BASES AND CIRCUITS OF FUZZIFYING MATROIDS

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Abstract. In this paper, as an application of fuzzy matroids, the fuzzifying greedy algorithm is proposed and an achievable example is given. Basis axioms and circuit axioms of fuzzifying matroids, which are the semantic extension for the basis axioms and circuit axioms of crisp matroids respectively, are presented. It is proved that a fuzzifying matroid is equivalent to a mapping which satisfies the basis axioms or circuit axioms.

1. Introduction

Matroids were first introduced by Whitney in 1935. Linear independence in vector spaces and the sets of edges without any circuit in a graph are both matroids. Matroids theories are widely used in combinatorial mathematics, information security and so on.

In 1988, Goetschel and Voxman introduced the notion of fuzzy matroids [3](called G-V fuzzy matroid). Since then, several studies established the corresponding axioms of fuzzy circuits, fuzzy bases, fuzzy rank functions and fuzzy closure operators [4, 5, 6, 7, 10, 11, 12, 14, 15, 16]. Unfortunately, a G-V fuzzy matroid and its fuzzy rank function are not one-to-one corresponding.

From a completely different point of view, Shi introduced the notion of M-fuzzifying matroids [17] and further he introduced the notion of (L, M)-fuzzy matroids in [18], where L, M are completely distributive lattices. M-fuzzifying rank functions were also defined in [17]. It is proved that there is a one-to-one corresponding relation between M-fuzzifying matroids and M-fuzzifying rank functions. From then on, M-fuzzifying matroids are characterized by M-fuzzifying closure operators, M-fuzzifying base-maps, M-fuzzifying circuit-maps, M-fuzzifying dependent sets, M-fuzzifying nullities, M-fuzzifying α-flats, M-fuzzifying derived operators and difference derived operators, M-fuzzifying P-closure operators, M-fuzzifying submodular functions [13, 19, 20, 21, 22, 24, 25, 26] and so on. The minors and free product of M-fuzzifying matroids are discussed in [8, 9] respectively.

When $M = [0, 1]$, $[0, 1]$-fuzzifying matroids is named as fuzzifying matroids for short. Although the fuzzifying base-maps and fuzzifying circuit-maps of fuzzifying matroids are presented in [25, 26], they are not the semantic extension for the basis axioms and circuit axioms of crisp matroids.

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In this paper, our aim is to establish the basis axioms and circuit axioms of fuzzifying matroids, which are the semantic extension for the basis axioms and circuit axioms of crisp matroids, respectively. We shall prove that there is a one-to-one corresponding relation between fuzzifying matroids and mappings which satisfy the basis axioms or circuit axioms.

2. **Preliminaries**

Let E be a set. A fuzzy subset of E is a mapping $A : E \to [0,1]$, and the family of all fuzzy subsets of E is denoted by $[0,1]^E$. For arbitrary $A \in [0,1]^E$ and $a \in [0,1]$, we denote $A_{[a]} = \{x \in E : A(x) \geq a\}$ and $A_{(a)} = \{x \in E : A(x) > a\}$.

Throughout this paper, let E be a non-empty finite set and $(\cdot)'$ be an order-reversing involution mapping from $[0,1]$ to $[0,1]$.

2.1. **Fuzzy Natural Number.**

The terminology and basic notions in this part are from [17].

Let $\mathbb{N}$ be the set of all natural numbers. A fuzzy natural number is an antitone map $\lambda : \mathbb{N} \to [0,1]$ satisfying $\lambda(0) = 1$ and $\bigwedge_{n\in\mathbb{N}} \lambda(n) = 0$. The set of all fuzzy natural numbers is denoted by $\mathbb{N}([0,1])$. For arbitrary $m \in \mathbb{N}$, we define $\underline{m} \in \mathbb{N}([0,1])$ such that $\underline{m}(t) = 1$ when $t \leq m$ and $\underline{m}(t) = 0$ when $t \geq m+1$[17]. In the sequel, we shall not distinguish m from $\underline{m}$.

