UNCERTAINTY DATA CREATING INTERVAL-VALUED FUZZY RELATION IN DECISION MAKING MODEL WITH GENERAL PREFERENCE STRUCTURE

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Abstract. The paper introduces a new approach to preference structure, where from a weak preference relation derive the following relations: strict preference, indifference and incomparability, which by aggregations and negations are created and examined. We decomposing a preference relation into a strict preference, an indifference, and an incomparability relation. This approach allows one to quantify different types of uncertainty in selecting alternatives. In presented preference structure we use interval-valued fuzzy relations, which can be interpreted as a tool that may help to model in a better way imperfect information, especially under imperfectly defined facts and imprecise knowledge. Preference structures are of great interest nowadays because of their applications, so we propose at the end the algorithm of decision making by use new preference structure.

1. Introduction

In this paper we are especially interested in exploiting techniques for decomposing a fuzzy (weak) preference relation into a preference structure consisting of a strict preference P, an indifference I, and an incomparability relation J. If obtained values in processes of preference are imprecise then we use interval-valued fuzzy relation. Interval-Valued Fuzzy Relations (IVFRs) [40] form a generalization of the concept of a fuzzy relation [39] and represent uncertainties, systematic or random uncertainties. Fuzzy sets and relations are applied in diverse areas, e.g. in group decision making [13, 26, 29, 38]. In recent applications to image processing [5] or classification [34, 35] it has been proven that, under some circumstances, the use of IVFSs together with the total order provide results that are better than their fuzzy counterparts. Many decision making processes take place in an environment in which the information is not precisely known. As a consequence, experts may feel more comfortable using an interval number rather than an exact crisp numerical value to represent their preference. The concept of a preference relation has been studied by many authors, both in crisp or fuzzy environments [14, 32]. But preference values may be imprecise then interval-valued fuzzy preference relations (IVFRs) can be considered as an appropriate representation format to capture experts' uncertain preference information. Diverse properties of IVFRs, also in the

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case of interval-valued fuzzy reciprocal relations, have been studied by a range of authors [22, 27]. The assumption of reciprocity is often used for a preference relation both in the interval-valued [5] and classical fuzzy environment [14].

Our main goal this paper is to examine certain aspect of decision making problem based on preference relations built by aggregations and reciprocity property built by negation function, which means that instead of using classical negation in definition of reciprocity, we apply different negations. This reciprocity appears in preference relations as a natural assumption. We present generalisation of the concept of defining model of three relations: strict preference, indifference and incomparability, corresponding to preference relation.

This work is composed of the following parts. Firstly, concepts and results useful in further considerations are reminded (section II). Next, results connected with preference structure and some properties of interval-valued fuzzy relations are presented (section III). Finally, we present the algorithm of decision making problem with new strict preference, indifference and incomparability relations and example, showing its operation.

2. **Basic definitions**

Decision making problem is based on selection of alternatives and may use fuzzy logic in order to model the linguistic terms used by the system. A key step for the subsequent success of fuzzy systems is the definition of the membership functions representing the problem information as well as possible. Sometimes, it is really difficult to determine the membership functions because the same concept can be defined in different ways by different persons. This problem led Zadeh to suggest the notion of type-2 fuzzy sets as an extension of fuzzy sets. A particular case of type-2 fuzzy sets are the Interval-Valued Fuzzy Sets (IVFSs) that assign as membership degree of the elements to the set an interval instead of a number.

If obtained values in processes of preference are imprecise then we use interval-valued fuzzy relation. We may create interval-valued membership functions from fuzzy membership functions using, for example, ignorance function such as in [5]. Considering the a decision making problem, namely a problem of deciding on set alternatives $X = \{x_1, ..., x_n\}$ for $n > 2$, where relation R_{ij} can be interpreted as a preference of an alternative x_i over an alternative x_j: the higher the R_{ij}, the more preferred is x_i, the more likely x_i appears in comparison with an alternative x_j. Correspondingly, the matrix $R = (R_{ij})$ obtained by collecting the outputs of the compare alternatives can be interpreted as a preference relation. More specifically, suppose that R can be considered as a weak preference relation, which means that R_{ij} is interpreted as $x_i \succeq x_j$, that is, "x_i is at least as likely/good as x_j".

Regardless of the particular decomposition scheme employed, the crucial point is that the relations I and J do have a very interesting meaning in the context of selection alternatives: Indifference corresponds to the conflict involved in a selection situation, while incomparability reflects the corresponding degree of ignorance. More generally, one may speak of a conflict if there is evidence in favor of two or more alternatives simultaneously, while a situation of ignorance occurs if none of the alternatives is supported. The more predictions appear possible, i.e. the higher the diversity of predictions, the higher the degree of ignorance.

Necessary for our future considerations will be the negation function. So, we give the definition of negation functions on the unit interval $[0, 1]$.
A fuzzy negation function [24] is a decreasing function $N : [0, 1] \to [0, 1]$ verifying the boundary conditions $N(0) = 1$ and $N(1) = 0$. Strictly decreasing and continuous negation functions are known as strict negations, whereas involutive negation functions (i.e. those for all $x \in [0, 1]$ verifying $N(N(x)) = x$) are known as strong negations (and constitute a subclass of strict negations).
Typical examples of negation functions are:
$\lhd$ $N(x) = 1 - x$, which is a strong negation and is called the classical or standard negation;
$\lhd$ $N_S^\lambda(x) = \frac{1-x}{1+\lambda x}$, the Sugeno family of fuzzy (strong) negations, where $\lambda \in (-1, \infty)$, and for $\lambda = 0$ we get the classical fuzzy negation.

Crucial in our considerations is the concept of family of intervals. Thus, we recall the notion of operations and the order in the family of intervals.

$$L^I = \{[x_1, x_2] : \ x_1, x_2 \in [0, 1], \ x_1 \le x_2\}.$$

Note that L^I endowed with the partial order $[x_1, x_2] \le_{L^I} [y_1, y_2]$ if and only if $x_1 \le y_1$ and $x_2 \le y_2$ is a complete bounded lattice with the top element given by $\mathbf{1} = [1, 1]$ and the bottom element given by $\mathbf{0} = [0, 0]$. In this lattice, the supremum and the infimum of any two elements is defined respectively by
$[x_1, x_2] \vee [y_1, y_2] = [\max(x_1, y_1), \max(x_2, y_2)]$, $[x_1, x_2] \wedge [y_1, y_2] = [\min(x_1, y_1), \min(x_2, y_2)]$.

Now, we recall the definition of interval-valued (IV) negation function.

Definition 2.1 ([5])**.** An interval-valued (IV) negation is a function $N_{IV} : L^I \to L^I$ that is decreasing with respect to $\le_{L^I}$ with $N_{IV}(\mathbf{1}) = \mathbf{0}$ and $N_{IV}(\mathbf{0}) = \mathbf{1}$. An IV negation is said to be involutive if it fulfils $N_{IV}(N_{IV}(x)) = x$ for any $x \in L^I$ and is known as strong negation.

Below we present the notion of representable IV fuzzy negation. Other presentation, like for example the best interval representation, one may find in [6].

