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A NOTE ON THE ZIMMERMANN METHOD FOR SOLVING FUZZY LINEAR PROGRAMMING PROBLEMS

M. R. SAFI, H. R. MALEKI AND E. ZAEIMAZAD

ABSTRACT. There are several methods for solving fuzzy linear programming (FLP) problems. When the constraints and/or the objective function are fuzzy, the methods proposed by Zimmermann, Verdegay, Chanas and Werners are used more often than the others. In the Zimmerman method (ZM) the main objective function $\mathbf{cx}$ is added to the constraints as a fuzzy goal and the corresponding linear programming (LP) problem with a new objective (λ) is solved. When this new LP has alternative optimal solutions (AOS), ZM may not always present the "best" solution. Two cases may occur: $\mathbf{cx}$ may have different bounded values for the AOS or be unbounded. Since all of the AOS have the same λ, they have the same values for the new LP. Therefore, unless we check the value of $\mathbf{cx}$ for all AOS, it may be that we do not present the best solution to the decision maker (DM); it is possible that $\mathbf{cx}$ is unbounded but ZM presents a bounded solution as the optimal solution. In this note, we propose an algorithm for eliminating these difficulties.

1. Introduction

Following "*Decision Making in Fuzzy Environment*" proposed by Bellman and Zadeh [1] and "*On Fuzzy Mathematical Programming*" proposed by Tanaka, Okada and Asai [17], Zimmermann [22] first introduced fuzzy linear programming as conventional LP. He considered LP problems with a fuzzy goal and fuzzy constraints, used linear membership functions and the min operator as an aggregator for these functions, and assigned an equivalent LP problem to FLP. Zimmermann [23] and Werners [20] proposed an approach for determining suitable values for the aspiration level and admissible violation of the fuzzy goal, instead of leaving this decision to DM.

Since then FLP has developed in a number of directions with many successful applications. Among the others, the approach of Verdegay [19] and Chanas [5] which propose a parametric programming approach for solving FLP is the most often used. In their approach, one can obtain an optimal solution to the problem for every value of the parameter. Moreover, we obtain a complete fuzzy decision, thus

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allowing us to consider other possibilities of choice besides the maximizing variant. For a brief study of FLP with fuzzy goal, fuzzy constraints and fuzzy parameters, we refer to [2,6,9,10]. FLP with fuzzy variables and the application of ranking functions in FLP are discussed in the literature [3,4,12,13,14,16].

In this note, we discuss the case when ZM solves a problem and there are alternative optimal solutions (AOS). Since all of the AOS have the same λ, they have the same values for the corresponding LP. Therefore, unless we check the value of **cx** for all AOS, it may be that we do not present the best solution to DM. It is possible that, among the AOS, we present the first solution obtained by the software, whereas the others may have better values for **cx**. It may also be that cycling occurs during the process of finding the AOS, and hence the software cannot obtain all of the AOS. In this case, we may lose the best value for **cx**. Moreover, in ZM, if there exists an optimal solution for the corresponding LP, then it is bounded, whereas it is possible that **cx** is unbounded. Hence, when we present one of the AOS of the corresponding LP to DM, it is clearly not optimal. In this note we present an algorithm that eliminates these difficulties.

There exists yet another difficulty in ZM, in that it does not guarantee the efficiency of the solutions yielded by max-min operator. This problem has been addressed by some authors and procedures for dealing with the problem have been sugessted [7,8,11]. The two-phase approach proposed in [8] is discussed in this note.

We note that the approach of Verdegay [19] and Chanas [5] does not have these problems. However, solving parametric LP problems is difficult in practice.

The organization of this note is as follows. Section 2 reviews ZM. The approach of Verdegay [19] and Chanas [5] is discussed in Section 3 and the two-phase approach proposed in [8] is discussed in Section 4. In Section 5, some examples are solved by ZM, and the difficulties of this method are discussed. Our suggestions to improve ZM are proposed in Section 6, and the performance of our algorithm is discussed in Section 7.

