PRICING STOCK OPTIONS USING FUZZY SETS

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Abstract. We use the basic binomial option pricing method but allow some or all the parameters in the model to be uncertain and model this uncertainty using fuzzy numbers. We show that with the fuzzy model we can, with a reasonably small number of steps, consider almost all possible future stock prices; whereas the crisp model can consider only $n+1$ prices after n steps.

1. Introduction

We will be working with the binomial option pricing method. See Figure 1 (One Step) and Figure 4 (Two Step). They will be discussed in detail in the next two sections. See [5,7,15,16] for further discussion on this pricing model. The basic assumptions are that we have an European option and the stock pays no dividends during contract period. We purchase the stock option at time $t = 0$ for $\$V$ (the value of the option) per share and at time$t = T$we have the option to either buy the stock at $\$S$ (strike price) per share, or cancel the contract. Time is measured in years so foe example $T = 0.5$ means six months. In an American option we may terminate the contract (buy or cancel) at any time $t \in (0, T)$ but in an European option we can terminate (buy or cancel) only at time $t = T$. There is no cash flow in or out of the contract during the time interval $(0, T)$ so the stock pays no dividends in that time interval, there are no transaction costs and no taxes.

The binomial model contains a number of parameters whose values must be given in order that we may price the option (compute a fair value for V). Some, or all, of these values may be uncertain and we will model this uncertainty using fuzzy numbers. One reason for fuzzifying the model is to model uncertainty in the input parameters and in section 2.1 we present one way, using expert opinion, of obtaining the relevant fuzzy numbers. another reason for fuzzifying the model is discussed at the end of this section. We will use fuzzy numbers for all the parameters and then compute the fuzzy number $\overline{V}$ for the value of the fuzzy option at time zero. If a parameter like the spot value of the stock at time zero is known exactly, then one can use that crisp value in our fuzzy binomial pricing model. Initially we assume all parameters are fuzzy.

There has been some research on using fuzzy sets in pricing stock options [1,10,11, 13,17]. In [11] the authors fuzzified the up and down factors in the binomial pricing

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model by fuzzifying volatility, looked at the one and two step models, but did not formally discuss the n step model. A critical analysis of this paper may be found in [13]. We do not incorporate volatility into our models in this paper because it is usually not used in the binomial option pricing method. The authors in [1] develop a model similar to the one in this paper, but they consider only one step in the binomial pricing model. In [1] all parameters are triangular fuzzy numbers except for the interest rate is assumed to be constant volatility, and fuzzy risk neutral probabilities and the fuzzy payoff for one step are computed. In [13] the authors incorporate fuzzy volatility which makes the up/down factors fuzzy. However they use probabilities and expected values and we do not employ probability theory . In this paper we consider European options, assume constant volatility, use no probabilities, assume no cash flows during the life of the contract, work with only the binomial pricing method, fuzzify all/some of the parameters, and, using the extension principle, to compute the fuzzy value of the option after n steps. Numerical examples are presented. An interesting paper[17] use fuzzy interest rate, fuzzy volatility and fuzzy stock price as inputs to the Black-Scholes model, possibly a topic for future research.

The notations that we use in this paper are as follows.

We place a "bar" over a symbol to denote a fuzzy set. All our fuzzy sets will be fuzzy subsets of the real numbers so $\overline{M}$, $\overline{N}$, $\overline{A}$,... are all fuzzy subsets of the real numbers. A triangular fuzzy number $\overline{N}$ is defined by three numbers $a < b < c$ where the base of the triangle is on the interval $[a, c]$ and its vertex is at $x = b$. We write $\overline{N} = (a/b/c)$ for triangular fuzzy number $\overline{N}$. The membership function of fuzzy number $\overline{N}$ evaluated at x is written $\overline{N}(x)$ a number in $[0, 1]$. A triangular shaped fuzzy number has base on an interval $[a, c]$, vertex at $x = b$, but the sides are curves not straight lines. We write $\overline{N} \approx (a/b/c)$ for a triangular shaped fuzzy number $\overline{N}$. The α-cut of fuzzy number $\overline{N}$ is written as $\overline{N}[\alpha]$ and equals $\{x|\overline{N}(x) \geq \alpha\}$ for $0 < \alpha \leq 1$. We separately define $\overline{N}[0]$ as the closure of the union of all the $\overline{N}[\alpha]$, $0 < \alpha \leq 1$. We will call the $\alpha = 0$ cut the support of the fuzzy number. Alpha-cuts of fuzzy numbers are always closed and bounded, intervals, and we write $\overline{N}[\alpha] = [n_1(\alpha), n_2(\alpha)]$, for $0 \leq \alpha \leq 1$. For an introduction to fuzzy sets/logic the reader may consult [2,6,9,12].

In the next section we first review the crisp one step binomial price model and then fuzzify it. We do the same for the two step model in section 3 and the n step procedure in section 4. Our summary and plans for future research are presented in section 5 and the software used, with the relevant programs, comprises the last section.

