INTUITIONISTIC FUZZY BOUNDED LINEAR OPERATORS

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ABSTRACT. The object of this paper is to introduce the notion of intuitionistic fuzzy continuous mappings and intuitionistic fuzzy bounded linear operators from one intuitionistic fuzzy n-normed linear space to another. Relation between intuitionistic fuzzy continuity and intuitionistic fuzzy bounded linear operators are studied and some interesting results are obtained.

1. Introduction

The main purpose of this paper is to introduce the notion of intuitionistic fuzzy continuous mappings and intuitionistic fuzzy bounded linear operators in line with the classical results of fuzzy bounded linear operators [5, 6, 10, 11]. The theory of n-norms on a linear space was introduced by Gähler [13, 14] and subsequently developed by several authors [12, 16, 18, 23]. The theory of intuitionistic fuzzy sets was introduced by Atanassov [1, 2, 3] and the notions of fuzzy n-normed linear space and intuitionistic fuzzy n-normed linear space have been recently introduced by the present authors [21, 25]. The concepts of t-norm and t-co-norm were introduced by Schweizer and Sklar [24]. Fuzzy topology has very important applications in quantum particle physics, particularly in connection with both string and $\epsilon^{(\infty)}$ theory which were studied by Elnaschie [8, 9].

In this paper we introduce the notions of intuitionistic fuzzy continuities and intuitionistic fuzzy bounded linear operators and discuss interesting relationships that exist between them.

In this section we give some definitions and results which are necessary for the sequel. Throughout this paper for convenience we denote a given linear space X over a field F by $\underbrace{X \times ... \times X}_{n} = X^n$. Let $Z = (x_1, x_2, ..., x_n) \in X^n$.

Definition 2.1.[15] Let $n \in$N(natural numbers) and X be a real linear space of dimension $d \geq n$.(Here we allow d to be infinite). A real-valued function $||\bullet, ..., \bullet||$ on X^n is called an n-norm on X if it satisfies the following properties:

(1) $||x_1, x_2, ..., x_n|| = 0$ if any only if $x_1, x_2, ..., x_n$ are linearly dependent
(2) $||x_1, x_2, ..., x_n||$ is invariant under any permutation of $x_1, x_2, ..., x_n$
(3) $||x_1, x_2, ..., \alpha x_n|| = |\alpha| \; ||x_1, x_2, ..., x_n||$, for any real α
(4) $||x_1, x_2, , x_{n-1}, y + z|| \leq ||x_1, x_2, ..., x_{n-1}, y|| + ||x_1, x_2, ..., x_{n-1}, z||$,

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The pair $(X, ||\bullet, ..., \bullet||)$ is called an n-normed linear space.

Definition 2.2.[15] A sequence $\{x_k\}$ in an n-normed linear space (X,$||\bullet, ..., \bullet||$) is said to converge to an $x \in X$ (in the n-norm)whenever
$\lim_{k\to\infty} ||x_1, x_2, ..., x_{n-1}, x_k - x|| = 0$
for every $x_1, x_2, ..., x_{n-1} \in X$.

Definition 2.3.[21] Let X be a linear space over a real field F. A fuzzy subset N of $X^n \times R$(R- set of real numbers) is called a fuzzy n-norm on X
if and only if:
$(N1)$ For all $t \in R$ with $t \leq 0$, $N(Z, t) = 0$.
$(N2)$ For all $t \in R$ with $t > 0$,

$$N(x_1, x_2, ..., x_n, t) = 1 \Leftrightarrow x_1, x_2, ..., x_n \text{ are linearly dependent.}$$

$(N3)$ $N(x_1, x_2, ..., x_n, t)$ is invariant under any permutation of $x_1, x_2, ..., x_n$.
$(N4)$ For all $t \in$R with $t > 0$,

$$N(x_1, x_2, ..., cx_n, t) = N(x_1, x_2, ..., x_n, \frac{t}{|c|}), \text{ if} c \neq 0, c \in F(\text{field}).$$

$(N5)$ For all $s, t \in R$,

$$\begin{aligned} &N(x_1, x_2, ..., x_n + x_n^{'}, s + t) \\ &\quad \geq \min \left\{ N(x_1, x_2, ..., x_n, s), N(x_1, x_2, ..., x_n^{'}, t) \right\}. \end{aligned}$$

$(N6)$ $N(Z, \circ)$ is a non-decreasing function of R and $\lim_{t\to\infty} N(Z, t) = 1$.

The pair (X, N) is called a fuzzy n-normed linear space or in short f-n-NLS.

Definition 2.4.[24] A binary operation $* : [0, 1] \times [0, 1] \to [0, 1]$ is a continuous t-norm if $*$ satisfies the following conditions:

(1) $*$ is commutative and associative
(2) $*$ is continuous
(3) $a * 1 = a$ for all $a \in [0, 1]$
(4) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ and $a, b, c, d \in [0, 1]$.

Definition 2.5.[24] A binary operation $\diamond : [0, 1] \times [0, 1] \to [0, 1]$ is a continuous t-co-norm if $\diamond$ satisfies the following conditions:

(1) $\diamond$ is commutative and associative
(2) $\diamond$ is continuous
(3) $a \diamond 0 = a$ for all $a \in [0, 1]$
(4) $a \diamond b \leq c \diamond d$ whenever $a \leq c$ and $b \leq d$ and $a, b, c, d \in [0, 1]$.

Definition 2.6.[1] Let E be any set. An intuitionistic fuzzy set A of E is an object of the form $A = \{(x, \mu_A(x), \gamma_A(x))|x \in E\}$, where the functions
$\mu_A : E \to [0, 1]$ and $\gamma_A : E \to [0, 1]$ respectively denote the degree of membership

and non-membership of the element $x \in E$ and for every $x \in E$,
$0 \leq \mu(x) + \gamma(x) \leq 1$.