2. **The Zimmermann Method**

Consider the following general form of the symmetric FLP problem: [9]

$$\begin{aligned} \widetilde{\max}\quad z &= \mathbf{cx} \\ s.t.\qquad \mathbf{Ax} &\underset{\sim}{\leq} \mathbf{b} \qquad (2.1), \\ \mathbf{x} &\geq \mathbf{0} \end{aligned}$$

where, $\mathbf{A} \in \Re^{m\times n}, \mathbf{b} \in \Re^m, \mathbf{c}^T \in \Re^n, \mathbf{x} \in \Re^n$; also $\widetilde{\max}$ and $\underset{\sim}{\leq}$ denote the relaxed or fuzzy version of the ordinary max and ≤ respectively. For representing the fuzzy goal, let us stipulate that the objective function **cx** be essentially greater than or equal to an aspiration level b_0, chosen by the decision maker (DM). Then we consider the following problem:

$$
\begin{array}{llll}
\text{find} & \mathbf{x} & & \\
s.t. & \mathbf{cx} & \underset{\sim}{\geq} & b_0 \\
 & \mathbf{Ax} & \underset{\sim}{\leq} & \mathbf{b} \\
 & \mathbf{x} & \geq & \mathbf{0}.
\end{array} \qquad (2.2)
$$

For treating fuzzy inequalities, Zimmermann proposed linear membership function as follows:

$$
\tilde{A}^0(\mathbf{x}) = \begin{cases} 1 & if \quad \mathbf{cx} > b_0 \\ 1 - \dfrac{b_0 - \mathbf{cx}}{p_0} & if \quad b_0 - p_0 \leq \mathbf{cx} \leq b_0 \\ 0 & if \quad \mathbf{cx} < b_0 - p_0 \end{cases} \qquad (2.3),
$$

$$
\tilde{A}^i(\mathbf{x}) = \begin{cases} 1 & if \quad (\mathbf{Ax})_i < b_i \\ 1 - \dfrac{(\mathbf{Ax})_i - b_i}{p_i} & if \quad b_i \leq (\mathbf{Ax})_i \leq b_i + p_i \quad i = 1,2,\ldots,m \\ 0 & if \quad (\mathbf{Ax})_i > b_i + p_i \end{cases} \qquad (2.4),
$$

where for $i = 1,\ldots m$, $(\mathbf{Ax})_i$ is the ith row of $\mathbf{Ax}$, b_i is the ith element of $\mathbf{b}$ and for $i = 0,1,\ldots m$, p_i is a subjectively chosen constant by the DM expressing the limit of the admissible violation of the ith inequality.

Using the "min" operator of Bellman and Zadeh [1] together with the above linear membership functions, the problem of finding the maximum decision reduces to choosing $\mathbf{x}^*$ that maximizes the following objective function

$$
\mu_D(\mathbf{x}) = \min_{i=0,1,\ldots,m} \{\tilde{A}^i(\mathbf{x})\}.
$$

In other words, we have:

$$
\mu_D(\mathbf{x}^*) = \max_{\mathbf{x} \geq \mathbf{0}} \min_{i=0,1,\ldots,m} \{\tilde{A}^i(\mathbf{x})\} \qquad (2.5).
$$

By introducing the auxiliary variable λ, this problem can be transformed as follows:

$$
\begin{array}{lll}
\max & \lambda & \\
s.t. & \lambda \leq \tilde{A}^i(\mathbf{x}) & i = 0,1,\ldots,m \\
 & \mathbf{x} \geq \mathbf{0} & \lambda \in [0,1].
\end{array} \qquad (2.6)
$$

After some simplification, we have the equivalent LP problem: (in the sense that the optimal solution for 2.7 is also optimal for 2.6)

$$\begin{array}{ll} \max & \lambda \\ s.t. & \mathbf{cx} \geq b_0 - (1-\lambda)p_0 \\ & (\mathbf{Ax})_i \leq b_i + (1-\lambda)p_i \quad i = 1,2,\ldots,m \quad (2.7) \\ & \mathbf{x} \geq \mathbf{0}, \quad \lambda \in [0,1]. \end{array}$$

Remark 2.1. Zimmermann [23] and Werners [20] proposed solving the following two problems to determine suitable values for b_0 and p_0, instead of leaving the decision to DM.

$$\begin{array}{ll} \max & z = \mathbf{cx} \\ s.t. & \mathbf{Ax} \leq \mathbf{b} \quad (2.8), \\ & \mathbf{x} \geq \mathbf{0} \end{array}$$

$$\begin{array}{ll} \max & z = \mathbf{cx} \\ s.t. & \mathbf{Ax} \leq \mathbf{b}+\mathbf{p} \quad (2.9), \\ & \mathbf{x} \geq \mathbf{0} \end{array}$$

where $\mathbf{p} = (p_1, \ldots, p_m)$ is chosen by DM expressing the limit of the admissible violation of the constraints. Let z^1 and z^0 be the optimal values for (2.8) and (2.9), respectively. Set $b_0 = z^0$ and $p_0 = z^0 - z^1$.