As we will see in the next three sections the crisp binomial model can consider only $n + 1$ possible future stock prices after n steps and this has been considered a major fault of the binomial pricing model. Traders want to be able to consider all possible future prices for the stock. This led to taking $n \to \infty$ in the n- step binomial pricing model leading to the famous Black-Scholes pricing model [5,7]. However in our n- step fuzzy binomial pricing model in section 4 we show that it can consider all possible future stock prices in an interval $[\epsilon, M]$, for $\epsilon \approx 0$ and large

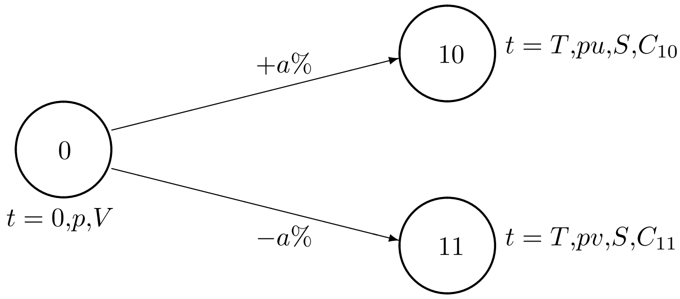

FIGURE 1. One Step Binomial Pricing Tree

M. This is the second motivation for using fuzzy sets. We do not need to look at $n \to \infty$ in the fuzzy model to be able to consider all possible future stock prices.

2. **One Step Model**

The one-step model is shown in Figure 1. In Figure 1, p is the share price (spot price) at time zero, V is the value of the option at $t = 0$, and a is the percent increase/decrease of the stock during the time period T. We will always write a as a decimal (e.g. $a = 0.05$, or $a = 0.10$). If $a = 0.05$, then the stock can rise 5% or fall 5%. Also S is the strike price, or the amount the option's contract says we will pay per share at time T. C_{10} is the value of the stock at time T if it increases by a% and C_{11} is the value if it decreases a%. Let $u = (1 + a)$ and $v = (1 - a)$. The value of the stock at time T is pu if increases by a% and pv when it decreases by a%. So

$$C_{10} = max(0, pu - S), \tag{1}$$

and

$$C_{11} = max(0, pv - S), \tag{2}$$

because if $pu \leq S$ the contract is cancelled, but if $pu > S$ we buy the stock now selling at pu per share at \$$S$ per share. Similar reasoning for C_{11}. If the price is going down you can make a profit.

We determine V. Assume that we buy x shares of this stock at time zero. The cost is xp. To do this we borrow \$$B$ at (risk free) interest rate r per year and make up the difference (if any) with cash \$$V$. So we must have

$$xp = B + V. \tag{3}$$

We determine x and B to be equivalent to our stock option. The two equations are

$$x(pu) - B\exp(rT) = C_{10}, \tag{4}$$

and

$$x(pv) - B\exp(rT) = C_{11}. \tag{5}$$

In equation (4) we sell our x shares at price \$$pu$ per share and pay back our loan of \$$B$, whose value is now $B\exp(rT)$, and the result must equal the proceeds C_{10}. In equation (5) we sell x shares at \$$pv$ per share, pay back our loan and the result equals C_{11}.

We will be using continuous interest. If we invest/borrow \$$A$ at interest rate r for time T and interest is compounded continuously, the amount at time T is $A\exp(rT)$ (future value). If we consider \$$A$ in the future at time T, its present value today is $A\exp(-rT)$. Reference [7] uses continuous interest and we will do the same.

Equations (4) and (5) have a unique solution in x and B as follows:

$$x = (C_{10} - C_{11})/(p(u-v)), \tag{6}$$

and

$$B = (xpv - C_{11})\exp(-rT). \tag{7}$$

Hence, by (3) have

$$V = f_1(p, a, S, r; T) = \exp(-rT)(\theta C_{10} + (1-\theta)C_{11}), \tag{8}$$

where

$$\theta = (\exp(rT) - v)/(u-v), \tag{9}$$

and

$$1 - \theta = (u - \exp(rT))/(u-v). \tag{10}$$

We notice that θ and $1-\theta$ are similar to probabilities since they are usually in $(0, 1)$ and their sum equals one.

2.1. **Fuzzy One Step.** Let p, a, S and r all be triangular fuzzy numbers. We first consider the case when all parameters fuzzy and then the case where p and S are not fuzzy. We can easily generalize to the case where all parameters are trapeziodal fuzzy numbers but we will use triangular fuzzy numbers in this paper. So let $\overline{p} = (p_1/p_2/p_3)$, $\overline{a} = (a_1/a_2/a_3)$, $\overline{S} = (s_1/s_2/s_3)$ and $\overline{r} = (r_1/r_2/r_3)$. Where do these fuzzy numbers come from?