For arbitrary $\lambda, \mu \in \mathbb{N}([0,1])$, we define $\lambda+\mu$ such that $(\lambda+\mu)(n) = \bigvee_{s+t=n}(\lambda(s)\wedge \mu(t))$ for arbitrary $n \in \mathbb{N}$. Obviously, $\underline{0} + \lambda = \lambda$. More properties of fuzzy natural numbers can be found in [17].

2.2. **Fuzzifying Matroid.**

Let E be a non-empty finite set and $\mathcal{A} \subseteq 2^E$. The following symbols are useful in the sequel.

$$Min(\mathcal{A}) = \{A \in \mathcal{A} : A \supseteq B \in \mathcal{A} \text{ implies } A = B\}.$$

$$Max(\mathcal{A}) = \{A \in \mathcal{A} : A \subseteq B \in \mathcal{A} \text{ implies } A = B\}.$$

$$Upp(\mathcal{A}) = \{X \in 2^E : \text{ there exists an } A \in \mathcal{A} \text{ such that } A \subseteq X\}.$$

$$Low(\mathcal{A}) = \{X \in 2^E : \text{ there exists an } A \in \mathcal{A} \text{ such that } X \subseteq A\}.$$

$$Opp(\mathcal{A}) = \{X \in 2^E : X \notin \mathcal{A}\}.$$

The definition of fuzzifying matroids was first given by Shi as follows.

Definition 2.1. [17, 18] Let $\mathcal{I} : 2^E \to [0,1]$ be a mapping. The pair $(E, \mathcal{I})$ is called a $[0,1]$-fuzzifying matroid if $\mathcal{I}$ satisfies the following conditions.

(FI1) $\mathcal{I}(\emptyset) = 1$.

(FI2) For arbitrary $A, B \in 2^E, A \supseteq B$ implies $\mathcal{I}(A) \leq \mathcal{I}(B)$.

(FI3) For arbitrary $A, B \in 2^E$ with $|A| < |B|$, $\bigvee\limits_{e\in B-A} \mathcal{I}(A \cup \{e\}) \geq \mathcal{I}(A) \wedge \mathcal{I}(B)$.

Clearly, $\bigvee\limits_{e\in B-A} \mathcal{I}(A \cup \{e\}) \geq \mathcal{I}(A) \wedge \mathcal{I}(B)$ in (FI3) is equivalent to that there exists an $e \in B - A$ such that $\mathcal{I}(A \cup \{e\}) \geq \mathcal{I}(A) \wedge \mathcal{I}(B)$ since E is a finite set. A $[0,1]$-fuzzifying matroid is also called a fuzzifying matroid for short.

Lemma 2.2. [17] *Let $\mathcal{I} : 2^E \to [0,1]$ be a mapping. Then the following conditions are equivalent.*

(i) *$(E,\mathcal{I})$ is a fuzzifying matroid.*
(ii) *For each $a \in (0,1], (E,\mathcal{I}_{[a]})$ is a matroid.*
(iii) *For each $b \in [0,1), (E,\mathcal{I}_{(b)})$ is a matroid.*

The fuzzifying base-maps and fuzzifying circuit-maps of fuzzifying matroids are established in [25, 26] as follows.

Definition 2.3. [26] Let $(E,\mathcal{I})$ be a fuzzifying matroid. A mapping $\mathcal{B} : 2^E \to [0,1]$ is called a quasi-base-map of $(E,\mathcal{I})$ if for all $A \subseteq E$, $\mathcal{I}(A) = \bigvee_{A\subseteq B} \mathcal{B}(B)$. The minimal quasi-base-map is called the base-map.

Theorem 2.4. [25, 26] *Let $(E,\mathcal{I})$ be a fuzzifying matroid. Define a mapping $\mathcal{B}_{\mathcal{I}} : 2^E \to [0,1]$ by*

$$\mathcal{B}_{\mathcal{I}}(B) = \begin{cases} \mathcal{I}(B), & \bigvee_{B\subset A} \mathcal{I}(A) \not\geq \mathcal{I}(B) \\ 0, & otherwise. \end{cases}$$

Then $\mathcal{B}_{\mathcal{I}}$ is the base-map of $(E,\mathcal{I})$.