3. The Verdegay and Chanas Approach

Consider the problem (2.1) with the membership functions (2.3) and (2.4) for the fuzzy goal and fuzzy constraints, respectively. Verdegay [19] and Chanas [5] show that using the technique of parametric programming one can analytically describe the set of solutions incorporating the whole range of possible values of the function

$$\mu_D(\mathbf{x}) = \min\{\tilde{A}^0(\mathbf{x}), \bigwedge_{i=1}^{m} \tilde{A}^i(\mathbf{x})\}.$$

where the notation "$\bigwedge$" denotes the min operator.
These identify the complete fuzzy decision space D. Verdegay and Chanas consider the following parametric programming problem:

$$\begin{array}{ll} \max & \mathbf{cx} \\ s.t. & (\mathbf{Ax})_i \leq b_i + \theta p_i \ ; \ i = 1,2,\ldots,m \quad (3.1), \\ & \mathbf{x} \geq \mathbf{0} \end{array}$$

where the parameter θ $(0 \leq \theta \leq 1)$ can be interpreted as the degree of constraints violation.

It is clear that for every admissible solution $\theta_{\mathbf{x}}$ of problem (3.1) with fixed parameter θ, the condition $\tilde{A}^i(\theta_{\mathbf{x}}) \geq 1-\theta,\ i = 1,2,\ldots,m,$ is valid. On the other hand, for every non-zero solution (if $p_i > 0,\ i = 1,2,\ldots,m$), there exist $1 \leq k \leq m$ such that

$\tilde{A}^k(\theta_{\mathbf{x}}) = 1-\theta$ and therefore the common degree of satisfaction of the constraints is $\mu_C(\theta_{\mathbf{x}}) = \bigwedge_{i=1}^{m} \tilde{A}^i(\mathbf{x}) = 1-\theta.$

Solving the problem by parametric programming techniques, we obtain the set of solutions maximizing the objective function as analytically dependent on parameter θ, i.e. for every θ, we obtain a solution (if there exist one) which satisfies jointly the constraints with degree $1-\theta$ and simultaneously attains the fuzzy goal with the highest degree possible. The maximum value of the objective function of problem (3.1) can now be represented as analytically dependent on parameter θ.

Having functions $\tilde{A}^0$ and μ_C we easily obtain a membership function $\mu_D = \mu_C \wedge \tilde{A}^0$ of the fuzzy decision.

As stated in [5], the following are two specifications of this approach:

"*1- One obtains a complete fuzzy decision allowing one to consider also other possibilities of choice besides the maximizing variant.*

2- The aspiration degree b_0 as well as the tolerance p_0 can be stated by DM only after the solving of the parametric problem which can be easier for the DM because of the additional information about existing realities."

As mentioned above, we can obtain an optimal solution (if there exist one) for each θ. Therefore, we can summarize the results in the Table 1 and then present them to DM.

θ	z^*	resources actually used			
		b_1	b_2	....	b_m
0.0					
0.1					
0.2					
…					
1.0					

TABLE 1. The solution of the parametric programming problem

Also, we can use parametric programming techniques and obtain intervals for θ such that the dependent value of $\mathbf{cx}$ is optimal in each interval.

4. The Two-phase Approach

S. M. Guu and Y. K. Wu [8] proposed a two-phase approach for solving problem (2.1), which concentrates on the fuzzy efficiency of solutions. They used the suggestion of Zimmermann [23] and Werners [20] (Remark 2.1) for obtaining b_0 and p_0. The two phases are as follows:

Phase I. Solve the problem:

$$\begin{array}{ll} \max & \lambda \\ s.t. & 1 \geq \tilde{A}^i(\mathbf{x}) \geq \lambda \geq 0 \quad i = 0,1,\ldots,m \qquad (4.1) \\ & \mathbf{x} \geq \mathbf{0}. \end{array}$$

and denote is an optimal solution for the problem by $(\mathbf{x}^*,\ \lambda^*)$.