The fuzzy parameters in our model can be estimated by experts. So let us briefly see how this may be accomplished. First assume we have only one expert and he/she is to estimate the value of some parameter r. We can solicit this estimate from the expert as is done in estimating job times in project scheduling [14]. Let r_1 = the "pessimistic" value of r, or the smallest possible value, let r_3 be the "optimistic" value of r, or the highest possible value, and let r_2 the most likely value of r. We then ask the expert to give values for r_1, r_2, r_3 and we construct the triangular fuzzy number $\overline{r} = (r_1/r_2/r_3)$ for r. If we have a group of N experts all to estimate the value of r we solicit the r_{1i}, r_{2i} and r_{3i}, $1 \leq i \leq N$, from them. Let r_1 be the average of the r_{1i}, r_2 is the mean of the r_{2i} and r_3 is the average of the r_{3i}. The simplest thing to do is to use $(r_1/r_2/r_3)$ for $\overline{r}$.

Now we have a choice: (1) fuzzify equations (4) and (5) and solve for $\overline{x}$ and $\overline{B}$ using $\alpha-$cuts and interval arithmetic ([2],[6],[9]) and then compute $\overline{V}$; or (2) fuzzify equation (8) and solve for $\overline{V}$ using the extension principle [2,6,9].In the first case the fuzzy equations are

$$\overline{x}\ \overline{p}\ \overline{u} - \overline{B}\exp(\overline{r}T) = \overline{C}_{10}, \tag{11}$$

and

$$\overline{x}\ \overline{p}\ \overline{v} - \overline{B}\exp(\overline{r}T) = \overline{C}_{11}, \tag{12}$$

where $\overline{u} = (1+\overline{a})$, $\overline{v} = (1-\overline{a})$ and we do not fuzzify time. Also $\overline{C}_{10}$=max(0,$\overline{p}\ \overline{u}-\overline{S}$), $\overline{C}_{11}$=max(0,$\overline{p}\ \overline{v}-\overline{S}$). There are two problems with this method are: (1) evaluating $\overline{C}_{10}$ and $\overline{C}_{11}$; (2) finding the alpha-cuts of $\overline{x}$ and $\overline{B}$ do not always produce fuzzy numbers for $\overline{x}$ and $\overline{B}$, respectively [4]; and (3) this method, when it works, usually produces a result for $\overline{V}$ which is more fuzzy (wider base) than the procedure in the second method [2,6,9]. Therefore we do not employ this way of getting the fuzzy value.

Fuzzifing equation (8) and using the extension principle gives

$$\overline{V}(\beta) = sup\{min(\overline{p}(z_1), \overline{a}(z_2), \overline{S}(z_3), \overline{r}(z_4)) | f_1(z_1, z_2, z_3, z_4; T) = \beta\} \tag{13}$$

However, we know how to find $\alpha-$cuts of $\overline{V}$. Let $\overline{V}[\alpha] = [v_1(\alpha), v_2(\alpha)]$, $\overline{p}[\alpha] = [p_1(\alpha), p_2(\alpha)]$, $\overline{a}[\alpha] = [a_1(\alpha), a_2(\alpha)]$, $\overline{S}[\alpha] = [S_1(\alpha), S_2(\alpha)]$, $\overline{r}[\alpha] = [r_1(\alpha), r_2(\alpha)]$. Then [3]

$$v_1(\alpha) = min\{f_1(p, a, S, r; T) |\ \mathbf{S}\ \}, \tag{14}$$

and

$$v_2(\alpha) = max\{f_1(p, a, S, r; T) |\ \mathbf{S}\ \}, \tag{15}$$

where the statement **S** is "$p \in [p_1(\alpha), p_2(\alpha)], a \in [a_1(\alpha), a_2(\alpha)]$, $S \in [S_1(\alpha), S_2(\alpha)], r \in [r_1(\alpha), r_2(\alpha)]$". If p is known to be 60 and $S = 62$, then **S** is "$p = 60, a \in [a_1(\alpha), a_2(\alpha)], S = 62, r \in [r_1(\alpha), r_2(\alpha)]$".

Example 2.1. Let $\overline{p} = (57/60/63)$, $\overline{a} = (0.04/0.05/0.06)$, $\overline{S} = (60/62/64)$ and $\overline{r} = (0.05/0.06/0.07)$. Then $\overline{p}[\alpha] = [57+3\alpha, 63-3\alpha]$, $\overline{a}[\alpha] = [0.04+0.01\alpha, 0.06-0.01\alpha]$, $\overline{S}[\alpha] = [60+2\alpha, 64-2\alpha]$ and $\overline{r}[\alpha] = [0.05+0.01\alpha, 0.07-0.01\alpha]$. We substitute these into equations (14) and (15) to find the alpha-cuts of $\overline{V}$. Assume that the contract time is six months,i.e. $T = 0.5$.

We solved the optimization problem using "Solver" [8,18]. The "little" Solver is a free add on to Microsoft Excel while the "big" Solver needs to be purchased [8].