Lemma 2.5. [26] *Let $\mathcal{B} : 2^E \longrightarrow [0,1]$ be a mapping satisfying the following conditions.*

(i) *For arbitrary $a \in (0,1]$, $Max\mathcal{B}_{[a]}$ is the family of bases of a crisp matroid on E.*
(ii) *For any $B \in 2^E$, if $\mathcal{B}(B) = a \neq 0$, then $B \in Max\mathcal{B}_{[a]}$.*

Then there is unique fuzzifying matroid with $\mathcal{B}$ as its base-map.

Definition 2.6. [26] Let $(E,\mathcal{I})$ be a fuzzifying matroid. A mapping $\mathcal{C} : 2^E \to [0,1]$ is called a quasi-circuit-map of $(E,\mathcal{I})$ if for all $A \subseteq E$, $\mathcal{I}(A) = (\bigvee_{C\subseteq A} \mathcal{C}(C))'$. The minimal quasi-circuit-map is called the circuit-map.

Theorem 2.7. [25, 26] *Let $(E,\mathcal{I})$ be a fuzzifying matroid. Define a mapping $\mathcal{C}_{\mathcal{I}} : 2^E \to [0,1]$ by:*

$$\mathcal{C}_{\mathcal{I}}(C) = \begin{cases} (\mathcal{I}(C))', & \bigvee_{B\subset C} (\mathcal{I}(B))' \not\geq (\mathcal{I}(C))' \\ 0, & otherwise. \end{cases}$$

Then $\mathcal{C}_{\mathcal{I}}$ is the circuit-map of $(E,\mathcal{I})$.

Lemma 2.8. [26] *Let $\mathcal{C} : 2^E \longrightarrow [0,1]$ be a mapping satisfying the following conditions.*

(i) *For arbitrary $a \in (0,1]$, $Min\mathcal{C}_{[a]}$ is the family of circuits of a crisp matroid on E.*
(ii) *For any $C \in 2^E$, if $\mathcal{C}(C) = a \neq 0$, then $C \in Min\mathcal{C}_{[a]}$.*

Then there is unique fuzzifying matroid with $\mathcal{C}$ as its circuit-map.

In the field of combinatorial optimization, there are many problems, such as knapsack problem, can be described as follows.

Let E be a non-empty finite set with a weight function $\omega : E \to \mathbb{R}^+$ and $\mathcal{I} \subseteq 2^E$ be a family of subsets of E such that $\mathcal{I} \neq \emptyset$ is a down set with the order $\subseteq$. For any $A \in 2^E$, $\omega(A)$ can be defined in a natural way with $\omega(A) = \Sigma_{e\in A}\omega(e)$. It needs to find a subset $A_0 \in \mathcal{I}$ with $\omega(A_0)$ is the maximum of $\{\omega(A) : A \in \mathcal{I}\}$.

For convenience, this problem that is denoted by $(E, \mathcal{I}, \omega)$ in this paper. As we all know, $(E, \mathcal{I}, \omega)$ is a NP-Hard problem and there is no algorithm with polynomial complexity to solve it. The following greedy algorithm is an effective way to get an approximate optimal solution of $(E, \mathcal{I}, \omega)$.

Greedy algorithm

Step 1. Let $A = \emptyset$.
Step 2. Choose a member $e \in E$ such that $\omega(e)$ is the maximum of $\{\omega(e) : e \notin A$ and $A \cup \{e\} \in \mathcal{I}\}$. If no such e exists, go to Step 4, otherwise, go to Step 3.
Step 3. Denote $A = A \cup \{e\}$ and go to Step 2.
Step 4. Return A and stop.

However, matroids have the following property.