Phase II. Let x^{**} be the optimal solution for the following problem:

$$\begin{array}{ll} \max & \sum_{i=0}^{m} \lambda_i \\ s.t. & 1 \geq \tilde{A}^i(\mathbf{x}) \geq \lambda_i \geq \tilde{A}^i(\mathbf{x}^*) \quad i = 0,1,\ldots,m \qquad (4.2) \\ & \mathbf{x} \geq \mathbf{0}. \end{array}$$

This solution is fuzzy efficient for problem (4.2) [8, Theorem 2.1] i.e. there does not exist an optimal solution $\mathbf{y}$ such that

$$\tilde{A}^i(\mathbf{x}^{**}) \leq \tilde{A}^i(\mathbf{y}) \ \ i = 0,1,\ldots,m \text{ and } \tilde{A}^k(\mathbf{x}^{**}) < \tilde{A}^k(\mathbf{y}) \text{ for some } 0 \leq k \leq m.$$

Remark 4.1. Trivially, if $\lambda^* = 1$ in Phase I, there is no need of phase II.

Guu and Wu [8] in the conclusion of their paper mentioned:
From computational consideration, the second phase of the model can be simplified. In fact, because the coefficients in the objective are all positive, the second phase of the problem can be restated as

$$\begin{array}{ll} \max & \sum_{i=0}^{m} \tilde{A}^i(\mathbf{x}) \\ s.t. & 1 \geq \tilde{A}^i(\mathbf{x}) \geq \tilde{A}^i(\mathbf{x}^*) \qquad i = 0,1,\ldots,m \quad (4.3) \\ & \mathbf{x} \in \mathbf{X}, \end{array}$$

where $\mathbf{X}$ denotes the crisp constraints in the original model.

5. **Examples and Results**

In this section, we illustrate the difficulties of ZM by several examples.

Example 5.1. In this example, which is taken from [21], the AOS have different values for the main objective $\mathbf{cx}$, and it may be that ZM does not lead to the best value for $\mathbf{cx}$. Consider the following FLP problem:

$$
\begin{array}{llrcrcll}
\widetilde{\max} & z = & x_1 & + & x_2 & & & \\
s.t. & & x_1 & + & 2x_2 & \lesssim & 10 & \\
& & -2x_1 & + & x_2 & \lesssim & 3 & \quad (5.1)\\
& & 2x_1 & + & x_2 & \lesssim & 12 & \\
& & x_1 & , & x_2 & \geq & 0. &
\end{array}
$$

Let $b_0 = 3,\ p_0 = 1,\ p_1 = 2,\ p_2 = 3$ and $p_3 = 3$. For obtaining a solution to (5.1) ZM solves the following equivalent problem:

$$
\begin{array}{llcrlcl}
\max & \lambda & & & & & \\
s.t. & x_1 & + & x_2 & \geq 3 & - & (1-\lambda) \\
& x_1 & + & 2x_2 & \leq 10 & + & 2(1-\lambda) \\
& -2x_1 & + & x_2 & \leq 3 & + & 3(1-\lambda) \qquad (5.2) \\
& 2x_1 & + & x_2 & \leq 12 & + & 3(1-\lambda) \\
& x_1 & , & x_2 & \geq 0 & , & \lambda \in [0,1].
\end{array}
$$

One of the optimal solutions for (5.2) is $\mathbf{x}_A^* = (3,\ 0)$ with $\lambda^* = 1$. This problem has four alternative optimal basic feasible solutions,

$$\mathbf{x}_B^* = (0,\ 3)\ ,\ \mathbf{x}_C^* = (.8,\ 4.6)\ ,\ \mathbf{x}_D^* = (6,\ 0) \text{ and } \mathbf{x}_E^* = (4.6667,\ 2.6667)\ ,$$

and the corresponding values of the main objective function $z = x_1 + x_2$ for these points are, $z_B^* = 3,\ z_C^* = 5.4,\ z_D^* = 6$ and $z_E^* = 7.3333$. Since the purpose of ZM is to obtain the best value for λ, it does not prefer one of the AOS to the others. Therefore, unless we check the value of z for all AOS, it is possible that we present (for example) $\mathbf{x}_A^*$ to DM as the optimal solution for (5.1), whereas the best value for z occurs at $\mathbf{x}_E^*$.

When solving this problem by WinQSB [15] there was a cycling between $\mathbf{x}_A^*$ and $\mathbf{x}_B^*$, and hence only these two solutions, as alternative basic optimal solutions, were obtained. Thus we lost the other three alternative basic optimal solutions that have better values for z.