Definition 2.2 (cf. [17])**.** N_{IV} is called the representable IV fuzzy negation if there exists a fuzzy negation N such that $N_{IV}([\underline{x}, \overline{x}]) = [N(\overline{x}), N(\underline{x})]$. We say that N is the associate fuzzy negation with the given IV fuzzy negation.

The following result links strong IV fuzzy negations (in the sense of representability) with strong fuzzy negations.

Theorem 2.3 (cf. [17])**.** *Let $x = [\underline{x}, \overline{x}] \in L^I$. N_{IV} is an involutive IV negation if and only if there exists a strong fuzzy negation N such that $N_{IV}([\underline{x}, \overline{x}]) = [N(\overline{x}), N(\underline{x})]$.*

Furthermore, we may define aggregation function on a set of intervals by the following way:

Definition 2.4 (cf. [12, 25])**.** An operation $\mathcal{A} : (L^I)^n \to L^I$ is called an interval-valued (IV) aggregation function if it is increasing with respect to the order $\leq_{L^I}$ and

$$\mathcal{A}(\underbrace{\mathbf{0}, ..., \mathbf{0}}_{n\times}) = \mathbf{0}, \quad \mathcal{A}(\underbrace{\mathbf{1}, ..., \mathbf{1}}_{n\times}) = \mathbf{1}.$$

A very used, in many applications, class of aggregation functions is that of representable aggregation functions.

Definition 2.5 (cf. [18, 19, 20])**.** Let $\mathcal{A} : (L^I)^2 \to L^I$ be an IV aggregation function. $\mathcal{A}$ is said to be a representable aggregation function if there exist two fuzzy aggregation functions $A_1, A_2 : [0,1]^2 \to [0,1]$, $A_1 \leq A_2$ such that, for every $[x_1, x_2], [y_1, y_2] \in L^I$ it holds that

$$\mathcal{A}([x_1, x_2], [y_1, y_2]) = [A_1(x_1, y_1), A_2(x_2, y_2)].$$

We observe that both $\wedge$ and $\vee$ in L^I define representable aggregation functions in L^I, with $A_1 = A_2 = \min$ in the first case and $A_1 = A_2 = \max$ in the second. Moreover, many other examples may be considered, such as:
$\lhd$ the representable arithmetic mean $\mathcal{A}_{mean}([x_1, x_2], [y_1, y_2]) = [\frac{x_1+y_1}{2}, \frac{x_2+y_2}{2}]$,
$\lhd$ the representable weighted mean with $w_1 + w_2 = 1$, $w_1, w_2 \in [0, 1]$
$\mathcal{A}_{wmean}([x_1, x_2], [y_1, y_2]) = [w_1x_1 + w_2y_1, w_1x_2 + w_2y_2]$,
$\lhd$ the representable geometric mean $\mathcal{A}_g([x_1, x_2], [y_1, y_2]) = [\sqrt{x_1y_1}, \sqrt{x_2y_2}]$,
$\lhd$ the representable weighted geometric mean with $w_1 + w_2 = 1$, $w_1, w_2 \in [0, 1]$
$\mathcal{A}_{wg}([x_1, x_2], [y_1, y_2]) = [x_1^{w_1}y_1^{w_2}, x_2^{w_1}y_2^{w_2}]$,
$\lhd$ the representable product-mean $\mathcal{A}_{P,mean}([x_1, x_2], [y_1, y_2]) = [x_1y_1, \frac{x_2+y_2}{2}]$.

In the subsequent part of this work we will use following properties of aggregation functions with respect to the order $\leq_{L^I}$.

Definition 2.6 (cf. [1, 31])**.** Let $\mathcal{A}, \mathcal{B} : (L^I)^2 \to L^I$ be IV aggregation functions, $x, y, z, t \in L^I$:
1. $\mathcal{A}$ is conjunctive (disjunctive, averaging), if $\mathcal{A} \leq_{L^I} \wedge$
$(\mathcal{A} \geq_{L^I} \vee,\ \wedge \leq_{L^I} \mathcal{A} \leq_{L^I} \vee)$,
2. $\mathcal{A}$ is conjunctor (disjunctor) if and only if it satisfies the condition
$\mathcal{A}(\mathbf{0}, \mathbf{1}) = \mathcal{A}(\mathbf{1}, \mathbf{0}) = \mathbf{0}$ $(\mathcal{A}(\mathbf{0}, \mathbf{1}) = \mathcal{A}(\mathbf{1}, \mathbf{0}) = \mathbf{1})$,
3. $\mathcal{A}$ is said to have $e \in L^I$ a neutral element, if $\mathcal{A}(x, e) = \mathcal{A}(e, x) = x$,
4. $\mathcal{A}$ is said to have $z \in L^I$ an absorbing element, if $\mathcal{A}(x, z) = \mathcal{A}(z, x) = z$,
5. $\mathcal{A}$ is symmetric, if $\mathcal{A}(x, y) = \mathcal{A}(y, x)$,
6. $\mathcal{A}$ is idempotent, if $\mathcal{A}(x, x) = x$,
7. $\mathcal{A}$ is bisymmetric, if $\mathcal{A}(\mathcal{A}(x, y), \mathcal{A}(z, t)) = \mathcal{A}(\mathcal{A}(x, z), \mathcal{A}(y, t))$,
8. $\mathcal{A}$ dominates $\mathcal{B}$ $(\mathcal{A} \gg \mathcal{B})$, if $\mathcal{A}(\mathcal{B}(x, y), \mathcal{B}(z, t)) \geq_{L^I} \mathcal{B}(\mathcal{A}(x, z), \mathcal{A}(y, t))$,
9. $\mathcal{A}$ satisfies the Non-Contradiction principle w.r.t. (NC(N)), if
$\mathcal{A}(x, N_{IV}(x)) = \mathbf{0}$,
10. $\mathcal{A}$ satisfies the Excluded-Middle principle w.r.t. (EM(N)), if
$\mathcal{A}(x, N_{IV}(x)) = \mathbf{1}$.

3. **Interval-valued fuzzy preference structure**

3.1. **Preference structure.** Firstly, we recall concept of fuzzy preference relation.