α	$\overline{V}[\alpha]$	$\overline{V}[\alpha]$
	All Parameters Fuzzy	a and r Fuzzy
0	$[0, 5.22]$	$[0.32, 1.23]$
0.25	$[0, 4.11]$	$[0.44, 1.12]$
0.5	$[0, 3.01]$	$[0.55, 1.01]$
0.75	$[0, 1.90]$	$[0.67, 0.89]$
1	0.78	0.78

Table 1. Alpha-Cuts of Fuzzy Value in Example 2.1

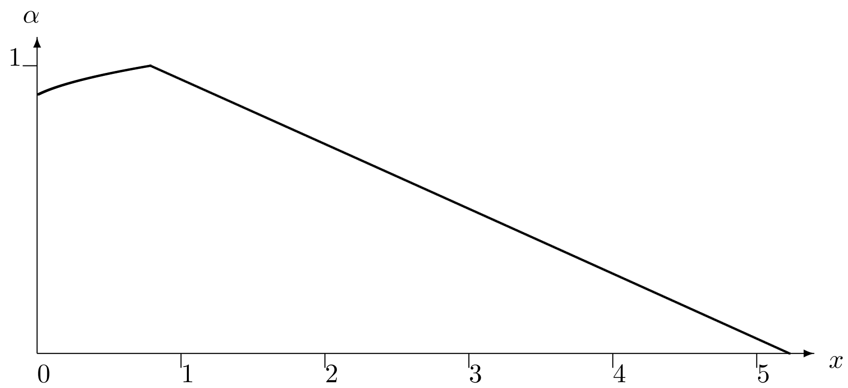

Figure 2. Fuzzy Value $\overline{V}$ in Example 2.1 (p,a,S,r) Fuzzy

The "big" Solver obviously solves larger optimization problems. Since this software is probably not well known we give the program in Excel for solving equations (14) and (15) at the end of the paper. The solutions for selected α values are in Table 1. An approximate graph of $\overline{V}$ is in Figure 2. The graph is approximate because we have forced the sides only through the end points of the $\alpha = 0, 0.5, 1$ cuts. The numbers on the $x-$axis are the possible values of the option at time zero.

In Figure 1 let us look at $\overline{p}\,\overline{u}$ and $\overline{p}\,\overline{v}$, at the end of step one, for the fuzzy values given above, where $\overline{u}[0] = 1 + \overline{a}[0]$ and $\overline{v}[0] = 1 - \overline{a}[0]$. We get the future stock price in $\overline{p}[0]\overline{u}[0] \cup \overline{p}[0]\overline{v}[0] = [53.58, 66.78]$ using the supports of the fuzzy numbers. So this fuzzy model considers all prices at the end of step one in that interval.

Now let us assume that the spot price on the stock is known precisely ($p = 60$) and the stock price in the option's contract is also known exactly ($S = 62$). We can find alpha-cuts of the value of the option from equations (14) and (15) by simply substituting $p = 60$ and $S = 62$. The results are also shown in Table 1 with an approximate (sides only through $\alpha = 0, 0.5, 1$ cuts) graph of $\overline{V}$ in Figure 3.

If $p = 60$ this fuzzy model considers all stock prices at the end of step one in the set $[56.4, 57.6] \cup [62.4, 63.6]$.

We notice that $\overline{V}$ is less fuzzy in Figure 3 than in Figure 2. This is because in Figure 3 only $\overline{a}$ and $\overline{r}$ are fuzzy but all the parameters are fuzzy in Figure 2. Consider the $\alpha = 0$ cut when all parameters are fuzzy in Table 1. Given $p = 59$ in