Theorem 2.9. [2, 23] *Let E be a non-empty finite set and $\mathcal{I} \subseteq 2^E$ such that $\mathcal{I} \neq \emptyset$ is a down set. Then the greedy algorithm can get the optimal solution of the problem $(E, \mathcal{I}, \omega)$ for any weight function ω if and only if $(E, \mathcal{I})$ is a matroid.*

3. Applications of Fuzzifying Matroids

In this section, we introduce the applications of fuzzifying matroids and obtain the fuzzifying greedy algorithm.

An assignment function from $(0, 1]$ to $\mathbb{R}$ is called a gradual number in $\mathbb{R}$ [1]. In addition, a gradual number is called non-negative if its assignment function is a mapping from $(0, 1]$ to $\mathbb{R}^+$ [1]. Let r_1, r_2 be two gradual numbers. Then the addition of r_1 and r_2 is defined as follows [1].

$$\forall t \in (0, 1], (r_1 + r_2)(t) = r_1(t) + r_2(t).$$

Let E be a non-empty finite set. A function ω from E to all non-negative gradual numbers, that is, for any $e \in E$, $\omega(e)$ is a gradual number, is called a non-negative weight function. Also for any $A \in 2^E$, $\omega(A)$ is defined by $\omega(A) = \Sigma_{e\in A}\omega(e)$. Let $\mathcal{I} : 2^E \to [0, 1]$ be a mapping such that $\mathcal{I}(\emptyset) = 1$ and $\mathcal{I}(A) \geq \mathcal{I}(B)$ for any $A \subseteq B$ and $t \in (0, 1]$ be any fixed value. The following algorithm is called fuzzifying greedy algorithm.

Fuzzifying greedy algorithm

Step 1. Let $A = \emptyset$.
Step 2. Choose a member $e \in E$ such that $\omega(e)(t)$ is the maximum of $\{\omega(e)(t) : e \notin A$ and $\mathcal{I}(A \cup \{e\}) \geq t\}$. If no such e exists, go to Step 4, otherwise, go to Step 3.
Step 3. Denote $A = A \cup \{e\}$ and go to Step 2.
Step 4. Return A and stop.

Theorem 3.1. *Let E be a non-empty finite set and $\mathcal{I} : 2^E \to [0,1]$ be a mapping such that $\mathcal{I}(\emptyset) = 1$ and $\mathcal{I}(A) \geq \mathcal{I}(B)$ for any $A \subseteq B$. Then the fuzzifying greedy algorithm can get a subset $A \subseteq E$ such that $\mathcal{I}(A) \geq t$ and $\omega(A)(t) = max\{\omega(B)(t) : \mathcal{I}(B) \geq t\}$ for any $t \in (0,1]$ and any non-negative weight function ω if and only if $(E, \mathcal{I})$ is a fuzzifying matroid.*

Proof. Let $(E, \mathcal{I})$ be a fuzzifying matroid, ω be a non-negative weight function and $t \in (0,1]$. We define $\overline{\omega}(e) = \omega(e)(t)$ for any $e \in E$. Then $\overline{\omega}$ is a weight function and $\overline{\omega}(A) = \omega(A)(t)$ for any $A \in 2^E$. Thus $\omega(e)(t)$ is the maximum of $\{\omega(e)(t) : e \notin A$ and $\mathcal{I}(A \cup \{e\}) \geq t\}$ if and only if $\overline{\omega}(e)$ is the maximum of $\{\overline{\omega}(e) : e \notin A$ and $A \cup \{e\} \in \mathcal{I}_{[t]}\}$. So the subset A returned by fuzzifying greedy algorithm is equivalent to the subset returned by greedy algorithm of $(E, \mathcal{I}_{[t]}, \overline{\omega})$. Since $(E, \mathcal{I}_{[t]})$ is a matroid by Lemma 2.2, then we have $\mathcal{I}(A) \geq t$ and

$$\omega(A)(t) = \overline{\omega}(A) = max\{\overline{\omega}(B) : B \in \mathcal{I}_{[t]}\} = max\{\omega(B)(t) : \mathcal{I}(B) \geq t\}.$$