A fuzzy preference relation R on a set of alternatives $X = \{x_1, ..., x_n\}$ is a fuzzy subset of the Cartesian product $X \times X$, that is $R : X \times X \to [0, 1]$ ([15, 16, 21, 28] or [32]) for each pair of alternatives x_i and x_j, R_{ij} represents a degree of (weak) preference of x_i over x_j, namely the degree to which x_i is considered as least as good as x_j. The preference relation may be conveniently represented by the $n \times n$ matrix $R = (R_{ij})$ for all $i, j \in \{1, ..., n\}$. From a weak preference relation R, Fodor and Roubens [21] or [16] derive the following relations:
1. Strict preference P_{ij} is a measure of strict preference of x_i over x_j, indicating that x_i is (weakly) preferred to x_j but x_j is not (weakly) preferred to x_i.
2. Indifference I_{ij} is a measure of the simultaneous fulfillment of R_{ij} and R_{ji}. Roughly speaking, x_i and x_j are considered equal in the sense that both x_i is as good as x_j and the other way around.
3. Incomparability J_{ij} is a measure of the incomparability of x_i and x_j. More specifically, Fodor and Roubens [21] proposed and examined the following expressions of the above relations in terms of a t-norm T, t-conorm S and a strict negation N:

$$P_{ij} = T(R_{ij}, N(R_{ji})),\ I_{ij} = T(R_{ij}, R_{ji}),\ J_{ij} = T(N(R_{ij}), N(R_{ji})), \tag{1}$$

$$R_{ij} = S(P_{ij}, I_{ij}),\ NR_{ji} = S(P_{ij}, J_{ij}) \tag{2}$$

for all $i, j \in \{1, ..., n\}$ and conclude that no exist De Morgan triple (T, S, N) satisfy presented conditions. Because of this negative results Fodor and Roubens [21] proposed an axiomatic construction of preference structure from a large fuzzy preference relation R (satisfied by Łukasiewicz t-norm and t-conorm).
We will propose generalization this structure in two aspects: Operations and Data. Because many real situations get imprecise information, so we need adequate preference structure, i.e. we would like to consider interval-valued fuzzy preference relation. Firstly, we recall the notion of interval-valued fuzzy relation.

Definition 3.1 (cf. [33, 40])**.** An interval-valued fuzzy relation (IVFR) R between universes X, Y is a mapping $R : X \times Y \to L^I$ such that

$$R(x, y) = [\underline{R}(x, y), \overline{R}(x, y)] \ \ for\ all\ pairs\ \ (x, y) \in X \times Y.$$

The class of all IVFRs between universes X, Y is denoted by $IVFR(X \times Y)$, or $IVFR(X)$ for $X = Y$.

Note that, if we consider the order defined in L^I, we see that the family $(IVFR(X \times Y), \vee, \wedge)$ is a complete and distributive lattice (see [9] for a study on the concept of lattices).

An approach that adds flexibility to represent uncertainty in decision making problems consists of using interval-valued fuzzy relations ([23, 36, 37]). An interval-valued fuzzy preference relation R on X is defined as an interval-valued fuzzy subset of $X \times X$, that is, $R : X \times X \to L^I$, which have been studied deeply (see [5, 7, 36, 37, 38] or [8] created with grouping and overlap functions). The interval $R_{ij} = [\underline{R}_{ij}, \overline{R}_{ij}]$ denotes the degree to which elements x_i and x_j are related (representing the degree of preference of x_i over x_j) in the relation R for all x_i, $x_j \in X$. A preference structure can be characterized by weakly preference relation called the large preference relation. It has been mentioned, that it is possible to construct preference structure from a large preference relation R in the classical case, however, in fuzzy case this is not possible. This fact was proved by Alsina in [2] and later by Fodor and Roubens in [21].

We continue this examinations and we propose generalisation (1) to interval-valued fuzzy structure. In [5] for given an IVFR, $R = (R_{ij})$, was examined corresponding Interval-valued strict preference (P), interval-valued indifference (I) and interval-valued incomparability (J) by using IV t-norms and IV negations generated from the standard strict negation. We based on the corresponding ones methods given by Fodor and Roubens from 1994, and we propose generalization of these concepts, we use IV aggregations instead of IV t-norms and IV negations instead of classic negations. According to Independence of Irrelevant Alternatives and Positive Association Principle (see [21]) we propose three functions $\mathbf{p,i,j} : (L^I)^2 \to L^I$, such that

$$P_{ij} = \mathbf{p}(R_{ij}, R_{ji}),$$
$$I_{ij} = \mathbf{i}(R_{ij}, R_{ji}),$$
$$J_{ij} = \mathbf{j}(R_{ij}, R_{ji})$$

and these functions have the following differences:

$$\mathbf{p}(x, N(y)),\ \mathbf{i}(x, y)\ \ and\ \ \mathbf{j}(N(x), N(y))$$

are increasing with respect to both arguments and the preference structure adequate to (2) is characterized also by the following system of the functional equations, i.e. (2) be translated to:

$$\mathcal{C}(P_{ij}, I_{ij}) = R_{ij},$$
$$\mathcal{C}(P_{ij}, J_{ij}) = N(R_{ji}),$$

where IV aggregation function $\mathcal{C}$ is specific one for each preference structure.
If we denote $R_{ij} = x$ and $R_{ji} = y$ we can write our system in the following way:

$$\mathcal{C}(\mathbf{p}(x, y), \mathbf{i}(x, y)) = x, \tag{3}$$

$$\mathcal{C}(\mathbf{p}(x, y), \mathbf{j}(x, y)) = N(y). \tag{4}$$

We propose following method for built the preference structure by using two different IV aggregations $\mathcal{A}$ and $\mathcal{B}$:
Interval-valued strict preference

$$P_{ij} = \mathcal{A}(R_{ij}, N_{IV}(R_{ji})) = \mathcal{A}([\underline{R}_{ij}, \overline{R}_{ij}], N_{IV}([\underline{R}_{ji}, \overline{R}_{ji}])); \tag{5}$$

Interval-valued indifference

$$I_{ij} = \mathcal{B}(R_{ij}, R_{ji}) = \mathcal{B}([\underline{R}_{ij}, \overline{R}_{ij}], [\underline{R}_{ji}, \overline{R}_{ji}]); \tag{6}$$

Interval-valued incomparability

$$J_{ij} = \mathcal{B}(N_{IV}(R_{ij}), N_{IV}(R_{ji})) = \mathcal{B}(N_{IV}([\underline{R}_{ij}, \overline{R}_{ij}]), N_{IV}([\underline{R}_{ji}, \overline{R}_{ji}])) \tag{7}$$

for all $i, j \in \{1, ..., n\}$. If we used above IV aggregations $\mathcal{A}$ and $\mathcal{B}$ as representable and N_{IV} invotutive IV negation, then we obtain:

$$P_{ij} = [A_1(\underline{R}_{ij}, N(\overline{R}_{ji})), A_2(\overline{R}_{ij}, N(\underline{R}_{ji}))];$$

$$I_{ij} = [B_1(\underline{R}_{ij}, \underline{R}_{ji}), B_2(\overline{R}_{ij}, \overline{R}_{ji})];$$

$$J_{ij} = [B_1(N(\overline{R}_{ij}), N(\overline{R}_{ji})), B_2(N(\underline{R}_{ij}), N(\underline{R}_{ji}))].$$

It is generalisation by $\mathcal{A}$, $\mathcal{B}$ IV aggregations and N_{IV} IV negation of preference structure. Furthermore, we can say that alternative x_i strictly preferred to x_j ($x_i \succ x_j$):

$$x_i \succ x_j \Leftrightarrow \underline{P}_{ij} > \overline{P}_{ji}.$$

Then we observe that $\overline{R}_{ji} < \underline{R}_{ij} \Rightarrow x_i \succ x_j$. Directly, we obtain following result.

Proposition 3.2. *Let A_1, A_2 be fuzzy aggregation functions, N strong negation and $R \in IVFR(X)$ be preference relation. Then*

$$x_i \succ x_j \Leftrightarrow A_1(\underline{R}_{ij}, N(\overline{R}_{ji})) > A_2(\overline{R}_{ji}, N(\underline{R}_{ij}))$$

for any $x_i, x_j \in X$.