Remark 5.2. We can use the suggestion of Zimmermann and Werners (explained in Remark 2.1) for choosing b_0 and p_0. This suggestion eliminates the difficulties of Example 5.1 and we obtain the unique optimal solution $\mathbf{x}_E^* = (4.6667,\ 2.6667)$ with $\lambda^* = 1$ and $z_E^* = 7.3333$. However, besides eliminating the role of DM in

determining the value of b_0 and p_0, it entails solving two further LP problems before solving the main problem, and hence is time consuming.
On the other hand, if we use the two-phase approach, the problem does not have any feasible solution in Phase I. Deleting " $1 \geq \tilde{A}^i(\mathbf{x})$" from the constraints of (4.1) leads to ZM with the suggestion of Zimmermann and Werners and hence the same unique optimal solution $\mathbf{x}_E^* = (4.6667,\ 2.6667)$ with $\lambda^* = 1$ and $z_E^* = 7.3333$. Since $\lambda^* = 1$, the second phase is redundant (Remark 4.1).

The next example shows that even if $\lambda^* < 1$ we may have AOS leading to different values of z.

Example 5.3. Consider the following problem:

$$
\begin{array}{llrcrcl}
\tilde{\max} & z = & x_1 & + & x_2 & & \\
s.t. & & -x_1 & + & x_2 & \tilde{\leq} & 2 \\
 & & 6x_1 & - & 2x_2 & \tilde{\leq} & 30 \\
 & & -x_1 & + & 10x_2 & \tilde{\geq} & 10 \\
 & & x_1 & + & 3x_2 & \tilde{\leq} & 15 \\
 & & 3x_1 & - & x_2 & \tilde{\geq} & 17 \\
 & & x_1 & , & x_2 & \geq & 0 .
\end{array}
\tag{5.3}
$$

Let $b_0 = 3,\ p_0 = p_1 = p_2 = p_3 = 1$ and $p_4 = p_5 = 2$. Then the optimal solution is $\mathbf{x}_A^* = (5.6276,\ 1.4828)$ with $\lambda^* = 0.2$ and $z_A^* = 7.1104$. The problem has an alternative basic optimal solution $\mathbf{x}_B^* = (6.28,\ 3.44)$ with $z_B^* = 9.72$. For both these two solutions and all of their convex combinations we have $\tilde{A}^0(\mathbf{x}) = 1$.

Remark 5.4. Using the suggestion of Zimmermann and Werners yields $b_0 = z^1 = 9.9$. Since the feasible space of (5.3) in the crisp case is empty, we cannot obtain p_0 using the suggestion of Zimmermann and Werners. Let DM choose $p_0 = 1$. Again, we have AOS and the same problems as example 5.1.

Remark 5.5. If we use the two-phase approach, the problem does not have a feasible solution in Phase I. Deleting " $1 \geq \tilde{A}^i(\mathbf{x})$" from the constraints of (4.1) leads to the optimal solution $\mathbf{x}_C^* = (6.125,\ 2.975)$ with $\lambda^* = 0.2$ and $z^* = 9.1$. In Phase II, the feasible space, is again empty, and deleting " $1 \geq \tilde{A}^i(\mathbf{x})$" from the constraints of (4.2) we once again have $\mathbf{x}_C^*$ as the optimal solution. From the viewpoint of efficiency, we cannot prefer one of the points $\mathbf{x}_B^*$ and $\mathbf{x}_C^*$ to the other (Table 2), but the two-phase approach prefers $\mathbf{x}_C^*$ to $\mathbf{x}_B^*$ because of its objective function in Phase II, which consider the summation row in Table 2. However, $z_B^* > z_C^*$ and the two-phase approach does not lead to the best value for the main objective $z = \mathbf{cx}$.

	The degree of satisfaction	
	$\mathbf{x}_B^*$	$\mathbf{x}_C^*$
The fuzzy goal	0.82	.2
The first constraint	1	1
The second constraint	0.2	0.2
The third constraint	1	1
The fourth constraint	0.2	0.975
The fifth constraint	0.2	0.2
The summation	3.42	3.575

TABLE 2. Comparison of two optimal solutions from the efficiency viewpoint

In the next example, not only are there are different values for $\mathbf{cx}$, but they also have different degrees of membership in the main objective function.

