,y) = \phi^{op}\big(B(f(x,y))\big) = \phi^*\big(B(f(x,y))\big) = F^*(B)(x)$. This completes the proof. □

Remark 3.7. By Theorem 3.4 and Theorem 3.6, we may see that L-fuzzy bilinear operators are a very special case of powerset operators [7] in L-topological vector space and because if the vector structure of $X \times Y$, it is natural to require that f be a crisp bilinear operator. Based on this, we will henceforth use $(f,\phi)^{\rightarrow}$ instead of $\tilde{f}$, F and $(f,\phi)^{\leftarrow}$ instead of $(\tilde{f})^{-1}$, F^*. Moreover, if ϕ is the identity mapping, $(f, id)^{\rightarrow}$ is called the Zadeh$'$s type function and denoted $f_L^{\rightarrow}$ for short.

4. The Continuity of L-fuzzy Bilinear Operators

Definition 4.1. Let (L^X,δ_X), (L^Y,δ_Y) and (L_1^Z,δ_Z) be L-topological vector spaces, $(f,\phi)^{\rightarrow} : \mathrm{Pt}(L^{X\times Y}) \to \mathrm{Pt}(L_1^Z)$ be an L-fuzzy bilinear operator and $(x,y)_\lambda \in M^*(L^{X\times Y})$. If for each R-neighborhood W of $(f(x,y))_{\phi(\lambda)}$, $(f,\phi)^{\leftarrow}(W)$ is an R-neighborhood of $(x,y)_\lambda$, then we sat that $(f,\phi)^{\rightarrow}$ is L-continuous at $(x,y)_\lambda$.

Definition 4.2. Let (L^X,δ_X), (L^Y,δ_Y) and (L_1^Z,δ_Z) be L-topological vector spaces, $(f,\phi)^{\rightarrow} : \mathrm{Pt}(L^{X\times Y}) \to \mathrm{Pt}(L_1^Z)$ be an L-fuzzy bilinear operator. Then $(f,\phi)^{\rightarrow}$ is called L-continuous, if for each $U \in \delta_Z$, $(f,\phi)^{\leftarrow}(U) \in \delta_X \times \delta_Y$.

The following proposition easily follows from [Theorem 2.4.5, 6] and [Theorem 5.1.7, 6].

Proposition 4.3. *Let (L^X,δ_X), (L^Y,δ_Y) and (L_1^Z,δ_Z) be L-topological vector Spaces and $(f,\phi)^{\rightarrow} : \mathrm{Pt}(L^{X\times Y}) \to \mathrm{Pt}(L_1^Z)$ be an L-fuzzy bilinear operator. Then the following are equivalent:*

(1) $(f,\phi)^{\rightarrow}$ is L-continuous;

(2) $(f,\phi)^{\rightarrow}$ is L-continuous at $(x,y)_\lambda$ for each $(x,y)_\lambda \in M^*(L^{X\times Y})$;

(3) For each molecule net $S = \{(x^n,y^n)_{\lambda(n)}\}_{n\in D}$ in $M^*(L^{X\times Y})$, if S converges to $(x,y)_\lambda$ then $(f,\phi)^{\rightarrow}(S)$ converges to $(f(x,y))_{\phi(\lambda)}$.

Lemma 4.4. *Let (L^X,δ) be a locally convex L-convex topological vector space [14] and $\Re = \{p_d \mid d \in \Gamma\}$ be a family of L-probabilistic seminorms on X that generates δ. Then a molecule net $\{x^n_{\lambda(n)} \mid n \in D\}$ in L^X is convergent to x_λ with respect to δ iff for each $t > 0, p_d \in \Re$ and $\mu \not\geq \lambda$, there exists $n_1 \in D$ such that $\lambda(n) \not\leq \Big(p_d(x^n - x)(t)\Big)$ and $\lambda(n) \not\leq \mu$ with $n \geq n_1, n \in D$.*

Proof. Necessity. For each $t > 0, p_d \in \Re$ and $\mu \not\geq \lambda$, put $U_{p_d,t}(x) = p_d(x)(t)$. Then [Theorem 3.1, 13], $(U_{p_d,t})' \vee \mu^*$ is an R-neighborhood of θ_λ. From $\{x^n_{\lambda(n)}\} \to x_\lambda$, it follows that there exists $n_1 \in D$ such that $\forall n \geq n_1, x^n_{\lambda(n)} \not\leq x + (U_{p_d,t})' \vee \mu^*$, i.e. $x^n_{\lambda(n)} - x \not\leq (U_{p_d,t})' \vee \mu^*$. Hence $\lambda(n) \not\leq \Big(p_d(x^n - x)(t)\Big)'$ and $\lambda(n) \not\leq \mu$ with $n \geq n_1, n \in D$.