Proposition 3.3. *For an interval-valued fuzzy preference structure (P, I, J) the following equalities of boundary elements are fulfilled:*
$\boldsymbol{p}(\boldsymbol{0}, \boldsymbol{1}) = \boldsymbol{i}(\boldsymbol{0}, \boldsymbol{0}) = \boldsymbol{j}(\boldsymbol{1}, \boldsymbol{1}) = \boldsymbol{0}$ *and* $\boldsymbol{p}(\boldsymbol{1}, \boldsymbol{0}) = \boldsymbol{i}(\boldsymbol{1}, \boldsymbol{1}) = \boldsymbol{j}(\boldsymbol{0}, \boldsymbol{0}) = \boldsymbol{1}$.

Proposition 3.4. *If (3) and (4) are satisfied by $\boldsymbol{p}, \boldsymbol{i}$ and $\boldsymbol{j}$, then*

$$\boldsymbol{i}(\boldsymbol{0}, y) = \boldsymbol{p}(\boldsymbol{0}, y) = \boldsymbol{j}(x, \boldsymbol{1}) = \boldsymbol{p}(x, \boldsymbol{1}) = \boldsymbol{0},$$

where IV aggregation function $\mathcal{C}$ fulfils $\mathcal{C}(s, t) = \boldsymbol{0} \Leftrightarrow s = \boldsymbol{0}$ and $t = \boldsymbol{0}$ and $\mathcal{C}(s, t) = \boldsymbol{1} \Leftrightarrow s = \boldsymbol{1}$ and $t = \boldsymbol{1}$.

Equations (3) and (4) are generalisation for equations presented by Fodor and Roubens in [21] or [2]. Especially, the equation (3) is generalisation of the Alsina equation [2] from the classical relationship in set theory

$$(A \cap B) \cup (A \cap coB) = A,$$

which holds for any two subsets A and B of a universe X and can be generalized to fuzzy sets by defining the intersection and union of fuzzy sets based on a t-norm T and t-conorm S and the complement of fuzzy sets based on a strong negation N which satisfy the functional equation

$$\forall_{(x,y)\in[0,1]^2}(S(T(x, y), T(x, N(y))) = x).$$

The solution of this equation was examined and presented in [2]. Moreover, in [21] this equation was study for two different t-norms or dually t-conorms. In [11] authors used arbitrary unirms or nullnorms instead of t-norms and t-conorms, but all cases lead back to the known t-norm and t-conorm solutions. And more generally in [10] replacing only the t-conorm by aggregation function have been studied. We continue this ideas and study yet more generalisation, where instead t-norms and t-conorms we use different aggregation functions. We observe following special solutions of the equation (3).

Proposition 3.5. *The equation (3) holds if one of the following conditions is fulfilled:*
1. $\mathcal{A}, \mathcal{B}$ are projections into the first coordinate and $\mathcal{C}$ is idempotent;
2. $\mathcal{B}$ is the projection into the first coordinate and $\mathcal{A}, \mathcal{C}$ satisfy the absorbtion law;
3. $\mathcal{A}$ is the projection into the first coordinate and $\mathcal{B}, \mathcal{C}$ satisfy the absorbtion law.

Proof. We prove the second condition. Let $\mathcal{B}$ be the projection into the first coordinate and $\mathcal{A}, \mathcal{C}$ satisfy the absorbtion law. Then

$$\mathcal{C}(\mathcal{A}(x, N_{IV}(y)), \mathcal{B}(x, y)) = \mathcal{C}(\mathcal{A}(x, N_{IV}(y)), x) = x.$$

Thus (3) is fulfilled by (5) and (6). Proofs of first and third conditions are similar. □

Proposition 3.6. *Let IV aggregation functions $\mathcal{A}, \mathcal{B} \leq_{L^I} \wedge$. Then the following conditions are equivalent:*
1. $\mathcal{C}$ is IV idempotent aggregation function;
2. $\mathcal{A}, \mathcal{B}, \mathcal{C}$ satisfy (3).

Proof. Let IV aggregation functions $\mathcal{A}, \mathcal{B} \leq_{L^I} \wedge$ and $\mathcal{C}$ be IV idempotent aggregation function, then

$$\mathcal{C}(\mathcal{A}(x, N_{IV}(y)), \mathcal{B}(x, y)) \leq_{L^I} \mathcal{C}(x \wedge N_{IV}(y), x \wedge y) \leq \mathcal{C}(x, x) = x.$$

Thus $\mathcal{A}, \mathcal{B}, \mathcal{C}$ satisfy (3). If we assume (3), then by properties of operation minimum we obtain idempotency of $\mathcal{C}$. □

Directly from the equations (3) and (5-6) appears the corresponding following composition:

Definition 3.7. Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be IV aggregation functions and N_{IV} be IV negation function. The operation $\mathcal{D} : (L^I)^2 \to L^I$ associated with aggregation functions $\mathcal{A}, \mathcal{B}, \mathcal{C}$ we define as follows

$$\mathcal{D}(x, y) = \mathcal{C}(\mathcal{A}(x, N_{IV}(y)), \mathcal{B}(x, y)). \tag{8}$$

We can also write this operation as follows:

$$\mathcal{D}(x, y) = \mathcal{C}(P(x, y), I(x, y)).$$

Example 3.8. Let $x, y \in L^I$.

$$\mathcal{A}(x, y) = \begin{cases} \mathbf{1}, & if \;\; y = \mathbf{1} \\ \mathcal{A}_{mean}, & otherwise \end{cases}$$

for N_{IV} IV negation function,

$$\mathcal{B}(x, y) = \begin{cases} \mathbf{1}, & if \;\; x \geq_{L^I} [\frac{1}{2}, \frac{1}{2}] \; or \; y \geq_{L^I} [\frac{1}{2}, \frac{1}{2}] \\ \mathbf{0}, & if \;\; x = y = \mathbf{0} \\ \alpha, & otherwise \end{cases}$$

and

$$\mathcal{C}(x, y) = \begin{cases} \mathbf{1}, & if \;\; x = \mathbf{1} \; or \; y = \mathbf{1} \\ \mathcal{A}_{mean}, & otherwise. \end{cases}$$

Then

$$\mathcal{D}(x,y) = \begin{cases} \frac{1}{4}x + \frac{1}{4}N_{IV}(y) + \frac{1}{2}\alpha, & if \ \ x, y <_{L^I} [\frac{1}{2}, \frac{1}{2}], y \neq \mathbf{0} \\ \mathbf{1}, & otherwise \end{cases}$$

for $\alpha \in (0,1)$.

The operation $\mathcal{D}$ has following properties.