Conversely, it suffices to prove that $(E, \mathcal{I}_{[t]})$ is a matroid for any $t \in (0,1]$. Let $\omega : E \to \mathbb{R}^+$ be a weight function. We define $\overline{\omega}(e)(a) = \omega(e)$ for any $a \in (0,1]$. Then $\overline{\omega}$ is a non-negative weight function and $\overline{\omega}(A)(t) = \omega(A)$ for any $A \in 2^E$. Then it is easy to check that the subset A returned by fuzzifying greedy algorithm is equivalent to the subset returned by greedy algorithm of $(E, \mathcal{I}_{[t]}, \omega)$. Thus

$$\omega(A) = \overline{\omega}(A)(t) = max\{\overline{\omega}(B)(t) : \mathcal{I}(B) \geq t\} = max\{\omega(B) : B \in \mathcal{I}_{[t]}\}.$$

So by Theorem 2.9, we get that $(E, \mathcal{I}_{[t]})$ is a matroid. □

Example 3.2. Let $E = \{x, y, z\}$ and for any $t \in [0,1]$, $\omega(x)(t) = min\{2, \frac{1}{t}\}$, $\omega(y)(t) = 1.5$ and

$$\omega(z)(t) = \begin{cases} 2.5, t \in [0, 0.5]; \\ 0.5, t \in (0.5, 1]. \end{cases}$$

Let $\mathcal{I} : 2^E \to [0,1]$ be a mapping, where

$$\mathcal{I}(\emptyset) = 1;\ \mathcal{I}(\{x\}) = 0.7;\ \mathcal{I}(\{y\}) = 1;\ \mathcal{I}(\{z\}) = 0.7;$$
$$\mathcal{I}(\{x, y\}) = 0.7;\ \mathcal{I}(\{y, z\}) = 0.5;\ \mathcal{I}(\{x, z\}) = 0.3;\ \mathcal{I}(E) = 0.$$

It is easy to see that $(E, \mathcal{I})$ is a fuzzifying matroid. By fuzzifying greedy algorithm, we get that the optimal solution of this problem is

$$\begin{cases} \{x, y, z\}, t = 0; \\ \{x, z\}, t \in (0, 0.3]; \\ \{y, z\}, t \in (0.3, 0.5]; \\ \{x, y\}, t \in (0.5, 0.7]; \\ \{y\}, t \in (0.7, 1]. \end{cases}$$

Also, the optimal value if:

$$\omega^*(t) = \begin{cases} 6, t = 0; \\ 4.5, t \in (0, 0.3]; \\ 4, t \in (0.3, 0.5]; \\ 1.5 + \frac{1}{t}, t \in (0.5, 0.7]; \\ 1.5, t \in (0.7, 1]. \end{cases}$$

4. **Basis Axioms of Fuzzifying Matroids**

In this section, we give the basis axioms of fuzzifying matroids. Recall that $\mathcal{B} \subseteq 2^E$ is the family of bases of a matroid $(E, \mathcal{I})$ if and only if $\mathcal{B}$ satisfies the following conditions.

(B1) $\mathcal{B} \neq \emptyset$.

(B2) If $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 - B_2$, then there exists $y \in B_2 - B_1$ such that $(B_1 - \{x\}) \cup \{y\} \in \mathcal{B}$.

In particular, if $(E, \mathcal{I})$ is a matroid, then $\mathcal{B}_{\mathcal{I}} = Max(\mathcal{I})$ is the family of bases and satisfies (B1) and (B2). Thus (B1) and (B2) are called the basis axioms of $(E, \mathcal{I})$. Now we generalize the basis axioms to fuzzifying matroids.