Sufficiency. By assumption, for each R-neighborhood $W = \bigvee_{i=1}^{m} (U_{p_{d_i},t})' \vee \mu^*$ of θ_λ, there exist $n_i \in D$ such that $\forall n \geq n_i, \lambda(n) \not\leq \Big(p_{d_i}(x^n - x)(t)\Big)'$ and $\lambda(n) \not\leq \mu, i = 1, 2, \cdots, n$. Let $n_0 \geq n_1, \cdots, n_m, n_0 \in D$, then $x^n_{\lambda(n)} \not\leq x + (U_{p_{d_i},t})'$ and $\lambda(n) \not\leq \mu$. From $\lambda(n) \in M(L)$, we have $x^n_{\lambda(n)} \not\leq x + W$. Therefore $\{x^n_{\lambda(n)}\} \to x_\lambda$. □

Theorem 4.5. *Let (L^X, δ_X), (L^Y, δ_Y) and (L_1^Z, δ_Z) be locally convex L-topological vector spaces and $(f, \phi)^{\to} : \mathrm{Pt}(L^{X\times Y}) \to \mathrm{Pt}(L_1^Z)$ be an L-continuous fuzzy bilinear operator. Then for each $x \in X$, $y \in Y$, $(f, \phi)^{\to}|_{x_1} : L^Y \to L_1^Z$ and $(f, \phi)^{\to}|_{y_1} : L^X \to L_1^Z$ are L-continuous fuzzy linear operators.*

Proof. It suffices to prove that $(f, \phi)^{\to}|_{y_1}$ is an L-continuous linear operator for each $y \in Y$. For each molecule net $\{x^n_{\lambda(n)}\}_{n\in D}$ in L^X, suppose that $x^n_{\lambda(n)} \xrightarrow{\delta_X} x_\lambda$. By Lemma 4.4, $y_{\lambda(n)} \xrightarrow{\delta_Y} y_\lambda$. So $(x^n, y)_{\lambda(n)} \xrightarrow{\delta_X\times\delta_Y} (x, y)_\lambda$. Since $(f, \phi)^{\to}$ is an L-continuous fuzzy bilinear operator, from Proposition 4.3, $(f, \phi)^{\to}(S) = (f(x^n, y))_{\phi(\lambda(n))} = (f, \phi)^{\to}|_{y_1}(x^n_{\lambda(n)}) \xrightarrow{\delta_Z} (f(x, y))_{\phi(\lambda)} = (f, \phi)^{\to}|_{y_1}(x_\lambda)$. This shows that $(f, \phi)^{\to}|_{y_1} : L^X \to L_1^Z$ is an L-continuous fuzzy linear operator. □

Theorem 4.6. *Suppose that (L^X, δ_X), (L^Y, δ_Y) and (L_1^Z, δ_Z) are locally convex L-topological vector spaces, $(f, \phi)^{\to} : \mathrm{Pt}(L^{X\times Y}) \to \mathrm{Pt}(L_1^Z)$ is an L-fuzzy bilinear operator. Then $(f, \phi)^{\to}$ is L-continuous iff $(f, \phi)^{\to}$ is L-continuous at $(\theta, \theta^1)_\lambda$ for all $\lambda \in M(L)$.*

Proof. By Proposition 4.3, the necessity is obvious.