Definition 2.7.[25] An intuitionistic fuzzy n-normed linear space (i-f-n-NLS) is an object of the form

$$A = \{(Z, N(Z,t), M(Z,t)) | Z \in X^n\},$$

where X is a linear space over a real field F, $*$ is a continuous t-norm, $\diamond$ is a continuous t-co-norm and N, M are fuzzy sets on $X^n \times (0, \infty)$ such that
N denotes the degree of membership, M denotes the degree of non-membership of $(Z,t) \in X^n \times (0,\infty)$ and N and M satisfy the following conditions:

(i) $N(Z,t) + M(Z,t) \leq 1$;
(ii) $N(Z,t) > 0$;
(iii) $N(x_1, x_2, ..., x_n, t) = 1$ if and only if $x_1, x_2, ..., x_n$ are linearly dependent;
(iv) $N(x_1, x_2, ..., x_n, t)$ is invariant under permutation of $x_1, x_2, ..., x_n$;
(v) $N(x_1, x_2, ..., cx_n, t) = N(x_1, x_2, ..., x_n, \frac{t}{|c|})$ if $c \neq 0$, $c \in F$;
(vi) $N(x_1, x_2, ..., x_n, s) * N(x_1, x_2, ..., x_n', t) \leq N(x_1, x_2, ..., x_n + x_n', s + t)$;
(vii) $N(Z, \circ) : (0, \infty) \to [0,1]$ is continuous;
$(viii)$ $M(Z,t) > 0$;
(ix) $M(x_1, x_2, ..., x_n, t) = 0$ if and only if $x_1, x_2, ..., x_n$ are linearly dependent;
(x) $M(x_1, x_2, ..., x_n, t)$ is invariant under permutation of $x_1, x_2, ..., x_n$;
(xi) $M(x_1, x_2, ..., cx_n, t) = M(x_1, x_2, ..., x_n, \frac{t}{|c|})$, if $c \neq 0$, $c \in F$;
(xii) $M(x_1, x_2, ..., x_n, s) \diamond M(x_1, x_2, ..., x_n', t) \geq M(x_1, x_2, ..., x_n + x_n', s + t)$;
$(xiii)$ $M(Z, \circ) : (0, \infty) \to [0,1]$ is continuous.

Remark 2.8. For convenience we denote the intuitionistic fuzzy n-normed linear space by $A = (X, N, M, *, \diamond)$.

2. Intuitionistic Fuzzy Continuous Mappings

In this section we introduce different types of intuitionistic fuzzy continuous mappings over intuitionistic fuzzy n-normed linear spaces and derive some related results.

In what follows on let $A = (X, N_1, M_1, *, \diamond)$ and $B = (Y, N_2, M_2, *', \diamond')$ will denote two intuitionistic fuzzy n-normed linear spaces, where X and Y are linear spaces over the same real field.

Definition 3.1. A mapping T from one intuitionistic fuzzy n-normed linear space A to another intuitionistic fuzzy n-normed linear space B is said to be intuitionistic fuzzy continuous at $Z_0 = (x_{01}, x_{02}, ..., x_{0n}) \in X^n$ if, for given $\epsilon > 0$, $\alpha \in (0,1)$, there exist $\delta = \delta(\alpha, \epsilon) > 0$ and $\beta = \beta(\alpha, \epsilon) > 0$ such that for all $Z \in X^n$,
$N_1((x_1, x_2, ..., x_n) - (x_{01}, x_{02}, ..., x_{0n}), \delta) > \beta$
and $M_1((x_1, x_2, ..., x_n) - (x_{01}, x_{02}, ..., x_{0n}), \delta) < 1 - \beta$
$\Rightarrow N_2(T(x_1, x_2, ..., x_n) - T(x_{01}, x_{02}, ..., x_{0n}), \epsilon) > \alpha$

and $M_2(T(x_1, x_2, ..., x_n) - T(x_{01}, x_{02}, ..., x_{0n}), \epsilon) < 1 - \alpha$.
Briefly,
$N_1(Z - Z_0, \delta) > \beta$ and $M_1(Z - Z_0, \delta) < 1 - \beta$
$\Rightarrow N_2(T(Z) - T(Z_0), \epsilon) > \alpha$ and $M_2(T(Z) - T(Z_0), \epsilon) < 1 - \alpha$.

If T is intuitionistic fuzzy continuous at each point of X^n, then T is said to be intuitionistic fuzzy continuous on X^n.

Definition 3.2. A mapping T from one intuitionistic fuzzy n-normed linear space A to another intuitionistic fuzzy n-normed linear space B is said to be strongly intuitionistic fuzzy continuous at $Z_0 \in X^n$ if, for given $\epsilon > 0$, there exists $\delta > 0$ such that for all $Z \in X^n$,
$N_2(T(Z) - T(Z_0), \epsilon) \geq N_1(Z - Z_0, \delta)$ and
$M_2(T(Z) - T(Z_0), \epsilon) \leq M_1(Z - Z_0, \delta)$.

If T is strongly intuitionistic fuzzy continuous at each point of X^n, then T is said to be strongly intuitionistic fuzzy continuous on X^n.

Definition 3.3. A mapping T from one intuitionistic fuzzy n-normed linear space A to another intuitionistic fuzzy n-normed linear space B is said to be weakly intuitionistic fuzzy continuous at $Z_0 \in X^n$ if, for given $\epsilon > 0$ and $\alpha \in (0, 1)$, there exists $\delta = \delta(\alpha, \epsilon) > 0$ such that for all $Z \in X^n$,
$N_1(Z - Z_0, \delta) \geq \alpha$ and $M_1(Z - Z_0, \delta) \leq 1 - \alpha$
$\Rightarrow N_2(T(Z) - T(Z_0), \epsilon) \geq \alpha$ and $M_2(T(Z) - T(Z_0), \epsilon) \leq 1 - \alpha$.

If T is weakly intuitionistic fuzzy continuous at each point of X^n, then we say that T is weakly intuitionistic fuzzy continuous on X^n.

Definition 3.4. A mapping T from one intuitionistic fuzzy n-normed linear space A to another intuitionistic fuzzy n-normed linear space B is said to be sequentially intuitionistic fuzzy continuous at $Z_0 \in X^n$, if for any sequence $\{Z_k, k \geq 1\}$, with $Z_k \to Z_0$ implies $T(Z_k) \to T(Z_0)$.
In other words,
$\lim_{k\to\infty} N_1(Z_k - Z_0, t) = 1$ and $\lim_{k\to\infty} M_1(Z_k - Z_0, t) = 0$, for all $t > 0$.
$\Rightarrow \lim_{k\to\infty} N_2(T(Z_k) - T(Z_0), t) = 1$ and $\lim_{k\to\infty} M_2(T(Z_k) - T(Z_0), t) = 0$,
for all $t > 0$.