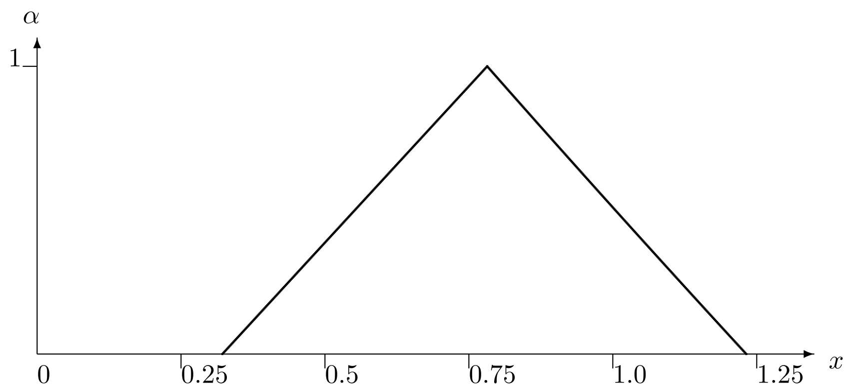

FIGURE 3. Fuzzy Value $\overline{V}$ in Example 2.1 (only a,r) Fuzzy

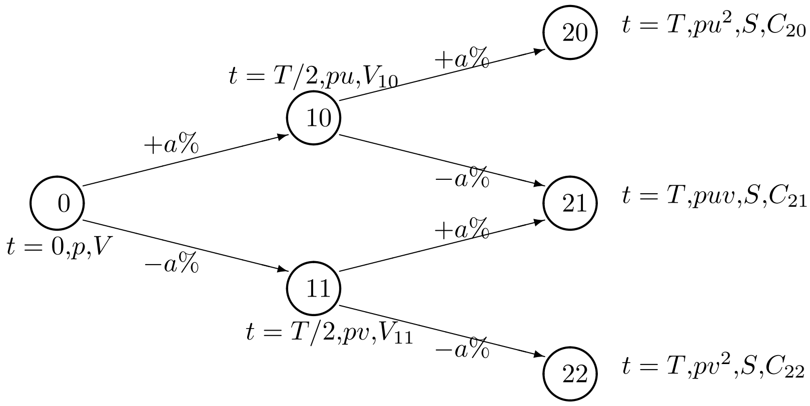

FIGURE 4. Two Step Binomial Pricing Tree

$\overline{p}[0]$, $a = 0.054$ in $\overline{a}[0]$, $S = 61$ in $\overline{S}[0]$, $r = 0.063$ in $\overline{r}[0]$ and $T = 0.5$, the function f_1 in equation (8) computes a value for $V = 0.915$ in $\overline{V}[0] = [0, 5.22]$. In fact, as p, a, S and r range through all their values in their $\alpha = 0$ cut, we obtain all the values of V in the interval $[0, 5.22]$. This is the correct interpretation of how we computed $\overline{V}$. Now suppose we have $p = 59$, $a = 0.054$, $S = 61$, $r = 0.063$ and $T = 0.5$ and choose $V = 0.5$ in its alpha zero cut. The option is now priced wrong and a target for arbitrageurs. However, this interpretation is wrong because the value that is associated with these parameters is $V = 0.915$ and not $V = 0.5$. This discussion is equally applicable in the next two sections where we look at the two step model and the n step model.

3. Two Step Model

The two step method is shown in Figure 4. The end of the first step is the same as in Figure 1. The value of the stock at the end of step two is pu^2, puv or pv^2, with values $C_{20} = max(0, pu^2 - S)$, $C_{21} = max(0, puv - S)$ and $C_{22} = max(0, pv^2 - S)$, respectively. Let V_{10} (V_{11}) be the value at node 10 (11). The time for both steps is

T so the time for each individual step is $t = T/2$. We find these values V_{1j}, $j = 0, 1$, as in the one step case. For$\theta((1-\theta))$ defined in equation (9)((10))

$$V_{10} = \exp(-rt)(\theta C_{20} + (1-\theta)C_{21}), \tag{16}$$

and

$$V_{11} = \exp(-rt)(\theta C_{21} + (1-\theta)C_{22}), \tag{17}$$

The value of the option V at time zero is then

$$V = \exp(-rt)(\theta V_{10} + (1-\theta)V_{11}), \tag{18}$$

which equals

$$V = f_2(p, a, S, r; T) = \exp(-rT)(\theta^2 C_{20} + 2\theta(1-\theta)C_{21} + (1-\theta)^2 C_{22}). \tag{19}$$

Notice that in this equation we use T.

3.1. **Fuzzy Two Step.** As in the previous section we first fuzzify all the parameters except time. The $\alpha-$cuts of $\overline{V}$ will be found the same way as in section 2

$$v_1(\alpha) = min\{f_2(p, a, S, r; T) | \ \mathbf{S}\ \}, \tag{20}$$

and

$$v_2(\alpha) = max\{f_2(p, a, S, r; T) | \ \mathbf{S}\ \}, \tag{21}$$

where the statement **S** is "$p \in [p_1(\alpha), p_2(\alpha)], a \in [a_1(\alpha), a_2(\alpha)]$, $S \in [S_1(\alpha), S_2(\alpha)], r \in [r_1(\alpha), r_2(\alpha)]$". If $p = 60$ and $S = 62$, then **S** is "$p = 60, a \in [a_1(\alpha), a_2(\alpha)], S = 62, r \in [r_1(\alpha), r_2(\alpha)]$".

Example 3.1. We first assume all parameters are fuzzy. The data for the fuzzy sets is the same as in Example 2.1. The contract time is six months so $T = 0.5$ and $t = 0.25$. We solved the optimization problem using our "little" Solver. We give the program for solving equations (20) and (21) in Excel at the end of this paper. The solutions for selected α values are in Table 2. An approximate graph of $\overline{V}$ (only through three alpha-cuts) is in Figure 5.

In Figure 4,(for the fuzzy values in Example 2.1.) let us look at $\overline{p}\ \overline{u}^2$, $\overline{p}\ \overline{u}\ \overline{v}$ and $\overline{p}\ \overline{v}^2$, at the end of step two. We get the future stock price in $\overline{p}[0](\overline{u}[0])^2 \cup \overline{p}[0]\overline{u}[0]\overline{v}[0] \cup \overline{p}[0](\overline{v}[0])^2 = [50.36, 70.79]$ using the supports of the fuzzy numbers. So this fuzzy model considers all prices at the end of step two in that interval.

Now let us assume that the spot price on the stock is known precisely ($p = 60$) and the stock price in the options contract is also known exactly ($S = 62$). We can find alpha-cuts of the value of the option from equations (20) and (21) by simply substituting $p = 60$ and $S = 62$. The results are also shown in Table 2 with an approximate graph of $\overline{V}$ (only through three alpha-cuts) in Figure 6.