Lemma 4.1. *Let $(E, \mathcal{I})$ be a fuzzifying matroid, and $\mathcal{B}_{\mathcal{I}}$ be the fuzzifying base-map.*

(i) *For $B \in 2^E$, $\mathcal{B}_{\mathcal{I}}(B) \neq 0$ iff $\mathcal{I}(A) < \mathcal{I}(B)$ for arbitrary $A \supset B$.*

(ii) *For $B \in 2^E$, $\mathcal{B}_{\mathcal{I}}(B) \neq 0$ iff $\mathcal{I}(A) < \mathcal{I}(B)$ for arbitrary $|A| > |B|$.*

(iii) *For arbitrary $B \in 2^E$, there exists a $B_0 \supseteq B$ such that $\mathcal{B}_{\mathcal{I}}(B_0) = \mathcal{I}(B)$.*

Proof. (i) Let $\mathcal{B}_{\mathcal{I}}(B) \neq 0$ and $B \subset A$. By the definition of $\mathcal{B}_{\mathcal{I}}$, we have $\mathcal{B}_{\mathcal{I}}(B) = \mathcal{I}(B)$. Then $\mathcal{I}(A) < \mathcal{I}(B)$ follows from $\bigvee_{B \subset A} \mathcal{I}(A) \ngeq \mathcal{I}(B)$.

Conversely, if $\mathcal{I}(A) < \mathcal{I}(B)$ for an arbitrary A with $A \supset B$, then $\bigvee_{B \subset A} \mathcal{I}(A) \ngeq \mathcal{I}(B)$ and $\mathcal{I}(B) > 0$. So $\mathcal{B}_{\mathcal{I}}(B) = \mathcal{I}(B) > 0$.

(ii) Let $\mathcal{B}_{\mathcal{I}}(B) \neq 0$ and $|B| < |A|$. Then $\mathcal{B}_{\mathcal{I}}(B) = \mathcal{I}(B)$. If $B \subset A$, then we get $\mathcal{I}(A) < \mathcal{I}(B)$ by (i). So we suppose that $B \not\subset A$.

By (FI3), we know that there exists an element $e \in A - B$ such that $\mathcal{I}(B \cup \{e\}) \geq \mathcal{I}(B) \wedge \mathcal{I}(A)$. If $\mathcal{I}(A) \geq \mathcal{I}(B)$, then $\mathcal{I}(B \cup \{e\}) = \mathcal{I}(B)$. This implies $\bigvee_{B \subset A} \mathcal{I}(A) \geq \mathcal{I}(B)$. Hence $\mathcal{B}_{\mathcal{I}}(B) = 0$, which is a contradiction. Therefore $\mathcal{I}(A) < \mathcal{I}(B)$.

the Converse, is trivial by (i).

(iii) For arbitrary $B \in 2^E$, it is clear that $\{A : B \subseteq A \text{ and } \mathcal{I}(B) = \mathcal{I}(A)\} \neq \emptyset$. Let $B_0 \in \{A : B \subseteq A \text{ and } \mathcal{I}(B) = \mathcal{I}(A)\}$ such that $|B_0|$ is maximum. By the construction of B_0, we get $\mathcal{I}(A) < \mathcal{I}(B) = \mathcal{I}(B_0)$ for arbitrary $A \supset B_0$, which means $\bigvee_{B_0 \subset A} \mathcal{I}(A) < \mathcal{I}(B_0)$. Thus $\mathcal{B}_{\mathcal{I}}(B_0) = \mathcal{I}(B_0) = \mathcal{I}(B)$. □

Remark 4.2. Lemma 4.1 is valid just for fuzzifying matroids. When $[0, 1]$ is replaced by a completely distributive lattice, we didn't get any ideal result. This is why we only study fuzzifying matroids instead of M-fuzzifying matroids.

Theorem 4.3. *Let $(E, \mathcal{I})$ be a fuzzifying matroid, and $\mathcal{B}_{\mathcal{I}}$ be the fuzzifying base-map. Then $\mathcal{B}_{\mathcal{I}}$ satisfies the following conditions.*

(FB1) *There exists $B \in 2^E$ such that $\mathcal{B}_{\mathcal{I}}(B) = 1$.*

(FB2) *If $\mathcal{B}_{\mathcal{I}}(B_1) \neq 0$ and $|B_1| < |B_2|$, then $\mathcal{B}_{\mathcal{I}}(B_1) > \mathcal{B}_{\mathcal{I}}(B_2)$ and there exists $B_3 \in 2^E$ such that $B_1 \subset B_3$ and $\mathcal{B}_{\mathcal{I}}(B_3) \geq \mathcal{B}_{\mathcal{I}}(B_2)$.*