If T is sequentially intuitionistic fuzzy continuous at each point of X^n, then T is said to be sequentially intuitionistic fuzzy continuous on X^n.

Theorem 3.5. Let $T : A \to B$ be a mapping, where A and B are intuitionistic fuzzy n-normed linear spaces. If T is strongly intuitionistic fuzzy continuous then it is sequentially intuitionistic fuzzy continuous, but not conversely.

Proof. First we suppose that T is strongly intuitionistic fuzzy continuous at Z_0. Then for each $\epsilon > 0$ there exists $\delta > 0$ such that for all Z,

$N_2(T(Z)-T(Z_0),\epsilon) \geq N_1(Z-Z_0,\delta)$ and
$M_2(T(Z)-T(Z_0),\epsilon) \leq M_1(Z-Z_0,\delta)$. (3.1)

Let $\{Z_k, k \geq 1\}$ be a sequence such that $Z_k \to Z_0$.

That is, $\lim\limits_{k\to\infty} N_1(Z_k - Z_0, t) = 1$ and $\lim\limits_{k\to\infty} M_1(Z_k - Z_0, t) = 0$ for all $t > 0$.

Now from (3.1) we have,

$N_2(T(Z_k)-T(Z_0),\epsilon) \geq N_1(Z_k-Z_0,\delta)$ and
$M_2(T(Z_k)-T(Z_0),\epsilon) \leq M_1(Z_k-Z_0,\delta)$
$\Rightarrow \lim\limits_{k\to\infty} N_2(T(Z_n)-T(Z_0),\epsilon) \geq \lim\limits_{k\to\infty} N_1(Z_k-Z_0,\delta)$ and
$\lim\limits_{k\to\infty} M_2(T(Z_k)-T(Z_0),\epsilon) \leq \lim\limits_{k\to\infty} M_1(Z_k-Z_0,\delta)$
$\Rightarrow \lim\limits_{k\to\infty} N_2(T(Z_k)-T(Z_0),\epsilon) = 1$ and $\lim\limits_{k\to\infty} M_2(T(Z_k)-T(Z_0),\epsilon) = 0$.

Since ϵ is a small arbitrary positive number, it follows that $T(Z_k) \to T(Z_0)$. □

The following example shows that sequentially intuitionistic fuzzy continuity of T does not imply strong intuitionistic fuzzy continuity of T.

Example 3.6. Let (X,$||\bullet,\bullet,...,\bullet||$) be an n-normed linear space.
Define $a * b$=min$\{a,b\}$ and $a \diamond b$=max$\{a,b\}$ for all $a,b \in [0,1]$, $t > 0$, $c > 0$,

$N_1(x_1,x_2,...,x_n,t) = \frac{t}{t+||x_1,x_2,...,x_n||}$, $M_1(x_1,x_2,...,x_n,t) = \frac{||x_1,x_2,...,x_n||}{t+||x_1,x_2,...,x_n||}$.

$N_2(x_1,x_2,...,x_n,t) = \frac{t}{t+c||x_1,x_2,...,x_n||}$, $M_2(x_1,x_2,...,x_n,t) = \frac{c||x_1,x_2,...,x_n||}{t+c||x_1,x_2,...,x_n||}$

Briefly, $N_1(Z,t) = \frac{t}{t+||Z||}$, $M_1(Z,t) = \frac{||Z||}{t+||Z||}$, $N_2(Z,t) = \frac{t}{t+c||Z||}$ and $M_2(Z,t) = \frac{k||Z||}{t+c||Z||}$.

Then

$$A = \{(Z, N_1(Z,t), M_1(Z,t)) | Z \in X^n\}$$

and

$$B = \{(Z, N_2(Z,t), M_2(Z,t)) | Z \in X^n\}$$

are intuitionistic fuzzy n-normed linear spaces.

Now consider a function $T(Z) = \frac{Z^4}{1+Z^2}$

Choose a sequence $\{Z_k, k \geq 1\}$ such that $Z_k \to Z_0$.

Now for all $t > 0$ we have,

$\lim\limits_{k\to\infty} N_1(Z_k-Z_0,t) = 1$ and $\lim\limits_{k\to\infty} M_1(Z_k-Z_0,t) = 0$.
$\Rightarrow \lim\limits_{k\to\infty} \frac{t}{t+||Z_k-Z_0||} = 1$ and $\lim\limits_{k\to\infty} \frac{||Z_k-Z_0||}{t+|Z_k-Z_0|} = 0$
$\Rightarrow \lim\limits_{k\to\infty} ||Z_K - Z_0|| = 0$. (3.2)

Now $N_2(T(Z_k) - T(Z_0), t)$

$$= \frac{t}{t+c||\frac{Z_k^4}{1+Z_k^2} - \frac{Z_0^4}{1+Z_o^2}||}$$
$$= \frac{t||1+Z_n^2||||1+Z_0^2||}{t||1+Z_n^2||||1+Z_0^2|+c||Z_k^4+Z_k^4 Z_0^2 - Z_0^4 - Z_k^2 Z_0^4||}$$
$$= \frac{t||1+Z_k^2||||1+Z_0^2||}{t||1+Z_k^2||||1+Z_0^2||+c||(Z_k-Z_0)(Z_k+Z_0)(Z_k^2+Z_0^2)+Z_k^2 Z_0^2(Z_k^2-Z_0^2||}$$
$$= \frac{t||1+Z_k^2||1+Z_0^2||}{t||1+Z_k^2||||1+Z_0^2||+c||Z_k-Z_0||||(Z_k+Z_0)(Z_k^2+Z_0^2)+Z_k^2 Z_0^2(Z_k+Z_0)||} \text{ and}$$