Example 5.6. Consider the following problem:

$$\begin{array}{llrcrcl}
\widetilde{\max} & z = & 2x_1 & + & 2.2x_2 & & \\
s.t. & & 2x_1 & + & x_2 & \lesssim & 8 \\
& & x_1 & + & x_2 & \gtrsim & 3 \\
& & -x_1 & + & x_2 & \lesssim & 2 \\
& & -x_1 & + & 10x_2 & \lesssim & 10 \\
& & 3x_1 & + & 3x_2 & \lesssim & 6 \\
& & x_1 & - & x_2 & \gtrsim & 2 \\
& & x_1 & , & x_2 & \geq & 0\,.
\end{array} \qquad (5.4)$$

If $b_0 = 7$, $p_0 = 10$, $p_1 = p_2 = p_4 = p_5 = 2$, and $p_3 = p_6 = 3$, then the optimal solution is $\mathbf{x}_A^* = (2.25,\ 0)$ with $\lambda^* = 0.6250$ and $z_A^* = 4.5$. The problem has an alternative basic optimal solution $\mathbf{x}_B^* = (1.5625,\ 0.6875)$ with $z_B^* = 4.6375$. These two solutions and all of their convex combinations have different values for z and different degrees of membership. The degree of membership in the main objective function for $\mathbf{x}_A^*$ is $\tilde{A}^0(\mathbf{x}_A^*) = 0.75$ and for $\mathbf{x}_B^*$ is $\tilde{A}^0(\mathbf{x}_B^*) = 0.7638$.

In the final example, ZM obtains an optimal solution with a finite value for z, whereas z is unbounded.

Example 5.7. Consider the following problem:

$$\begin{array}{llrcrl} \tilde{\max} & z = & x_1 & + & x_2 & \\ s.t. & & 2x_1 & - & 5x_2 & \lesssim 10 \\ & & 5x_1 & - & 2x_2 & \lesssim 30 \\ & & x_1 & , & x_2 & \geq 0. \end{array}$$

Let $b_0 = 6,\ p_0 = 1,\ p_1 = 2$ and $p_2 = 3.$ Then the optimal solution obtained by ZM is $\mathbf{x}_A^* = (5.7143,\ 0.2857)$ with $\lambda^* = 1$ and $z_A^* = 6$. The alternative basic optimal solutions are $\mathbf{x}_B^* = (6.1905,\ 0.4762)$ and $\mathbf{x}_C^* = (0,\ 6.0)$ with $z_B^* = 6.6667$ and $z_C^* = 6$. Unfortunately, none of them is the best value for z. In fact, the coefficients of x_2 in the objective function and the constraints show that we can increase x_2 without any restriction, and hence z is unbounded. However, ZM does not distinguish this case.

Remark 5.8. In this example, using the suggestion of Zimmermann and Werners to determine b_0 and p_0 eliminates the difficulty.

Remark 5.9. If we use the approach of Verdegay [19] and Chanas [5], we do not have the difficulties in the above examples, because in this approach, the objective in the corresponding LP is to maximize $\mathbf{cx}$. However, there are the practical difficulties of solving parametric LP problems, which are mentioned in the conclusion.

6. Improving the Zimmermann Method

Usually, DM wants to improve the value of $\mathbf{cx}$. He or she usually proposes an aspiration level b_0 but, if possible, prefers that this value be better than b_0. In Section5, we showed that if there exists a set of AOS, then it is possible that the solution obtained by ZM does not give the best value for z. Therefore, if $\mathbf{S}$ is the set of all AOS, we must optimize $z = \mathbf{cx}$ over $\mathbf{S}$. Now we propose the following algorithm for improving ZM and call it "*Improved Zimmermann Method*" (IZM):

The IZM Algorithm:

Step 1. Consider the following form of the FLP problem:

$$\begin{aligned} \tilde{\max} \quad z &= \mathbf{cx} \\ s.t. \quad (\mathbf{Ax})_i &\lesssim b_i \quad i=1,2,\dots,m \quad (6.1) \\ \mathbf{x} &\geq \mathbf{0}. \end{aligned}$$

Step 2. Take $b_0 \in \Re$ and $p_i > 0;\ i = 0,1,\dots,m$, from DM.