Proposition 3.9. *Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be IV aggregation functions and $\mathcal{D}$ be created by (8).*
1. $\mathcal{D}$ has the absorbing element $\boldsymbol{0}(\boldsymbol{1})$, if $\mathcal{A}, \mathcal{B}, \mathcal{C}$ have the absorbing element $\boldsymbol{0}(\boldsymbol{1})$.
2. $\mathcal{D}(\boldsymbol{0}, \boldsymbol{0}) = \boldsymbol{0}$ ($\mathcal{D}(\boldsymbol{1}, \boldsymbol{1}) = \boldsymbol{1}$), if $\mathcal{A}(\boldsymbol{0}, \boldsymbol{1}) = \boldsymbol{0}$ ($\mathcal{A}(\boldsymbol{1}, \boldsymbol{0}) = \boldsymbol{1}$).
3. $\mathcal{D}$ has NC(N) (EM(N)) property, if $\mathcal{B}$ has this property and $\mathcal{C}$ has the absorbing element $\boldsymbol{0}$ ($\boldsymbol{1}$).
4. $\mathcal{D}$ is isotonic with respect to the first variable.
5. $\mathcal{D} \gg \mathcal{E}$, if $\mathcal{A} \gg \mathcal{E}$, $\mathcal{B} \gg \mathcal{E}$ and $\mathcal{C} \gg \mathcal{E}$.
6. $\mathcal{D}$ is conjunctive (disjunctive), if $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are conjunctive (disjunctive).

Proof. The proof we may obtain by the Definition 2.6 and Definition 3.7. □

Now, we recall the crucial definition for this paper of reciprocity property based on negation.

Definition 3.10 ([30])**.** Let $card(X) = n$. An Interval-Valued Fuzzy Reciprocal Relation (IVFRR) R on the set X is a matrix $R = (R_{ij})_{n \times n}$ with $R_{ij} = [\underline{R}_{ij}, \overline{R}_{ij}]$, for all $i, j \in \{1, \ldots, n\}$, where $R_{ij} \in L^I$

$$r_{ii} = [0.5, 0.5], R_{ji} = N_{IV}(R_{ij}) = [N(\overline{R}_{ij}), N(\underline{R}_{ij})]$$

for $i \neq j$, where N is a fuzzy negation and N_{IV} is an IV negation function.

Presented the reciprocity property is based on negation. This notion is a generalization of the reciprocity property introduced in [37], where N was a standard negation. However, the assumption $R_{ji} = 1 - R_{ij}$ for $i, j \in \{1, \ldots, n\}$, which stems from the reciprocity property, is rather strong and frequently violated by decision makers in real-life situations. This is why we use a fuzzy negation instead of the classical negation $N(x) = 1 - x$. Especially, if $\overline{R}_{ij} = \underline{R}_{ij} = R_{ij}$ for $i, j \in \{1, \ldots, n\}$, then an IVFRR reduces to a reciprocal fuzzy relation (it is also worth mentioning that IVFRRs may be built from the fuzzy ones using the concept of ignorance function [5]). The interval R_{ij} indicates the interval-valued reciprocal degree or intensity of the alternative x_i over alternative x_j and $\underline{R}_{ij}$, $\overline{R}_{ij}$ are the lower and upper limits of R_{ij}, respectively.
The following results extends Theorem from [5]

Proposition 3.11 ([7])**.** *Let $R \in IVFR(X \times X)$ be reciprocal and let P_{ij} be its associated interval-valued strict fuzzy preference relation given. The following equivalence holds:*
$P_{ij} = R_{ij}$ for all $i, j \in \{1, ..., n\}$ if and only if IV aggregation $\mathcal{A}$ is idempotent.

Corollary 3.12. *$P_{ij} = R_{ij} \Leftrightarrow \mathcal{A}$ is averaging aggregation function.*

Proposition 3.13. *Let $R \in IVFR(X)$ be reciprocal and I, J be associated interval-valued indifference and incomparability fuzzy preference relations. If $\mathcal{B}$ is symmetric, then $I = J$.*

3.2. **Interval-valued fuzzy relations properties used in the preference structure.** Now, we consider the most important property in point of view of consistency of the group decision making, i.e. transitivity.

Proposition 3.14. *If reciprocal relation R is A_1-transitive and A_1 is bisymmetrical aggregation, then $\underline{P}$ is also A_1-transitive, i.e. $A_1(\underline{P}(x,y), \underline{P}(y,z) \leq \underline{P}(x,z)$.*

Proof. $A_1 \in [0,1]^m \to [0,1]$ is bisymmetrical aggregation iff

$$A_1(A_1(x_{11},...,x_{1m}),...,A_1(x_{m1},...,x_{mm})) =$$
$$A_1(A_1(x_{11},...,x_{m1}),...,A_1(x_{1m},...,x_{mm})).$$

Let $m = 2$. By bisymmetrically of A_1 and A_1-transitivity of reciprocal relation R we have

$$A_1(\underline{P}_{ij}, \underline{P}_{jk}) = A_1(A_1(\underline{r}_{ij}, N(\overline{r}_{ji})), A_1(\underline{r}_{jk}, N(\overline{r}_{kj}))) =$$
$$A_1(A_1(\underline{r}_{ij}, \underline{r}_{jk}), A_1(N(\overline{r}_{ji}), N(\overline{r}_{kj}))) = A_1(A_1(\underline{r}_{ij}, \underline{r}_{jk}), A_1(\underline{r}_{ij}, \underline{r}_{jk})) \leq$$
$$A_1(\underline{r}_{ik}, \underline{r}_{ik}) = A_1(\underline{r}_{ik}, N(\overline{r}_{ki})) = \underline{P}_{ik}.$$

□

We also will consider $\mathcal{A}$-transitivity of the relation $R \in IVFR(X)$, i.e.

$$\mathcal{A}(R(x,y), R(y,z)) \leq_{L^I} R(x,z)$$

for $x, y, z \in X$.

Proposition 3.15. *Let $R \in IVFR(X)$ be reciprocal and P be associated interval-valued strict fuzzy preference relation and $\mathcal{A}$ be bisymmetric IV aggregation function. If R is $\mathcal{A}$-transitive, then P is also $\mathcal{A}$-transitive.*

Proof. By reciprocity of R and $\mathcal{A}$-transitivity by bisymmetry of $\mathcal{A}$ we have:

$$\mathcal{A}(P(x,y), P(y,z)) = \mathcal{A}(\mathcal{A}(R(x,y), N_{IV}(R(y,x))), \mathcal{A}(R(y,z), N_{IV}(R(z,y))) =$$
$$\mathcal{A}(\mathcal{A}(R(x,y), R(x,y)), \mathcal{A}(R(y,z), R(y,z)) =$$
$$\mathcal{A}(\mathcal{A}(R(x,y), R(y,z)), \mathcal{A}(R(x,y), R(y,z)) \leq \mathcal{A}(R(x,z), R(x,z)) =$$
$$\mathcal{A}(R(x,z), N_{IV}R(z,x)) = P(x,z).$$

So P is $\mathcal{A}$-transitive. □

Similar to classic and fuzzy preference structure we asked the question about asymmetry property of strict preference relation. We also will consider the more general asymmetry property and more practical, i.e. general weak asymmetry property of relation $R \in IVFR(X)$ and $card(X) = n$:

- R is $\mathcal{A}$-asymmetric, if $\mathcal{A}(R_{ij}, R_{ji}) = \mathbf{0}$,
- R is weakly $\mathcal{A}$-asymmetric, if $\mathcal{A}(R_{ij}, R_{ji}) \leq_{L^I} [\frac{1}{2}, \frac{1}{2}]$ for all $i, j = \{1, ..., n\}$, $n \in \mathbb{N}$.