Sufficiency. Let $S = (x^n, y^n)_{\lambda(n)}$ be a molecule net in $L^{X\times Y}$ and S converges to $(x, y)_\lambda$ with respect to $\delta_X \times \delta_Y$. Then $x^n_{\lambda(n)} \xrightarrow{\delta_X} x_\lambda$ and $y^n_{\lambda(n)} \xrightarrow{\delta_Y} y_\lambda$. By Theorem 4.5, $(f, \phi)^{\to}|_{y_1}(x^n_{\lambda(n)})$ converges to $(f, \phi)^{\to}|_{y_1}(x_\lambda) = (f(x, y))_{\phi(\lambda)}$ and $(f, \phi)^{\to}|_{x_1}(y^n_{\lambda(n)}) \xrightarrow{\delta_Z} (f, \phi)^{\to}|_{x_1}(y_\lambda) = (f(x, y))_{\phi(\lambda)}$ for each R-neighborhood W of $\theta^2_{\phi(\lambda)}$ in (L_1^Z, δ_Z). By Theorem 3.2, 3, there is an R-neighborhood P of $\theta^2_{\phi(\lambda)}$ such that $P' + P' + P' \leq W'$. For P, there exist $n_1,\ n_2 \in D$ such that $(f, \phi)^{\to}|_{y_1}(x^n_{\lambda(n)}) \not\leq f(x, y) + P$, $\forall n \geq n_1,\ n \in D$, and $(f, \phi)^{\to}|_{x_1}(y^n_{\lambda(n)}) \not\leq f(x, y) + P$, $\forall n \geq n_2, n \in D$. Since D is a directed set, for $n_1,\ n_2 \in D$, there is a $n_3 \in D$ with $n_3 \geq n_1,\ \ n_3 \geq n_2$. Thus

$$(f(x^n - x, y))_{\phi(\lambda(n))} \not\leq P \quad \text{and} \quad (f(x, y^n - y))_{\phi(\lambda(n))} \not\leq P,\ \forall n \geq n_3, n \in D \quad (4.1)$$

Moreover, from $S = (x^n, y^n)_{\lambda(n)} \xrightarrow{\delta_X\times\delta_Y} (x, y)_\lambda$, we may deduce that $(x^n - x, y^n - y)_{\lambda(n)} \xrightarrow{\delta_X} (\theta, \theta^1)_\lambda$. Because of the continuity of $(f, \phi)^{\to}$ at $(\theta, \theta^1)_\lambda$, on P, there exists $n_4 \in D$ such that $(f, \phi)^{\to}((x^n - x, y^n - y)_{\lambda(n)} \not\leq P$ for all $n \geq n_4$, i.e.

$$(f(x^n - x, y^n - y))_{\phi(\lambda(n))} \not\leq P, \quad (4.2)$$

Let $n_0 \in D$ and $n_0 \geq n_3,\ n_0 \geq n_4$. The formulas (4.1) and (4.2) hold for all $n \geq n_0$. On the other hand, since $(f(x^n, y^n) - f(x,y))_{\phi(\lambda(n))} = (f(x^n - x, y^n - y))_{\phi(\lambda(n))} + (f(x, y^n - y))_{\phi(\lambda(n))} + (f(x^n - x, y))_{\phi(\lambda(n))}$ we have

$$(f(x^n, y^n) - f(x,y))_{\phi(\lambda(n))} \not\leq (P^{'} + P^{'} + P^{'})^{'}, \quad \forall n \geq n_0.$$

So $(f(x^n, y^n) - f(x,y))_{\phi(\lambda(n))} \not\leq W,\ \ \forall n \geq n_0$, i.e. $(f,\phi)^{\rightarrow}((x^n, y^n)_{\lambda(n)}) \not\leq f(x,y) + W,\ \ \forall n \geq n_0$. This means $(f,\phi)^{\rightarrow}$ is L-continuous at $(x,y)_\lambda$. From the arbitrariness of $(x,y)_\lambda$ and Proposition 1, sufficiency is proved. □

5. **Conclusion**

After presenting the definition of L-fuzzy bilinear operators, we discuss some of their properties. Then we obtain the following decomposition theorem of L-fuzzy bilinear operator : An L-fuzzy bilinear operator can be determined uniquely by the corresponding bilinear operator and a finitely meet-preserving order-homomorphism. We also prove that our definition of L-fuzzy bilinear operator is a special case of powerset operators in the sense of Rodabaugh. Finally we prove that an L-fuzzy bilinear operator is L-continuous iff it is L-continuous at $(\theta, \theta^1)_\lambda$ for all $\lambda \in M(L)$. Other properties of L-fuzzy continuous bilinear operators were also discussed.

6. **Future Research**

We claim that the theory of duality of classical topological vector spaces is a significant part of topological vector spaces and bilinear operators play an important role in discussion of the theory of duality. The motivation of this paper is to introduce the concept of L-fuzzy bilinear operators, and discuss their structure. In fact, a direction worth pursuing is to study the theory of duality in L-topological vector spaces after introducing the definition of L-fuzzy bilinear operators. The condition which makes L-fuzzy separately continuous bilinear operator be L-fuzzy continuous is also worthy of discussion. These will be discussed in the subsequently paper.

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