Step 3. Solve the following problem:

$$\begin{aligned} &\max \quad \lambda \\ &s.t. \quad \mathbf{cx} \geq b_0 - (1-\lambda)p_0 \\ &\quad (\mathbf{Ax})_i \leq b_i + (1-\lambda)p_i \quad i=1,2,\dots,m \quad (6.2) \\ &\quad \mathbf{x} \geq \mathbf{0}, \quad \lambda \in [0,1]. \end{aligned}$$

Step 4. If problem (6.2) does not have any feasible solution, then STOP. If it has AOS, then go to step 5. Else, let $(\mathbf{x}^*,\ \lambda^*)$ be the unique optimal solution of (6.2). Then $z^* = \mathbf{cx}^*$ is the best value for z with the degree of satisfaction $\tilde{A}^0(\mathbf{x}^*) = 1 - \dfrac{b_0 - \mathbf{cx}^*}{p_0}$. The degrees of satisfaction for the constraints are $\tilde{A}^i(\mathbf{x}^*) = 1 - \dfrac{(\mathbf{Ax}^*)_i - b_i}{p_i}$ $i = 1,2,\dots,m$; STOP.

Step 5. Let $(\mathbf{x}^*,\ \lambda^*)$ be an optimal solution for the problem (6.2), and then solve the following LP problem:

$$\begin{aligned} &\max \quad z = \mathbf{cx} \\ &s.t. \quad \mathbf{cx} \geq b_0 - (1-\lambda^*)p_0 \\ &\quad (\mathbf{Ax})_i \leq b_i + (1-\lambda^*)p_i \quad i = 1,2,\dots,m \quad (6.3) \\ &\quad \mathbf{x} \geq \mathbf{0}. \end{aligned}$$

If the problem (6.3) is unbounded, then the problem (6.1) does not have any bounded optimal solution. STOP.
Else, let $\mathbf{x}^{**}$ be the optimal solution of (6.3), then $z^{**} = \mathbf{cx}^{**}$ is the best value for z with the degree of satisfaction $\tilde{A}^0(\mathbf{x}^{**}) = 1 - \dfrac{b_0 - \mathbf{cx}^{**}}{p_0}$ for the main objective function

and the degree of satisfaction $\tilde{A}^i(\mathbf{x}^{**}) = 1 - \frac{(\mathbf{Ax}^{**})_i - b_i}{p_i}$ for the ith constraint, $i = 1,2,...,m$. SIOP ■

Remark 6.1. Let $\mathbf{S}^*$ be the set of all AOS in Step 5 and suppose it is not singleton. If the DM is interested in a fuzzy efficient solution in $\mathbf{S}^*$, he or she can carry out the following step:

Step 6. (Efficiency) Solve the following problem:

$$\begin{aligned} \max \quad & \sum_{i=0}^{m} \lambda_i \\ s.t. \quad & \tilde{A}^i(\mathbf{x}) \ge \lambda_i \ge \tilde{A}^i(\mathbf{x}^{**}) \quad i = 0,1,...,m \\ & \mathbf{cx} = \mathbf{cx}^{**} \\ & \mathbf{x} \ge \mathbf{0}. \end{aligned}$$

The optimal solution of this problem has the best value for $z = \mathbf{cx}$ and it is a fuzzy efficient solution in $\mathbf{S}^*$. In fact, this step is the second phase of two-phase approach [8] with a little difference.

Remark 6.2. It should be mentioned that the sixth step of above algorithm is somewhat similar to a step in the algorithm of R. N. Tiwari et. al [18].

The following examples are solved by the IZM algorithm.

Example 6.1. Consider Example 5.1. By Steps 1, 2 and 3 the optimal solution for (6.2) is $\mathbf{x}_A^* = (3,\ 0)$ with $\lambda^* = 1$ and $z^* = 3$. In addition, WinQSB gives another optimal basic feasible solution $\mathbf{x}_B^* = (0,\ 3)$ with the same value for the main objective. Solving the corresponding LP (6.3) in Step 5 leads to $\mathbf{x}^{**} = (4.6667,\ 2.6667)$ as the unique optimal solution with $z^{**} = 7.3333$ and $\tilde{A}^i(\mathbf{x}^{**}) = 1,\ i = 0,1,2,3$. This is the best solution for the main objective and the procedure ends.

Example 6.2. Consider Example 5.6. By Steps 1, 2 and 3 the optimal solution for (6.2) is $\mathbf{x}_A^* = (2.25,\ 0)$ with $\lambda^* = 0.6250$ and $z_A^* = 4.5$, $\tilde{A}^0(\mathbf{x}^*) = 0.75$. The problem has AOS. Solving (6.3) in Step 5 leads to $\mathbf{x}^{**} = (1.5625,\ 0.6875)$ as the unique optimal solution with $\tilde{A}^0(\mathbf{x}^{**}) = 0.625$ and $z^{**} = 4.6375$.