Proposition 3.16. *If R is $\mathcal{A}$-asymmetric, $\mathcal{A}$ is N-stable (i.e. $\mathcal{A}(N_{IV}(R_{ij}), N_{IV}(R_{ji})) = N_{IV}(\mathcal{A})(R_{ij}, R_{ji}))$ and bisymmetric IV aggregation function such that $\mathcal{A}(\boldsymbol{0}, \boldsymbol{1}) = \boldsymbol{0}$, then P is also $\mathcal{A}$-asymmetric.*

Proof. Let $\mathcal{A}$ be bisymmetric aggregation function, i.e.

$$\mathcal{A}(\mathcal{A}(x_{11}, ..., x_{1m}), ..., \mathcal{A}(x_{m1}, ..., x_{mm})) = \mathcal{A}(\mathcal{A}(x_{11}, ..., x_{m1}), ..., \mathcal{A}(x_{1m}, ..., x_{mm})).$$

Then for $m = 2$ we have

$$\mathcal{A}(P_{ij}, P_{ji}) = \mathcal{A}(\mathcal{A}(R_{ij}, N_{IV}(R_{ji})), \mathcal{A}(R_{ji}, N_{IV}(R_{ij})) =$$

$$\mathcal{A}(\mathcal{A}(R_{ij}, R_{ji}), \mathcal{A}(N_{IV}(R_{ji}), N_{IV}(R_{ij})) = \mathcal{A}(\mathcal{A}(R_{ij}, R_{ji}), N_{IV}\mathcal{A}(R_{ji}, R_{ij}) =$$

$$\mathcal{A}(\mathbf{0}, \mathbf{1}) = \mathbf{0}.$$

Thus P is $\mathcal{A}$-asymmetric. □

Proposition 3.17. *Let N_{IV} be IV negation function fulfil $N_{IV}(x) \leq_{L^I} x$. If R is weakly $\mathcal{A}$-asymmetric and $\mathcal{A}$ is N-stable and bisymmetric, subidempotent IV aggregation function, then P is also weakly $\mathcal{A}$-asymmetric.*

In [7] we check preservation reciprocity property by aggregation functions, now we study this property by operation $\mathcal{D}$. We will need concept of duality, i.e.

Definition 3.18 (cf. [31])**.** Let $F : [0, 1]^n \to [0, 1]$, N be a strong fuzzy negation. The N-dual of F is the function

(9) $$F^N(x_1, \ldots, x_n) = N(F(N(x_1), \ldots, N(x_n))), \ \ x_1, \ldots, x_n \in [0, 1].$$

Proposition 3.19. *Let $i \neq j$, $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be IV representable aggregation functions, such that $A_1 = A_2^N$,$B_1 = B_2^N$, $C_1 = C_2^N$ and N be IV strong negation function. Then operation $\mathcal{D}$ preserves reciprocity property of R.*

Proof. We will check whether it is happening:
$N_{IV}\overline{\mathcal{D}}(R_{ji}, R_{ij}) = \underline{\mathcal{D}}(R_{ij}, R_{ji})$.
Thus by duality of aggregations and reciprocity property of R we obtain

$$N\overline{\mathcal{D}}(R_{ji}, R_{ij}) = N(\mathcal{C}_2(\mathcal{A}_2(\overline{R}_{ji}, N\underline{R}_{ij}), \mathcal{B}_2(\overline{R}_{ji}, \overline{R}_{ij}))) =$$

$$N(\mathcal{C}_2(N\mathcal{A}_1(N\overline{R}_{ji}, NN\underline{R}_{ij}), N\mathcal{B}_1(N\overline{R}_{ji}, N\overline{R}_{ij}))) =$$

$$N(\mathcal{C}_2(N\mathcal{A}_1(\underline{R}_{ij}, N\overline{R}_{ji}), N\mathcal{B}_1(\underline{R}_{ij}, \underline{R}_{ji}))) =$$

$$\mathcal{C}_1(\mathcal{A}_1(\underline{R}_{ij}, N\overline{R}_{ji}), \mathcal{B}_1(\underline{R}_{ij}, \underline{R}_{ji})) = \underline{\mathcal{D}}(R_{ij}, R_{ji}).$$

What finished the proof.

□

4. **Illustrative example: decision making problem**

Our above results allow to perform the following applications. In a multi-expert decision making problem we have a set of alternatives described on set of criteria, a set of experts and each of the latter provides his/her opinions on the former set of alternatives by given criteria. It is well known that, depending on the context and/or the level of knowledge of the experts, in some decision making problems it may occur that it is difficult to express the preferences using precise numerical values. Moreover, it can also happen, when there are alternatives pairwise compared, that experts are not sure if they prefer an alternative or another. In these cases, they can choose values close to 0.5. For these reasons, some extensions of fuzzy sets are used.

We consider an interval-valued fuzzy relation on $X = \{x_1, \dots, x_n\}$ (set of alternatives) which represents the expert's opinion of each alternative over another one, i.e. preferences. The preferences will be represented with respect to a finite number of criteria, mathematically these are relations $R_1, \dots, R_n \in IVFR(X)$. As usual, the elements of the main diagonal of an interval-baled fuzzy preference relations will not be considered and therefore in the rest of the paper we are assuming that $i \neq j$. To find the solution alternative we apply modified voting method by generalized preference structure (P, I, J) and the order generated by $\leq_{L^I}$ and defined in the following way:

$$x \leq_{Lin} y \Leftrightarrow x \leq_{L^I} y \;\; or \;\; \overline{x} - \underline{x} \leq \overline{y} - \underline{y}. \tag{10}$$

It is worth to mention that at the beginning of algorithm it may be checked if $R_1, \dots, R_n \in IVFR(X)$ are reciprocal with respect to some fuzzy negation N. If the answer is positive we may apply the presented in this paper results in order to consider the appropriate aggregation function to aggregate these relations and obtain reciprocal aggregated result. We will present the algorithm in the case when we do not check reciprocity of input relations. In such situation the aggregated IVFR is normalized to obtain the given reciprocity with the use of the following formula

$$R_{ij} = \begin{cases} N_{IV}(R_{ji}) & if \;\; R_{ij} >_{Lin} R_{ji}, \\ R_{ij} & else. \end{cases} \tag{11}$$

The following algorithm gives an alternative who has the worst/best relationships in a considered X.

Algorithm. $\mathcal{D}$ **− composition**.

Inputs : $X = \{x_1, \dots, x_n\}$ set of alternatives; $\mathcal{A}$, $\mathcal{B}$ IV aggregation functions; $R_1, \dots, R_n \in IVFR(X)$; $\mathcal{D}$ created according to (8); Interval-valued fuzzy preference (reciprocal and transitive or not) relations; The order $\leq_{Lin}$.

Output : The best alternative: $x_{selection}$.