$M_2(T(Z_k) - T(Z_0), t)$

$$= \frac{c||\frac{Z_k^4}{1+Z_k^2} - \frac{Z_0^4}{1+Z_0^2}||}{t+c||\frac{Z_k^4}{1+Z_k^2} - \frac{Z_0^4}{1+Z_o^2}||}$$
$$= \frac{c||Z_k^4+Z_k^4 Z_0^2 - Z_0^4 - Z_0^4 Z_k^2||}{t||1+Z_n^2||||1+Z_0^2|+c||Z_k^4+Z_k^4 Z_0^2 - Z_0^4 - Z_0^4 Z_k^2||}$$
$$= \frac{c||Z_k-Z_0||||(Z_k+Z_0)(Z_k^2+Z_0^2)+Z_k^2 Z_0^2(Z_k+Z_0)||}{t+||1+Z_k^2||||1+Z_0^2||+c||Z_k-Z_0||||(Z_k+Z_0)(Z_k^2+Z_0^2)+Z_k^2 Z_0^2(Z_k+Z_0)||}.$$

Hence by (3.2) it follows that for all $t > 0$,

$$\lim_{k\to\infty} N_2(T(Z_k) - T(Z_0), t) = 1 \text{ and } \lim_{k\to\infty} M_2(T(Z_k) - T(Z_0), t) = 0$$
$$\Rightarrow T(Z_k) \to T(Z_0) \text{ in } B.$$

Let $\epsilon > 0$ be given. Then

$N_2(T(Z) - T(Z_0), \epsilon) \geq N_1(Z - Z_0, \delta)$

$$\Rightarrow \frac{\epsilon||1+Z^2||||1+Z_0^2||}{\epsilon||1+Z^2||||1+Z_0^2||+k||Z-Z_0||||(Z+Z_0)(Z^2+Z_0^2)+Z^2 Z_0^2(Z+Z_0)||} \geq \frac{\delta}{\delta+||Z-Z_0)||}$$

and $M_2(T(Z) - T(Z_0), \epsilon) \leq M_1(Z - Z_0, \delta)$

$$\Rightarrow \frac{k||Z-Z_0||||(Z+Z_0)(Z^2+Z_0^2)+Z^2 Z_0^2(Z+Z_0)||}{\epsilon+||1+Z^2||||1+Z_0^2|+k||Z-Z_0||||(Z+Z_0)(Z^2+Z_0^2)+Z^2 Z_0^2(Z+Z_0)||} \leq \frac{\delta}{\delta+||Z-Z_0||}$$

So, $c\delta||Z - Z_0||||Z + Z_0||Z^2 + Z_0^2 + Z^2 Z_0^2|| \leq \epsilon||1 + Z^2||||1 + Z_0^2||Z - Z_0||$

$$\Rightarrow \delta \leq \frac{\epsilon||1+Z^2||1+Z_0^2||}{k||Z+Z_0||||Z^2+Z_0^2+Z^2 Z_0^2||} \text{ (for } Z \neq Z_0). \tag{3.3}$$

We see that T is continuous at Z_0 if there exists $\delta > 0$ satisfying (3.3) for all $Z \neq Z_0$.

Let $\delta_1 = \inf \frac{||1+Z^2||||1+Z_0^2||}{||Z+Z_0||||Z^2+Z_0^2+Z^2 Z_0^2||}$ where the infimum is taken over all Z, $Z \neq Z_0$. Then $\delta = \frac{\epsilon}{k}\delta_1$ satisfies (3.3).

But $\delta_1 = 0$ which is impossible.

Therefore T is not strongly intuitionistic fuzzy continuous.

Theorem 3.7. Let $T : A \to B$ be a mapping, where A and B are intuitionistic fuzzy n-normed linear spaces. Then T is intuitionistic fuzzy continuous if and only if it is sequentially intuitionistic fuzzy continuous .

Proof. Suppose that T is intuitionistic fuzzy continuous at Z_0. Let $\{Z_k, k \geq 1\}$ be a sequence such that $Z_k \to Z_0$. Let $\epsilon > 0$, be given. Since T is intuitionistic fuzzy continuous at Z_0 there exists $\delta > 0$ and $\beta = \beta(\alpha, \epsilon) \in (0, 1)$ such that for all Z,

$$N_1(Z - Z_0, \delta) > \beta \text{ and } M_1(Z - Z_0, \delta) < 1 - \beta$$
$$\Rightarrow N_2(T(Z) - T(Z_0), \epsilon) > \alpha \text{ and } M_2(T(Z) - T(Z_0), \epsilon) < 1 - \alpha.$$

Since $Z_k \to Z_0$, there exists a positive integer k_0 such that

$$N_1(Z_k - Z_0, \delta) > \beta \text{ and } M_1(Z_k - Z_0, \delta) < 1 - \beta, \text{ for all } k \geq k_0.$$

Then

$$N_2(T(Z_k) - T(Z_0), \epsilon) > \alpha \text{ and } M_2(T(Z_k) - T(Z_0), \epsilon) < 1 - \alpha$$
$$\Rightarrow \lim_{k\to\infty} N_2(T(Z_k) - T(Z_0), t) = 1 \text{ and } \lim_{k\to\infty} M_2(T(Z_k) - T(Z_0), t) = 0.$$

Since $\epsilon > 0$ is arbitrary, hence $T(Z_k) \to T(Z_0)$.

Next we suppose that T is sequentially intuitionistic fuzzy continuous at Z_0.

If possible, suppose that T is not intuitionistic fuzzy continuous at Z_0.

Then there exist $\epsilon > 0$ and $\alpha > 0$ such that for any $\delta > 0$ and $\beta \in (0, 1)$ there exists $W \in X^n$ (depending on δ, β) such that

$$N_1(Z_0 - W, \delta) > \beta \text{ and } M_1(Z_0 - W, \delta) < 1 - \beta,$$
$$\text{but } N_2(T(Z_0) - T(W), \epsilon) \leq \alpha \text{ and } M_2(T(Z_0) - T(W), \epsilon) \geq 1 - \alpha. \tag{3.4}$$

Thus, for $\beta = 1 - \frac{1}{k+1}$, $\delta = \frac{1}{k+1}$, $k = 1, 2, ...$, there exists W_k such that

$$N_1(Z_0 - W_k, \tfrac{1}{k+1}) > 1 - \tfrac{1}{k+1} \text{ and } M_1(Z_0 - W_k, \tfrac{1}{k+1}) < \tfrac{1}{k+1},$$
$$\text{but } N_2(T(Z_0) - T(W_k), \epsilon) \leq \alpha \text{ and } M_2(T(Z_0) - T(W_k), \epsilon) \geq 1 - \alpha.$$

Taking $\delta > 0$, there exists k_0 such that $\frac{1}{k+1} < \delta$ for all $k \geq k_0$.