In Figure 4,(for the fuzzy value of $\overline{a}$ in Example 2.1.) we now have $60\overline{u}^2$, $60\overline{u}\ \overline{v}$ and $60\overline{v}^2$, at the end of step two. We get the price in $60(\overline{u}[0])^2 \cup 60\overline{u}[0]\overline{v}[0] \cup 60(\overline{v}[0])^2 = [53.02, 55.30] \cup [58.66, 61.06] \cup [64.90, 67.42]$. So in this fuzzy model all prices in that set are considered.

α	$\overline{V}[\alpha]$	$\overline{V}[\alpha]$
	All Parameters Fuzzy	a and r Fuzzy
0	$[0, 5.58]$	$[1.22, 2.19]$
0.25	$[0, 4.38]$	$[1.34, 2.07]$
0.5	$[0.37, 3.18]$	$[1.46, 1.95]$
0.75	$[1.04, 2.37]$	$[1.59, 1.83]$
1	1.71	1.71

TABLE 2. Alpha-Cuts of Fuzzy Value in Example 3.1

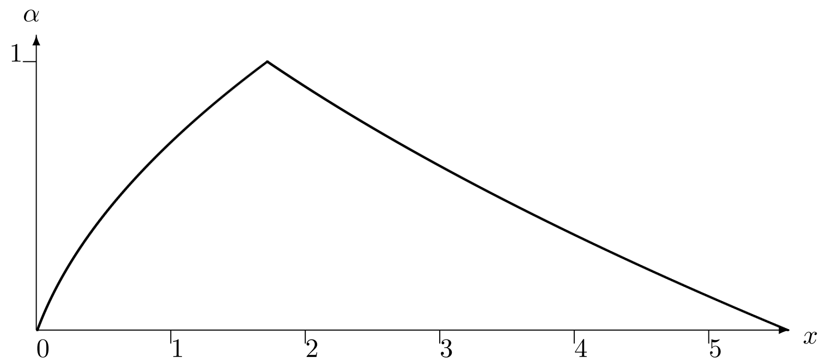

FIGURE 5. Fuzzy Value $\overline{V}$ in Example 3.1 (p,a,S,r) Fuzzy

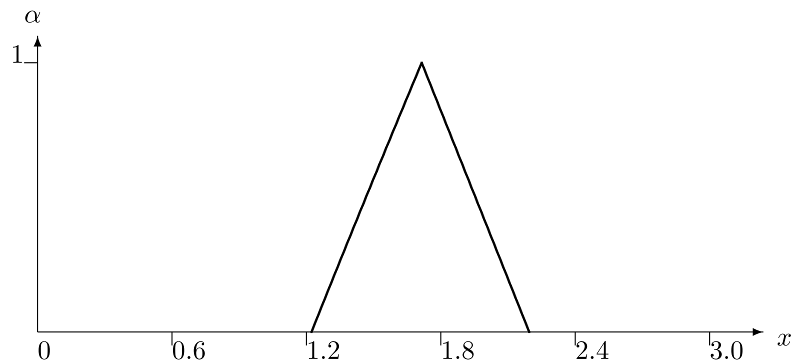

FIGURE 6. Fuzzy Value $\overline{V}$ in Example 3.1 (only a,r) Fuzzy

4. *n*-Step Model

In the previous section we we sense the beginning of a pattern. Now let $t = T/n$. At the end of n steps we have nodes labeled $n0$, $n1$,...,nn with the stock priced at $pu^{n-i}v^i$, $i = 0, 1, ..., n$ with values C_{ni}, $i = 0, ..., n$. Now $C_{ni} = max(0, pu^{n-i}v^i - S)$, $i = 0, ..., n$. Hence

$$V = F(p, a, S, r; T) = \exp(-rT) \sum_{i=0}^{n} \binom{n}{i} \theta^i (1-\theta)^i C_{ni}. \tag{22}$$

α	$\overline{V}[\alpha]$ All Parameters Fuzzy	$\overline{V}[\alpha]$ a and r Fuzzy
0	$[1.07, 7.58]$	$[2.87, 4.69]$
0.25	$[1.55, 6.47]$	$[3.10, 4.47]$
0.5	$[2.12, 5.46]$	$[3.33, 4.24]$
0.75	$[2.95, 4.62]$	$[3.56, 4.01]$
1	3.78	3.78

TABLE 3. Alpha-Cuts of Fuzzy Value in Example 4.1 Using $n = 10$

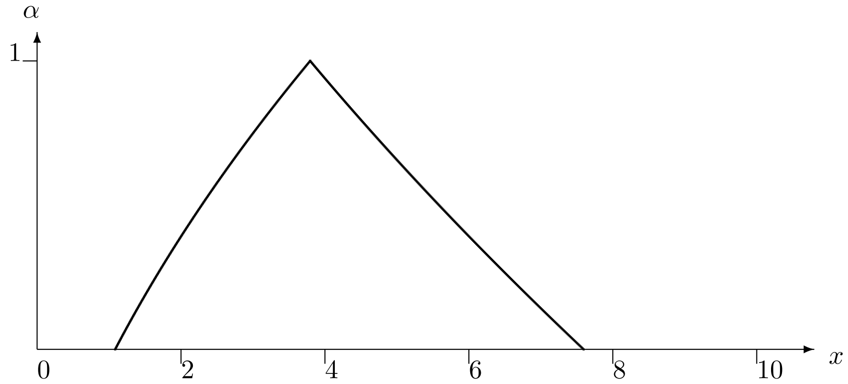

FIGURE 7. Fuzzy Value $\overline{V}$ in Example 4.1 (p,a,S,r) Fuzzy

4.1. **Fuzzy *n*- Step.** As in the previous sections we first fuzzify all the parameters except time. The $\alpha-$cuts of $\overline{V}$ will be found the same way as in sections 2 and 3