(**Step 1**) Aggregation of given relations $R_1, \dots, R_n \in IVFR(X)$ to obtain $R \in IVFR(X)$;

(**Step 2**) Normalization of relation R with the use of Eq. (11);

(**Step 3**) Build J interval-valued fuzzy relation according to (7);

(**Step 4**) Calculate

$$M_{ij} = \mathcal{A}(\mathcal{D}(R_{ij}, R_{ji}), J_{ij});$$

(**Step 5**) For $m = 1$ to $n + 1$
Find

$$x_{selection} = arg \max_i (\mathcal{B}(M_{ij})),$$

where $1 \leq j \neq i \leq n$, $\mathcal{B} \geq_{L^I} \vee$, using a linear order $\leq_{Lin}$;

If only one alternative is the "best" solution, then this alternative is the final solution of the decision making problem;

Else we chose the one with the smallest interval length as the final solution of the decision making problem (if they have the same lengths, then we change aggregation $\mathcal{B}$ and we repeat Step 5)

End

Example 4.1. Here we would like to consider the problem of evaluation of employees in the some company from Poland. The set of given workers we denote $X = \{x_1, ..., x_n\}$ for n people. The group of experts K give their preferences i.e. the opinions of each worker/candidate over another one in a corporation under a finite number of criteria and represent in set of interval-valued fuzzy relations (we omit the aspect of creation interval-valued membership functions from fuzzy membership functions represents preferences).

We provide here a numerical example which illustrates the presented algorithm for steps 3 - 5 for $cardX = 4$, $cardK = 4$ and one of criterium. Thus we have preference relations $R_1, \dots, R_4 \in IVFR(X)$, where $X = \{x_1, \dots, x_4\}$ (set of workers). After aggregation of interval-valued fuzzy relations $R_1, \dots, R_4$ we build preference relations P, I ($\mathcal{D}$) and J by (8) and(7) and the composition $\mathcal{D}$ and IV representable aggregation functions $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ from the Example 3.8 with N_{IV} is Sugeno IV negation with $\alpha = 0.5$. Moreover, we calculate M, where $\mathcal{A} = [A, A]$ and A is the weighted arithmetic mean with weights $0.8, 0.2$ for $\mathcal{D}, J$, respectively. The value of the weights increases the impact of Strict preference and Indifference relation in to Incomparability relation. Thus

$$M_{ij} = 0.8\mathcal{D}_{ij} + 0.2J_{ij}.$$

In the Java code our relations ($\mathcal{D}$ and M) we may presented as the code snippet:

```
for (int i = 0; i < R.length; i++) {
 for (int j = 0; j < R[0].length; j = j+2) {
```

$$R = \begin{bmatrix} - & [0.1166, 0.7464] & [0.109, 0.43] & [0.099, 0.827] \\ [0.2536, 0.8834] & - & [0.0374, 0.1188] & [0.0546, 0.1794] \\ [0.57, 0.891] & [0.8812, 0.9626] & - & [0.0872, 0.862] \\ [0.173, 0.901] & [0.8206, 0.9454] & [0.138, 0.9128] & - \end{bmatrix}.$$

$$J = \begin{bmatrix} - & [0.1222, 0.7435] & [0.1880, 0.5317] & [0.0913, 0.8083] \\ [0.1222, 0.7435] & - & [0.144917739, 0.2791491377] & [0.1670, 0.3421] \\ [0.1880, 0.5317] & [0.1449, 0.2791] & - & [0.0759, 0.8398] \\ [0.0913, 0.8083] & [0.1671, 0.3421] & [0.0759, 0.8398] & - \end{bmatrix}.$$

```
    if (i*2 != j && i*2 + 1 != j) {
      if (R[i][j] < 0.5 && N[j/2][2*i] < 0.5 &&
          R[i][j+1] < 0.5 && N[(j+1)/2][2*i+1] < 0.5) {
        D[i][j] = 0.25*R[i][j] + 0.25*N[j/2][2*i] + 0.5*alpha;
        D[i][j+1] = 0.25*R[i][j+1] + 0.25*N[(j+1)/2][2*i+1] + 0.5*alpha;
      } else {
        D[i][j] = 1;
        D[i][j+1] = 1; }
     }
   }
 }
for (int i = 0; i < D.length; i++) {
  for (int j = 0; j < D[0].length; j++) {
    if (i*2 != j && i*2 + 1 != j)
      M[i][j] = 0.8*D[i][j] + 0.2*J[i][j]; }
 }
```

We use relation R and J presented below.

Then after aggregation done on values in each row (using the maximum) we obtain x_i:
$x_1 = [0.8244427209, 0.9616735815]$; $x_2 = [0.8244427209, 0.9487190804]$;
$x_3 = [0.8376169418, 0.9679637478]$; $x_4 = [0.8334180769, 0.9679637478]$.

Thus by (10) we observe the following order:
$x_3 \succeq x_4 \succeq x_1 \succeq x_2$, i.e. the third candidate is the best one.

We observe interesting benefits of the proposed method. Because, by comparing the algorithm proposed above with algorithms using t-norms and t-conorms, we observe that the received interval-valued fuzzy set x_i in the new method they have a smaller width intervals, that is, it represent a lower degree of uncertainty and, as a result, gives you the possibility of better precision in the application.

5. **Conclusion**

We present certain aspect of decision making problem based on preference relations built by IV aggregations and reciprocity property built by negation functions. We propose new idea of reciprocity property and new preference structure. In future we would like to study more properties and classification of these reciprocity and preference structure. Moreover, we would like to use in preference structure IV negation defined with respect to linear order defined in [4] or [3].

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References

[1] J. Aczél, *Lectures on functional equations and their applications*, Academic press, New York, London, 1966.

[2] C. Alsina, *On a family of connectives for fuzzy sets*, Fuzzy Sets and Systems, **16**(1985), 231-235.

[3] M.J. Asiain, H. Bustince, B. Bedregal, Z. Takáć, M. Baczyński, D. Paternain, G. Dimuro, *About the Use of Admissible Order for Defining Implication Operators*, Cham: Springer International Publishing(2016), 353-362.

[4] M.J. Asiain, H. Bustince, R. Mesiar, A. Kolesarova, Z. Takac, *Negations with respect to admissible orders in the interval-valued fuzzy set theory*, IEEE Transactions on Fuzzy Systems, **PP(99)**(2017). DOI: 10.1109/TFUZZ.2017.2686372

[5] E. Barrenechea, J. Fernandez, M. Pagola, F. Chiclana, H. Bustince, *Construction of Interval-Valued Fuzzy Preference Relations from Ignorance Functions and Fuzzy Preference Relations: Application to Decision Making,* Knowledge-Based Systems, **58**(2014), 33–44.

[6] B. Bedregal, *On interval fuzzy negations*, Fuzzy Sets and Systems, **161(17)**(2010) 2290–2313.

[7] U. Bentkowska, B. Pękala, H. Bustince, J. Fernandez, A. Jurio, K. Balicki, *N-reciprocity property for interval-valued fuzzy relations with an application to group decision making problems in social networks*, International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems, submitted

[8] H. Bustince, M. Pagola, R. Mesiar, E. Hüllermeier, F. Herrera, *Grouping, Overlap and Generalized Bientropic Functions for Fuzzy Modeling of Pairwise Comparisons*, IEEE Transactions on fuzzy systems, **20(3)**(2012), 405–415.