Then

$$N_1(Z_0 - W_k, \delta) \geq N_1(Z_0 - W_k, \tfrac{1}{k+1}) \geq 1 - \tfrac{1}{k+1} \text{ and}$$
$$M_1(Z_0 - W_k, \delta) \leq M_1(Z_0 - W_k, \tfrac{1}{k+1}) \leq \tfrac{1}{k+1}$$
$$\Rightarrow \lim_{k\to\infty} N_1(Z_0 - W_k, \delta) = 1 \text{ and } \lim_{n\to\infty} M_2(Z_0 - W_k, \delta) = 0$$
$$\Rightarrow W_k \to Z_0.$$

However by (3.4), $N_2(T(Z_0) - T(W_k), \epsilon) \leq \alpha$ and $M_2(T(Z_0) - T(W_k), \epsilon) \geq 1 - \alpha$.

Hence $\lim_{k\to\infty} N_2(T(Z_0) - T(W_k), \epsilon) \neq 1$ and $\lim_{k\to\infty} M_2(T(Z_0) - T(W_k), \epsilon) \neq 0$.

Thus $T(W_k)$ does not converge to $T(Z_0)$ but $W_k \to Z_0$ which contradicts our assumption.

Hence T is intuitionistic fuzzy continuous at Z_0. □

3. **Intuitionistic Fuzzy Bounded Linear Operators**

We now introduce the notion of strongly intuitionistic fuzzy bounded linear operators and weakly intuitionistic fuzzy bounded linear operators on an intuitionistic fuzzy n-normed linear space.

Definition 4.1. Let $T : A \to B$ be a linear operator where A and B are intuitionistic fuzzy n-normed linear spaces. T is said to be strongly intuitionistic fuzzy bounded on X^n, if and only if there exists a positive number L such that for all $Z \in X^n$ and $s > 0$,
$N_2(T(Z), s) \geq N_1(Z, \frac{s}{L})$ and $M_2(T(Z), s) \leq M_1(Z, \frac{s}{L})$.

Definition 4.2. Let $T : A \to B$ be a linear operator where A and B are intuitionistic fuzzy n-normed linear spaces. T is said to be weakly intuitionistic fuzzy bounded on X^n, if for any $\alpha \in (0, 1)$ there exists $L_\alpha > 0$ such that for all $Z \in X^n$ and $t > 0$,
$N_1(Z, \frac{t}{L_\alpha}) \geq \alpha$ and $M_1(Z, \frac{t}{L_\alpha}) \leq 1 - \alpha$
$\Rightarrow N_2(T(Z), t) \geq \alpha$ and $M_2(T(Z), t) \leq 1 - \alpha$.

Theorem 4.3. For intuitionistic fuzzy n-normed linear spaces A and B, let $T : A \to B$ be a linear operator. If T is strongly intuitionistic fuzzy bounded then it is weakly intuitionistic fuzzy bounded, but not conversely.

Proof. First we suppose that T is strongly intuitionistic fuzzy bounded.
Then there exists $L > 0$ such that for all Z and $t > 0$, we have
$N_2(T(Z), t) \geq N_1(Z, \frac{t}{L})$ and $M_2(T(Z), s) \leq M_1(Z, \frac{t}{L})$.

Thus, for any $\alpha \in (0, 1)$, there exists $L_\alpha (= L > 0)$ such that
$N_1(Z, \frac{t}{L_\alpha}) \geq \alpha$ and $M_1(Z, \frac{t}{L_\alpha}) \leq 1 - \alpha$
$\Rightarrow N_2(T(Z), t) \geq \alpha$ and $M_2(T(Z), t) \leq 1 - \alpha$, for all $Z \in X^n$ and $t > 0$.

This implies that T is weakly intuitionistic fuzzy bounded. □

We now provide a counterexample to the converse.

Example 4.4. Let (X,$||\bullet, \bullet, ..., \bullet||$) be an n-normed linear space.
Define $a * b$=min$\{a, b\}$ and $a \diamond b$=max$\{a, b\}$ for all $a, b \in [0, 1]$,

$N_1(Z, t) = \frac{t^2 - ||Z||^2}{t^2 + ||Z||^2}$, $M_1(Z, t) = \frac{2||Z||}{t^2 + ||Z||^2}$, $t > ||Z||$ and
$N_2(Z, t) = \frac{t}{t + ||Z||}$, $M_2(Z, t) = \frac{||Z||}{t + ||Z||}$.

Then for $t > ||Z||$,

$N_1(Z, \frac{t}{L_\alpha}) \geq \alpha$
$\Rightarrow \frac{t^2(1-\alpha)^2 - ||Z||^2}{t^2(1-\alpha)^2 + ||Z||^2} \geq \alpha$
$\Rightarrow t^2(1 - \alpha)^2 - ||Z||^2 \geq \alpha t^2(1 - \alpha)^2 + \alpha||Z||^2$
$\Rightarrow t^2(1 - \alpha)^2 - \alpha t^2(1 - \alpha)^2 \geq ||Z||^2(1 + \alpha)$
$\Rightarrow ||Z||^2 \leq \frac{t^2(1-\alpha)^3}{1+\alpha}$

$\Rightarrow ||Z|| \leq \frac{t(1-\alpha)(1-\alpha)^{\frac{1}{2}}}{(1+\alpha)^{\frac{1}{2}}}$

$\Rightarrow t + ||Z|| \leq \frac{t((1-\alpha)(1-\alpha)^{\frac{1}{2}}+(1+\alpha)^{\frac{1}{2}})}{(1+\alpha)^{\frac{1}{2}}}$

$$\Rightarrow \frac{t}{t+||Z||} \geq \frac{(1+\alpha)^{\frac{1}{2}}}{(1-\alpha)(1-\alpha)^{\frac{1}{2}}+(1+\alpha)^{\frac{1}{2}}} \tag{4.1}$$