Example 6.3. Consider Example 5.8. By Steps 1, 2 and 3 the optimal solution for (6.2) is $\mathbf{x}^* = (5.7143,\ 0.2857)$ with $\lambda^* = 1$ and $z^* = 6$. The problem has AOS.

Solving (6.3) in Step 5 leads to an unbounded solution. Thus, the problem (6.1) does not have any bounded optimal solution and the procedure ends.

Example 6.4. [8] Consider the following problem:

$$\begin{array}{llll} \widetilde{\max} & z = & 4x_1+5x_2+9x_3+11x_4 & \\ & & x_1+x_2+x_3+x_4 & \lesssim 15 \\ & & 7x_1+5x_2+3x_3+2x_4 & \lesssim 80 \\ & & 3x_1+4x_2+10x_3+15x_4 & \lesssim 100 \\ & & x_1,x_2,x_3,x_4 & \geq 0, \end{array}$$

with $b_0=130,\ p_0=30.7143,\ p_1=5,\ p_2=40$ and $p_3=30.$ The Steps 1 to 5 lead to $\mathbf{x}^{**}=(8.5714,\ 0,\ 8.9286,\ 0)$ with $\lambda^*=0.5$ and $z^{**}=114.64286.$ The problem has AOS in Step 5. By Step 6 in Remark 6.1 we obtain $\mathbf{x}^{***}=(4.048,\ 5.655,\ 7.798,\ 0)$ with $z^{***}=z^{**}$.

Note that $\tilde{A}^0(\mathbf{x}^{**})=\tilde{A}^0(\mathbf{x}^{***})=\tilde{A}^1(\mathbf{x}^{**})=\tilde{A}^1(\mathbf{x}^{***})=\tilde{A}^3(\mathbf{x}^{**})=\tilde{A}^3(\mathbf{x}^{***})=0.5$, whereas $\tilde{A}^2(\mathbf{x}^{**})=0.8303572$ and $\tilde{A}^2(\mathbf{x}^{***})=1$. It is easy to see that not only does $\mathbf{x}^{***}$ achieve the optimal objective value, but it also attains a higher degree in the second fuzzy constraint. More precisely, $\mathbf{x}^{***}$ utilizes 80 units of the second resource, while $\mathbf{x}^{**}$ requires 86.784 units.

7. **Conclusion**

The above examples show that if DM proposes b_0, it is possible that ZM does not give the best value for the main objective $\mathbf{cx}$. In addition, if the main objective has an unbounded solution, ZM does not discover it. However, IZM algorithm is efficient in both cases. In fact, as shown in Table 3, IZM works better even than the method suggested by Zimmermann and Werners.

If we use the Verdegay and Chanas approach, we have the other difficulties. Using the dual simplex method for obtaining intervals for θ or obtaining the values for Table 3.2 is not easy, as stated in [10]:

"In any real-world problems, the number of constraints is always rather large (say 50), and so are the decision variables. Therefore, Chanas's approach for formulating the membership function of the fuzzy objective is not practical".

	Zimmermann and Werners suggestion	IZM
When the feasible space of (4.1) is empty	Fails	Efficient
Role of DM in proposing b_0 and p_0	None	Allowed
Number of problems that must be solved, when the feasible space (4.1) is bounded and it is not empty	Three	At most two

TABLE 3. Comparing ZM and IZM

On the other hand, the two-phase approach concentrates on fuzzy efficiency, and hence obtaining the best value for the main objective is not guaranteed (Remark 5.5).

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MOHAMMADREZA SAFI*, DEPARTMENT OF MATHEMATICES, UNIVERSITY OF SHAHID-BAHONAR KERMAN, KERMAN, IRAN
E-mail address: **safi_mohammadreza@yahoo.com**

HAMIDREZA MALEKI, DEPARTMENT OF BASIC SCIENCES, SHIRAZ UNIVERSITY OF TECHNOLOGY, SHIRAZ, IRAN
E-mail address: **maleki@sutech.ac.ir**

EFFAT ZAEIMAZAD, DEPARTMENT OF MATHEMATICES, UNIVERSITY OF SHAHID-BAHONAR KERMAN, KERMAN, IRAN
E-mail address: **effat_zaeimazad@yahoo.com**

* CORRESPONDING AUTHOR