(FB3) *$B_1, B_2 \in 2^E$. If $|B_1| = |B_2|$ and $B_1 \neq B_2$, then*

$$\bigwedge_{x \in B_1 - B_2} \left(\bigvee_{y \in B_2 - B_1} (\mathcal{B}_{\mathcal{I}}((B_1 - \{x\}) \cup \{y\})) \right) \geq \mathcal{B}_{\mathcal{I}}(B_1) \wedge \mathcal{B}_{\mathcal{I}}(B_2).$$

Proof. (FB1) By (FI1), we have $\emptyset \in \{A \in 2^E : \mathcal{I}(A) = 1\} \neq \emptyset$. Then let $B \in \{A \in 2^E : \mathcal{I}(A) = 1\}$ such that $|B|$ is maximum. It is easy to verify that $\mathcal{B}_\mathcal{I}(B) = 1$.

(FB2) Let $\mathcal{B}_\mathcal{I}(B_1) \neq 0$ and $|B_1| < |B_2|$. Then by Lemma 4.1(ii), it is clear that $\mathcal{B}_\mathcal{I}(B_1) = \mathcal{I}(B_1) > \mathcal{I}(B_2) \geq \mathcal{B}_\mathcal{I}(B_2)$. Since $|B_1| < |B_2|$, there exists $e \in B_2 - B_1$ with $\mathcal{I}(B_1 \cup \{e\}) \geq \mathcal{I}(B_1) \bigwedge \mathcal{I}(B_2) = \mathcal{I}(B_2)$. Thus there exists $B_3 \in 2^E$ such that $B_1 \subset (B_1 \cup \{e\}) \subseteq B_3$ and $\mathcal{B}_\mathcal{I}(B_3) = \mathcal{I}(B_1 \cup \{e\})$ by Lemma 4.1(iii), which implies $\mathcal{B}_\mathcal{I}(B_3) \geq \mathcal{B}_\mathcal{I}(B_2)$.

(FB3) If $\mathcal{B}_\mathcal{I}(B_1) = 0$ or $\mathcal{B}_\mathcal{I}(B_2) = 0$, then it is easy to verify this inequality. Now we suppose that $\mathcal{B}_\mathcal{I}(B_i) \neq 0$ for $i = 1, 2$ and let $x \in B_1 - B_2$.

We denote $a = \mathcal{I}(B_1) \wedge \mathcal{I}(B_2) = \mathcal{B}_\mathcal{I}(B_1) \wedge \mathcal{B}_\mathcal{I}(B_2)$ and consider $(E, \mathcal{I}_{[a]})$. Clearly, $(E, \mathcal{I}_{[a]})$ is a crisp matroid and let $\mathcal{B}_{\mathcal{I}_{[a]}}$ be the family of bases for $(E, \mathcal{I}_{[a]})$. By Lemma 4.1(ii), it is easy to verify that $B_1,\ B_2 \in \mathcal{B}_{\mathcal{I}_{[a]}}$ since $|B_1| = |B_2|$. Hence there exists $y \in B_2 - B_1$ such that $(B_1 - \{x\}) \cup \{y\} \in \mathcal{B}_{\mathcal{I}_{[a]}}$, which implies that $\mathcal{I}((B_1 - \{x\}) \cup \{y\}) \geq a$ and $\mathcal{I}(A) < a$ for an arbitrary $A \supset (B_1 - \{x\}) \cup \{y\}$. Therefore $\mathcal{B}_\mathcal{I}((B_1 - \{x\}) \cup \{y\}) = \mathcal{I}((B_1 - \{x\}) \cup \{y\}) \geq a = \mathcal{B}_\mathcal{I}(B_1) \wedge \mathcal{B}_\mathcal{I}(B_2)$. Hence

$$\bigwedge_{x \in B_1 - B_2} \left(\bigvee_{y \in B_2 - B_1} (\mathcal{B}_\mathcal{I}((B_1 - \{x\}) \cup \{y\})) \right) \geq \mathcal{B}_\mathcal{I}(B_1) \wedge \mathcal{B}_\mathcal{I}(B_2).$$