$$v_1(\alpha) = min\{F(p, a, S, r; T)|\ \mathbf{S}\ \}, \tag{23}$$

and

$$v_2(\alpha) = max\{F(p, a, S, r; T)|\ \mathbf{S}\ \}, \tag{24}$$

where the statement **S** is "$p \in [p_1(\alpha), p_2(\alpha)], a \in [a_1(\alpha), a_2(\alpha)]$, $S \in [S_1(\alpha), S_2(\alpha)], r \in [r_1(\alpha), r_2(\alpha)]$". If $p = 60$ and $S = 62$, then **S** is "$p = 60, a \in [a_1(\alpha), a_2(\alpha)], S = 62, r \in [r_1(\alpha), r_2(\alpha)]$".

Example 4.1. The data for the fuzzy sets is the same as in Examples 2.1 and 3.1. Contract $T = 0.5$ so $t = T/10 = 0.05$ using $n = 10$. For large n, say $n \geq 100$, this problem becomes too much for our "little" solver. So let us use $n = 10$. The results when all parameters are fuzzy, and for the case where p and S are not fuzzy, are in Table 3. Approximate graphs (only through three $\alpha-$cuts) are in Figures 7 and 8.

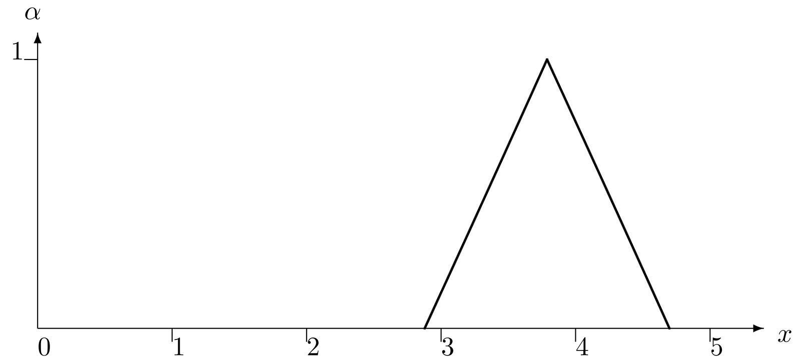

FIGURE 8. Fuzzy Value $\overline{V}$ in Example 4.1 (only a,r) Fuzzy

n	Price in Interval
10	$[30.7, 112.8]$
100	$[0.12, 21(10)^3]$
1000	$[7.6(10)^{-26}, 1.27(10)^{27}]$

TABLE 4. Values for the Stock Price, n Steps Fuzzy Model

Assume that all parameters are fuzzy. Let us look at $\overline{p}\ \overline{u}^{n-i}\overline{v}^i$, $i = 0, ..., n$ at the end of step n, for the fuzzy values in Examples 2.1 and 3.1. We get the price in

$$\bigcup_{i=0}^{n} \overline{p}[0](\overline{u}[0])^{n-i}(\overline{v}[0])^i, \tag{25}$$

using the supports of the fuzzy numbers. For the data in the previous examples this interval is shown in Table 4. So,with this model for $n = 1000$ and this fuzz we essentially get all possible prices. The same is probably true for $n = 100$. So for $n = 100$ the fuzzy model probably considers all possible future stock prices.

Next assume that $p = 60$. Put for $n = 1000(100)$ $\overline{p}[0] = 60$ into equation (25), producing set Ω. In this fuzzy model we get the stock price, after 1000(100) steps in the interval $[0, T]$, to be all values in the set Ω.

5. **Summary and Future Research**

Let $\overline{V}_{n,\gamma}$ be the fuzzy value when all parameters are fuzzy ($\gamma = 4$) and when only two (a,r) are fuzzy ($\gamma = 2$) for a n step model. We have already noticed that $\overline{V}_{n,4}$ is more fuzzy than $\overline{V}_{n,2}$. That is, The alpha-cuts of $\overline{V}_{n,2}$ are contained in the alpha-cuts of $\overline{V}_{n,4}$. Also, from the Tables and Figures, $\overline{V}_{n,\gamma}$ becomes more fuzzy as n grows for $\gamma = 2, 4$. Uncertainty (fuzziness) grows as one performs more and more computations. Moreover,we see that the fuzzy sets $\overline{V}_{n,\gamma}$ shift to the right as n grows. What is

$$\lim_{n\to\infty} \overline{V}_{n,\gamma} \tag{26}$$

and how is it related to the non-fuzzy (and fuzzy) Black-Scholes model? This is topic for future research.

It is also important to see that in the fuzzy model,for a reasonably small n, after n steps, all possible future stock prices can be considered (Table 4).