[9] G. Birkhoff, *Lattice Theory.* AMS Coll. Publ. 25, Providence, 1967.

[10] T. Calvo, *On mixed De Morgan triplets*, Fuzzy Sets and Systems, **50**(1992), 47-50.

[11] T. Calvo, B. De Baets, J. Fodor, *The functional equations of Frank and Alsina for uninorms and nullnorms*, Fuzzy Sets and Systems, **120**(2001), 385-394.

[12] T. Calvo, A. Kolesárová, M. Komorniková, R. Mesiar, *Aggregation operators: properties, classes and construction methods,* in: Calvo, T., et al. (eds.) Aggregation Operators New trends and Applications, Physica-Verlag, Heidelberg(2002), 3–104.

[13] H. Chen, L. Zhou, *An approach to group decision making with interval fuzzy preference relations based on induced generalized continuous ordered weighted averaging operator*, Expert Systems with Applications, **38**(2011), 13432–13440.

[14] F. Chiclana, E. Herrera-Viedma, S. Alonso, R.A.M. Pereira, *Preferences and consistency issues in group decision making*, in Bustince, H., at al. (eds.) Fuzzy Sets and Their Extensions: Representation, Aggregation and Models, Springer-Verlag, Berlin, (2008), 219–237.

[15] F. Chiclana, F. Herrera, E. Herrera-Viedma, *Integrating three representation models in fuzzy multipurpose decision making based on fuzzy preference relations*, Fuzzy Sets and Systems, **97(1)**(1998), 33-48.

[16] B. De Baets, B. Van de Walle, E. Kerre, *Fuzzy preference structures without incomparability*, Fuzzy Sets and Systems, **76(3)**(1995), 333-348.

[17] G. Deschrijver, C. Cornelis, E.E. Kerre, *On the Representation of Intuitonistic Fuzzy t-Norms and t-Conorms*, IEEE Trans. Fuzzy Syst., **12**(2004), 45–61.

[18] G. Deschrijver, E. Kerre, *Aggregation Operators in Interval-valued Fuzzy and Atanassovs Intuitionistic Fuzzy Set Theory*, in: Bustince, H., et al. (eds.) Fuzzy Sets and their Extensions: Representation, Aggregation and Models, Springer, (2008), 183–203.

[19] G. Deschrijver, *Quasi-arithmetic means and OWA functions in interval-valued and Atanassovs intuitionistic fuzzy set theory*, in: Galichet, S., et al. (eds.) Proceedings of EUSFLAT-LFA 2011, 18-22.07.2011, Aix-les-Bains, France, (2011), 506–513.

[20] P. Drygaś, B. Pękala, *Properties of Decomposable Operations on some Extension of the Fuzzy Set Theory*, in: Atanassov, K.T., Hryniewicz, O., Kacprzyk, J. et al. (eds.) Advances in Fuzzy Sets, Intuitionistic Fuzzy Sets, Generalized Nets and Related Topics, EXIT, Warsaw, (2008), 105–118.

[21] J. Fodor, M. Roubens, *Fuzzy Preference Modelling and Multicriteria Decision Support in Theory and Decision Library*, Kluwer Academic Publishers, 1994.

[22] S. Freson, B. De Baets, H. De Meyer, *Closing reciprocal relations w.r.t. stochastic transitivity*, Fuzzy Sets and Systems, **241**(2014), 2–26.

[23] M.B. Gorzałczany, *A Method of Inference in Approximate Reasoning Based on Interval-Valued Fuzzy Sets*, Fuzzy Sets Syst., **21(1)**(1987), 1–17.

[24] E.P. Klement, R. Mesiar, E. Pap, *Triangular Norms*, Kluwer Acad. Publ., Dordrecht, 2000.

[25] M. Komorníková, R. Mesiar, *Aggregation functions on bounded partially ordered sets and their classification*, Fuzzy Sets and Systems, **175**(2011), 48–56.

[26] F. Liu, W.-G. Zhang, L.-H. Zhang, *A group decision making model based on a generalized ordered weighted geometric average operator with interval preference matrices*, Fuzzy Sets and Systems, **246**(2014), 1–18.

[27] B. Llamazares, B. De Baets, *Fuzzy Strict Preference Relations Compatible with Fuzzy Orderings*, International Uncertainty Fuzziness and Knowledge-Based Systems, **18(1)**(2010), 13–24.

[28] S.V. Ovchinnikov, M. Roubens, *On Strict Preference Relations*, Fuzzy Sets and Systems, **43**(1991), 319-326.

[29] D. Paternain, A. Jurio, E. Barrenechea, H. Bustince, B. Bedregal, E. Szmidt, *An alternative to fuzzy methods in decision-making problems*, Expert Systems with Applications, **39(9)**(2012), 7729–7735.

[30] B. Pękala, U. Bentkowska, *Generalized reciprocity property for interval-valued fuzzy setting in some aspect of social network*, IWIFSGN'2016 Fifteenth International Workshop on Intuitionistic Fuzzy Sets and Generalized Nets, Warsaw, Poland, October 12–14, Springer, 2016.

[31] A. Pradera, G. Beliakov, H. Bustince, B. De Baets, *A review of the relationships between implication, negation and aggregation functions from the point of view of material implication*, Inf. Sci., **329**(2016), 357–380.

[32] M. Roubens, P. Vincke, *Preference Modelling*, Springer-Verlag, Berlin, 1985.

[33] R. Sambuc, *Fonctions ϕ-floues: Application á l'aide au diagnostic en pathologie thyroidienne*, Ph.D. Thesis, Université de Marseille, France, 1975 (in French).

[34] J. Sanz, A. Fernandez, H. Bustince, F. Herrera, *Improving the performance of fuzzy rule-based classification systems with interval-valued fuzzy sets and genetic amplitude tuning*, Information Sciences, **180(19)**(2010), 3674–3685.

[35] J. Sanz, A. Fernandez, H. Bustince, F. Herrera, *A genetic tuning to improve the performance of fuzzy rule-based classification systems with intervalvalued fuzzy sets: degree of ignorance and lateral position*, International Journal of Approximate Reasoning, **52(6)**(2011), 751–766.

[36] I.B. Türksen, T. Bilgiç, *Interval-valued strict preference relations with Zadeh triples*, Fuzzy Sets and Systems, **78**(1996), 183–195.

[37] Z. Xu, *On compatibility of interval fuzzy preference relations*, Fuzzy Optimization and Decision Making, **3**(2004), 217–225.

[38] G.-L. Xu, F. Liu, *An approach to group decision making based on interval multiplicative and fuzzy preference relations by using projection*, Applied Mathematical Modelling, **37**(2013), 3929–3943.

[39] L.A. Zadeh, *Fuzzy sets*, Information and Control, **8**(1965), 338–353.

[40] L.A. Zadeh, *The Concept of a Linguistic Variable and its Application to Approximate Reasoning-I*, Information Sciences, **8**(1975), 199–249.

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