Now,

$$\frac{(1+\alpha)^{\frac{1}{2}}}{(1-\alpha)(1-\alpha)^{\frac{1}{2}}+(1+\alpha)^{\frac{1}{2}}} \geq \alpha$$
$$\Rightarrow 1+\alpha+\alpha^3 \geq \alpha^2$$

Thus by (4.1), we have
$\frac{t}{t+||Z||} \geq \alpha$, for all $\alpha \in (0,1)$.
So,

$$N_1(\ Z, \frac{t}{L_\alpha}) \geq \alpha$$
$$\Rightarrow N_2(T(Z),t) \geq \alpha.$$

Similarly, for

$$M_1(\ Z, \frac{t}{L_\alpha}) \leq 1-\alpha$$
$$\Rightarrow M_2(T(Z),t) \leq 1-\alpha.$$

Thus $N_1(Z, \frac{t}{L_\alpha}) \geq \alpha$ and $M_1(Z, \frac{t}{L_\alpha}) \leq 1-\alpha$

$$\Rightarrow N_2(T(Z),t) \geq \alpha \text{ and } M_2(T(Z),t) \leq 1-\alpha.$$

Hence T is weakly intuitionistic fuzzy bounded.

Now, for $t > ||Z||$,

$N_2(T(Z),t) \geq N_1(Z, \frac{t}{L})$ and $M_2(T(Z),t) \leq M_1(Z, \frac{t}{L})$.

$$\Leftrightarrow \ L^2 \geq \frac{t^2}{2t||Z||+||Z||^2}$$
$$\Leftrightarrow L \geq (\frac{t^2}{(2t||Z||+||Z||^2)})^{\frac{1}{2}}$$
$$\Leftrightarrow L = \infty \text{ as } t \to \infty.$$

Hence T is not strongly intuitionistic fuzzy bounded.

Theorem 4.5. Let $T : A \to B$ be a linear operator where A and B are intuitionistic fuzzy n-normed linear spaces. Then
(a) T is strongly intuitionistic fuzzy continuous everywhere on X^n if T is strongly intuitionistic fuzzy continuous at a point Z_0.
(b) T is strongly intuitionistic fuzzy continuous if and only if T is strongly intuitionistic fuzzy bounded.

Proof. (a) Since T is strongly intuitionistic fuzzy continuous at Z_0, for each $\epsilon > 0$ there exists $\delta > 0$ such that, for all Z we have:
$N_2(T(Z)-T(Z_0),\epsilon) \geq N_1(Z-Z_0,\delta)$ and $M_2(T(Z)-T(Z_0),\epsilon) \leq M_1(Z-Z_0,\delta)$.

Let $W \in X^n$, and replace Z by $Z + Z_0 - W$. Then

$$\begin{aligned} N_2(\ &T(Z + Z_0 - W) - T(Z_0), \epsilon) \geq N_1(Z + Z_0 - W - Z_0, \delta) \text{ and} \\ &M_2(T(Z + Z_0 - W) - T(Z_0), \epsilon) \leq M_1(Z + Z_0 - W - Z_0, \delta). \\ &\Rightarrow N_2(T(Z) + T(Z_0) - T(W) - T(Z_0), \epsilon) \geq N_1(Z - W, \delta) \\ &\quad \text{and } M_2(T(Z) + T(Z_0) - T(W) - T(Z_0), \epsilon) \leq M_1(Z - W, \delta) \\ &\Rightarrow N_2(T(Z) - T(W), \epsilon) \geq N_1(Z - W, \delta) \\ &\quad \text{and } M_2(T(Z) - T(W), \epsilon) \leq M_1(Z - W, \delta). \end{aligned}$$

Since W is arbitrary, it follows that T is strongly intuitionistic fuzzy continuous on X^n.

(b) Assume T is strongly intuitionistic fuzzy bounded.

Then there exists $L > 0$ such that

$N_2(T(Z), \epsilon) \geq N_1(Z, \frac{\epsilon}{L})$ and $M_2(T(Z), \epsilon) \leq M_1(Z, \frac{\epsilon}{L})$ for all $\epsilon > 0$.
$\Rightarrow N_2(T(Z) - T(\underline{0}), \epsilon) \geq N_1(Z - \underline{0}, \delta)$ and $M_2(T(Z) - T(\underline{0}), \epsilon) \leq M_1(Z - \underline{0}, \delta)$,
where $\delta = \frac{\epsilon}{L}$ and $\underline{0} = (0, 0, ..., 0) \in X^n$.
$\Rightarrow T$ is strongly intuitionistic fuzzy continuous on X^n.

Conversely, suppose that T is strongly intuitionistic fuzzy continuous on X^n.

Using continuity of T at $Z = \underline{0}$ for $\epsilon = 1$, there exists $\delta > 0$ such that
$N_2(T(Z) - T(\underline{0}), 1) \geq N_1(Z - \underline{0}, \delta)$ and $M_2(T(Z) - T(\underline{0}), 1) \leq M_1(Z - \underline{0}, \delta)$.

Suppose that $Z \neq \underline{0}$ and $t > 0$. Putting $U = \frac{Z}{t}$ we obtain

$$\begin{aligned} N_2(\ &T(Z), t) \\ &= N_2(tT(U), t) \\ &= N_2(T(U), 1) \\ &\geq N_1(U, \delta) \\ &= N_1(\frac{Z}{t}, \delta) \\ &= N_1(Z, \frac{t}{M}), \text{ where } L = \frac{1}{\delta}. \end{aligned}$$

So, $N_2(T(Z), t) \geq N_1(Z, \frac{t}{M})$ and

$$\begin{aligned} M_2(\ &T(Z), t) \\ &= M_2(tT(U), t) \\ &= M_2(T(U), 1) \\ &\leq M_1(U, \delta) \\ &= M_1(\frac{Z}{t}, \delta) \\ &= M_1(Z, \frac{t}{M}) \text{ where } L = \frac{1}{\delta}. \end{aligned}$$

Hence $M_2(T(Z), t) \leq M_1(Z, \frac{t}{L})$.

If $Z = \underline{0}$ and $t > 0$, then
$T(\underline{0}) = \underline{0}$ and $N_2(\underline{0}, t) = N_1(\underline{0}, \frac{t}{M}) = 1$ and
$M_2(\underline{0}, t) = M_1(\underline{0}, \frac{t}{M}) = 0$.