We also hope to continue this research into: (1) other options; and (2) other derivatives (forwards, futures, swaps).

6. Excel Programs

This section contains Solver commands for the optimization problems in Examples 2.1 and 3.1.

SOLVER is an optimization package which is an add-in to Microsoft Excel. It is free and if your Excel does not have it, then contact Microsoft or [8,18]. We used SOLVER because of the wide availability of Excel.

Excel is a spreadsheet whose columns are labeled A,B,C,... and rows are labeled 1,2,3,... So "cell" B4 means the cell in the fourth row and B column. When we say $H2 = \mathcal{K}$ we mean put into cell H2 the formula/expression/data $\mathcal{K}$.

6.1. **Example 2.1.**

Here $t = T$ =duration of the contract. First assume that all parameters are fuzzy.

(1) $A1 = 60$ (initial value p),$A2 = 0.05$ (initial value a), $A3 = 62$ (initial value S). $A4 = 0.06$ (initial value r)
(2) $B1 = (exp(A4 * t) - (1 - A2))/(2 * A2)$,$B2 = (1 - B1)$ (theta values)
(3) $C1 = MAX(0, A1 * (1 + A2) - A3)$,$C2 = MAX(0, A1 * (1 - A2) - A3)$ (C_{1j} values)
(4) $D1 = B1 * C1 + B2 * C2$
(5) $E1 = exp(-A4 * t) * D1$

Now open the SOLVER window. Do the following: (1) target cell= $E1$ (what to max/min); (2) changing cells $A1 : A4$ (the variables); and (3) constraints are for $\alpha = 0$: $57 \leq A1 \leq 63$, $0.04 \leq A2 \leq 0.06$, $60 \leq A3 \leq 64$, $0.05 \leq A4 \leq 0.07$. Choose max or min and click the solve button. In the options box we used automatic scaling, estimates=tangent, derivatives=forward, and search=Newton.

If $p = 60$ and $S = 62$ just set $A1 = 60$ and $A3 = 62$ in the constraints.

6.2. **Example 3.1.**

Here $t = T/2$. First assume that all parameters are fuzzy. The A column and the B column are the same as for Example 2.1.

(1) $C1 = MAX(0, A1 * (1 + A2)^2 - A3)$,$C2 = MAX(0, A1 * (1 + A2)(1 - A2) - A3)$, $C3 = MAX(0, A1 * (1 - A2)^2 - A3)$ (C_{2j} values)
(2) $D1 = B1^2 * C1 + 2 * B1 * B2 * C2 + B2^2 * C3$
(3) $E1 = exp(-A4 * T) * D1$

The rest is the same as in Example 2.1

If $p = 60$ and $S = 62$ set $A1 = 60$ and $A3 = 62$ in the constraints.

6.3. **Example 4.1.**

Here $t = T/10$. First assume all parameters are fuzzy. The A and B columns are the same as in Example 2.1.

(1) $C1 = MAX(0, A1 * (1 + A2)^{10} - A3)$

(2) $C2 = MAX(0, A1 * (1 + A2)^9 * (1 - A2) - A3)$
(3) $C3 = MAX(0, A1 * (1 + A2)^8 * (1 - A2)^2 - A3)$
(4) $C4 = MAX(0, A1 * (1 + A2)^7 * (1 - A2)^3 - A3)$
(5) $C5 = MAX(0, A1 * (1 + A2)^6 * (1 - A2)^4 - A3)$
(6) $C6 = MAX(0, A1 * (1 + A2)^5 * (1 - A2)^5 - A3)$
(7) $C7 = MAX(0, A1 * (1 + A2)^4 * (1 - A2)^6 - A3)$
(8) $C8 = MAX(0, A1 * (1 + A2)^3 * (1 - A2)^7 - A3)$
(9) $C9 = MAX(0, A1 * (1 + A2)^2 * (1 - A2)^8 - A3)$
(10) $C10 = MAX(0, A1 * (1 + A2) * (1 - A2)^9 - A3)$
(11) $C11 = MAX(0, A1 * (1 - A2)^{10} - A3)$
(12) $D1 = B1^{10} * C1 + 10 * B1^9 * B2 * C2 + 45 * B1^8 * B2^2 * C3 + 120 * B1^7 * B2^3 * C4 + 210 * B1^6 * B2^4 * C5 + 252 * B1^5 * B2^5 * C6 + 210 * B1^4 * B2^6 * C7 + 120 * B1^3 * B2^7 * C8 + 45 * B1^2 * B2^8 * C9 + 10 * B1^B 2^9 * C10 + B2^{10} * C11$
(13) $E1 = EXP(-A4 * T) * D1$

The rest is the same as Example 2.1.
If $p = 60$ and $S = 62$ set $A1 = 60$ and $A3 = 62$ in the constraints.

6.4. **Problems.** The major problems with SOLVER are: (1) it can get out of the feasible set; and (2) it can stop at a local max/min. If Solver gives an error message (left the feasible set) just abort that run and start again with new initial values. To guard against finding local max/mins you will need to run SOLVER many times (the more the better) with different initial conditions.

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