So T is strongly intuitionistic fuzzy bounded. □

Theorem 4.6. Let $T : A \to B$ be a linear operator where A and B are intuitionistic fuzzy n-normed linear spaces. If T is sequentially intuitionistic fuzzy continuous at a point Z_0 then it is sequentially intuitionistic fuzzy continuous on X^n.

Proof. Let Z be an arbitrary point and $\{Z_k, k \geq 1\}$ be a sequence such that $Z_k \to Z$.

Then $\lim_{k\to\infty} N_1(Z_k - Z, t) = 1$ and $\lim_{k\to\infty} M_1(Z_k - Z, t) = 0$
$\Rightarrow \lim_{k\to\infty} N_1((Z_k - Z + Z_0) - Z_0, t) = 1$ and
$\lim_{k\to\infty} M_1((Z_k - Z + Z_0) - Z_0, t) = 0$ for all $t > 0$.

T is continuous at Z_0 we have:
$\lim_{k\to\infty} N_2(T(Z_k - Z + Z_0) - T(Z_0), t) = 1$ and
$\lim_{k\to\infty} M_2(T(Z_k - Z + Z_0) - T(Z_0), t) = 0$ for all $t > 0$
$\Rightarrow \lim_{k\to\infty} N_2(T(Z_k) - T(Z) + T(Z_0) - T(Z_0), t) = 1$ and
$\lim_{k\to\infty} M_2(T(Z_k) - T(Z) + T(Z_0) - T(Z_0), t) = 0$ for all $t > 0$
$\Rightarrow \lim_{k\to\infty} N_2(T(Z_k) - T(Z), t) = 1$ and $\lim_{k\to\infty} M_2(T(Z_k) - T(Z), t) = 0$
for all $t > 0$

Thus $\lim_{k\to\infty} N_1(Z_k - Z, t) = 1$ and $\lim_{k\to\infty} M_1(Z_k - z, t) = 0$ for all $t > 0$
$\Rightarrow \lim_{k\to\infty} N_2(T(Z_k) - T(z), t) = 1$ and $\lim_{k\to\infty} M_2(T(Z_k) - T(Z), t) = 0$,
for all $t > 0$.
This shows that T is sequentially intuitionistic fuzzy continuous on X^n. □

Theorem 4.7. Let $T : A \to B$ be a linear operator where A and B are intuitionistic fuzzy n-normed linear spaces. Then
(a) T is weakly intuitionistic fuzzy continuous on X^n if T is weakly intuitionistic fuzzy continuous at a point Z_0.
(b) T is weakly intuitionistic fuzzy continuous if and only if T is weakly intuitionistic fuzzy bounded.

Proof. (a) Since T is weakly intuitionistic fuzzy continuous at Z_0, hence for each $\epsilon > 0$ and $\alpha \in (0, 1)$, there exists $\delta(\alpha, \epsilon) > 0$, such that for all $Z \in X^n$,

$N_1(Z - Z_0, \delta) \geq \alpha$ and $M_1(Z - Z_0, \delta) \leq 1 - \alpha$
$\Rightarrow N_2(T(Z) - T(Z_0), \epsilon) \geq \alpha$ and $M_2(T(Z) - T(Z_0), \epsilon) \leq 1 - \alpha$.

Taking any $Y \in X^n$, and replacing Z by $Z + Z_0 - Y$ we get

$N_1(Z + Z_0 - Y - Z_0, \delta) \geq \alpha$ and $M_1(Z + Z_0 - Y - Z_0, \delta) \leq 1 - \alpha$
$\Rightarrow N_2(T(Z + Z_0 - Y) - T(Z_0), \epsilon) \geq \alpha$ and $M_2(T(Z + Z_0 - Y) - T(Z_0), \epsilon) \leq 1 - \alpha$.

That is $N_1(Z - Y, \delta) \geq \alpha$ and $M_1(Z - Y, \delta) \leq 1 - \alpha$
$\Rightarrow N_2(T(Z) - T(Y), \epsilon) \geq \alpha$ and $M_2(T(Z) - T(Y), \epsilon) \leq 1 - \alpha$.

Since Y is arbitrary, it follows that T is strongly intuitionistic fuzzy continuous on X^n.

(b) First we suppose that T is weakly intuitionistic fuzzy bounded.

Then for any $\alpha \in (0,1)$, there exists $M_\alpha > 0$ such that for all $t > 0$ we have

$$\begin{aligned} N_1(\ Z, \frac{t}{L_\alpha}) \geq \alpha \text{ and } M_1(Z, \frac{t}{L_\alpha}) \leq 1-\alpha \\ &\Rightarrow N_2(T(Z), t) \geq \alpha \text{ and } M_2(T(Z), t) \leq 1-\alpha. \\ &\Rightarrow N_1(Z-\underline{0}, \frac{\epsilon}{L_\alpha}) \geq \alpha \text{ and } M_1(Z-\underline{0}, \frac{\epsilon}{L_\alpha}) \leq 1-\alpha \\ &\Rightarrow N_2(T(Z)-T(\underline{0}), \epsilon) \geq \alpha \text{ and } M_2(T(Z)-T(\underline{0}), \epsilon) \leq 1-\alpha \text{ for } \epsilon > 0 \\ &\Rightarrow N_1(Z-\underline{0}, \delta) \geq \alpha \text{ and } M_1(Z-\underline{0}, \delta) \leq 1-\alpha \text{where } \delta = \frac{\epsilon}{L_\alpha} \\ &\Rightarrow N_2(T(Z)-T(\underline{0}), \epsilon) \geq \alpha \text{ and } M_2(T(Z)-T(\underline{0}), \epsilon) \leq 1-\alpha. \end{aligned}$$

$\Rightarrow T$ is weakly intuitionistic fuzzy continuous on X^n.

Conversely, suppose that T is weakly intuitionistic fuzzy continuous on X^n.

By continuity of T at $Z = \underline{0}$ for $\epsilon = 1$, it follows that for all $\alpha \in (0,1)$ there exists $\delta(\alpha, 1) > 0$ such that for all Z,

$$\begin{aligned} N_1(\ Z-\underline{0}, \delta) \geq \alpha \text{ and } M_1(Z-\underline{0}, \delta) \leq 1-\alpha \\ &\Rightarrow N_2(T(Z)-T(\underline{0}), 1) \geq \alpha \text{ and } M_2(T(Z)-T(\underline{0}), 1) \leq 1-\alpha \\ &\Rightarrow N_1(Z, \delta) \geq \alpha \text{ and } M_1(Z, \delta) \leq 1-\alpha \\ &\Rightarrow N_2(T(Z), 1) \geq \alpha \text{ and } M_2(T(Z), 1) \leq 1-\alpha. \end{aligned}$$

Suppose that $Z \neq \underline{0}$ and $t > 0$. Putting $Z = \frac{U}{t}$, we have:

$$\begin{aligned} N_1(\frac{U}{t}, \delta)\ \geq \alpha \text{ and } M_1(\frac{U}{t}, \delta) \leq 1-\alpha \\ &\Rightarrow N_2(T(\frac{U}{t}), 1) \geq \alpha \text{ and} M_2(T(\frac{U}{t}), 1) \leq 1-\alpha \\ &\Rightarrow N_1(U, t\delta) \geq \alpha \text{ and } M_1(U, t\delta) \leq 1-\alpha \\ &\Rightarrow N_2(T(U), t) \geq \alpha \text{ and} M_2(T(U), t) \leq 1-\alpha \\ &\Rightarrow N_1(U, \frac{t}{L_\alpha}) \geq \alpha \text{ and } M_1(U, \frac{t}{L_\alpha}) \leq 1-\alpha \\ &\Rightarrow N_2(T(U), t) \geq \alpha \text{ and } M_2(T(U), t) \leq 1-\alpha, \\ &\quad \text{where } L_\alpha = \frac{1}{\delta(\alpha, 1)} \\ &\Rightarrow T \text{ is weakly intuitionistic fuzzy bounded.} \end{aligned}$$

If $Z = \underline{0}$ and $t > 0$, then for $L_\alpha > 0$,

$N_1(Z, \frac{t}{L_\alpha}, t) = N_2(T(Z), t) = 1$ and
$M_1(Z, \frac{t}{L_\alpha}, t) = M_2(T(Z), t) = 0.$

Thus T is weakly intuitionistic fuzzy bounded. □

References

[1] K. T. Atanassov, *Intuitionistic fuzzy sets,* Fuzzy Sets and Systems, **20 (1)**(1986), 87-96.
[2] K. T. Atanassov, *More on intuitionistic fuzzy sets,* Fuzzy Sets and Systems, **33**(1989), 37-46.

[3] K. T. Atanassov, *Intuitionistic fuzzy sets,* Physica-Verlag Heidelberg, Newyork, 1999.
[4] T. Bag and S. K. Samanta, *Finite dimensional fuzzy normed linear spaces,* The Journal of Fuzzy Mathematics, **11(3)**(2003), 687-705.
[5] T. Bag and S. K. Samanta, *Fuzzy bounded linear operators,* Fuzzy Sets and Systems, **151**(2005), 513-547.
[6] S. C. Chang and J. N. Mordesen, *Fuzzy linear operators and fuzzy normed linear spaces,* Bull. Cal. Math. Soc., **86**(1994), 429-436.
[7] G. Deschrijver and E. Kerre , *On the Cartesian product of the intuitionistic fuzzy sets,* The Journal of Fuzzy Mathematics, **11 (3)**(2003), 537-547.
[8] MS. Elnaschie, *On the uncertainty of Cantorian geometry and two-slit experiment,* Chaos, Soliton and Fractals, **9 (3)**(1998), 517-529.
[9] MS. Elnaschie, *On the verications of heterotic strings theory and* ϵ^{∞} *theory,* Chaos, Soliton and Fractals,**11 (2)** (2000), 2397-2407.
[10] C. Felbin, *Finite dimensional fuzzy normed linear spaces,* Fuzzy Sets and Systems, **48** (1992), 239-248.
[11] C. Felbin, *The Completion of fuzzy normed linear space,* Journal of Mathematical Analysis and Applications, **174 (2)**(1993), 428-440.
[12] C. Felbin, *Finite dimensional fuzzy normed linear spaces II,* Journal of Analysis, **7** (1999), 117-131.
[13] S. Gähler, *Lineare 2-Normierte Räume,* Math. Nachr., **28** (1965), 1-43.
[14] S. Gähler, *Unter Suchungen Über Veralla gemeinerte m-metrische Räume I,* Math. Nachr., (1969), 165-189.
[15] H. Gunawan and M. Mashadi, *on n-Normed spaces,* International J. Math. & Math. Sci., **27 (10)**(2001), 631-639.
[16] A. K. Katsaras, *Fuzzy topological vector spaces II,* Fuzzy Sets and Systems, **12** (1984), 143-154.
[17] S. S. Kim and Y. J. Cho, *Strict convexity in linear n-normed spaces,* Demonstratio Math., **29 (4)** (1996), 739-744.
[18] S. V. Krishna and K. K. M. Sharma, *Separation of fuzzy normed linear spaces,* Fuzzy Sets and Systems, **63** (1994), 207-217.
[19] R. Malceski, *Strong n-convex n-normed spaces,* Mat. Bilten, **21** (1997), 81-102.
[20] A. Misiak, *n-inner product spaces,* Math. Nachr., **140** (1989), 299-319.
[21] AL. Narayanan and S. Vijayabalaji, *Fuzzy n-normed linear space,* **24**(2005), 3963-3977.
[22] J. H. Park, *Intuitionistic fuzzy metric space,* Chaos, Solitons and Fractals, **22**(2004), 1039-1046.
[23] G. S. Rhie, B. M. Choi and S. K. Dong, *On the completeness of fuzzy normed linear spaces,* Math. Japonica, **45 (1)** (1997), 33-37.
[24] B. Schweizer and A. Sklar, *Statistical metric spaces,* Pacific J. Math., **10**(1960), 314-334.
[25] S. Vijayabalaji, N. Thillaigovindan and Y. B. Jun, *Intuitionistic fuzzy n-normed linear space,* Submitted to Bulletin of Korean Mathematical Society, Korea.

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