□

Remark 4.4. Let $(E, \mathcal{I})$ be a crisp matroid. Then (FB1) can be reduced to (B1). Since the bases of a crisp matroid have the same cardinality, we get that (FB2) is trivial. Finally, it is easy to verify that (FB3) is consist of (B2). Thus (FB1), (FB2) and (FB3) are the semantic extension for the basis axioms of crisp matroids.

Theorem 4.5. *Let $\mathcal{B} : 2^E \to [0,1]$ be a mapping which satisfies* (FB1), (FB2) *and* (FB3). *We define a mapping $\mathcal{I}_\mathcal{B} : 2^E \to [0,1]$ such that*

$$\mathcal{I}_\mathcal{B}(A) = \bigvee \{\mathcal{B}(B) : A \subseteq B\}.$$

Then $(E, \mathcal{I}_\mathcal{B})$ is a fuzzifying matroid.

Proof. According to Lemma 2.2, it suffices to prove that $(\mathcal{I}_\mathcal{B})_{[a]}$ is a crisp matroid for an arbitrary $a \in (0,1]$.

By the definition of $\mathcal{I}_\mathcal{B}$, it is easy to show that $A_1 \subseteq A_2$ implies $\mathcal{I}_\mathcal{B}(A_1) \geq \mathcal{I}_\mathcal{B}(A_2)$. Thus $(\mathcal{I}_\mathcal{B})_{[a]} = Low(Max((\mathcal{I}_\mathcal{B})_{[a]}))$. By the relation between crisp matroids and crisp basis axioms, we just need to prove that $Max((\mathcal{I}_\mathcal{B})_{[a]})$ satisfies (B1) and (B2).

It is easy to verify that $Max((\mathcal{I}_\mathcal{B})_{[a]}) = Max(\mathcal{B}_{[a]})$. Thus we need to prove $Max(\mathcal{B}_{[a]})$ satisfies (B1) and (B2).

(B1) By (FB1), it is clear that $Max(\mathcal{B}_{[a]}) \neq \emptyset$.

(B2) Let $B_1, B_2 \in Max(\mathcal{B}_{[a]})$ and $B_1 \neq B_2$. That is to say, $\mathcal{B}(B_i) \geq a$ and $\mathcal{B}(B) < a$ for an arbitrary $B \supset B_1$ or $B \supset B_2$.

Now we prove $|B_1| = |B_2|$. If not, without loss of generality, let $|B_1| < |B_2|$. By (FB2), there exists a B_3 such that $B_1 \subset B_3$ and $\mathcal{B}(B_3) \geq \mathcal{B}(B_2) \geq a$. Thus we get $B_3 \in \mathcal{B}_{[a]}$, which is a contradiction. So $|B_1| = |B_2|$.

Let $x \in B_1 - B_2$. Then according to (FB3), there exists a $y \in B_2 - B_1$ such that $\mathcal{B}((B_1 - \{x\}) \cup \{y\}) \geq \mathcal{B}(B_1) \wedge \mathcal{B}(B_2) \geq a$. Hence $(B_1 - \{x\}) \cup \{y\} \in \mathcal{B}_{[a]}$. By Lemma

4.1, it is easy to see that $(B_1-\{x\})\cup\{y\}\in Max(\mathcal{B}[a])$ since $|(B_1-\{x\})\cup\{y\}| = |B_1|$. □

Corollary 4.6. *Let $\mathcal{B}: 2^E\to[0,1]$ be a mapping which satisfies* (FB1), (FB2) *and* (FB3) *and $a\in(0,1]$. Then $(\mathcal{I}_{\mathcal{B}})_{[a]} = \mathcal{I}_{Max(\mathcal{B}_{[a]})}$*

Proof. According to the proof of Theorem 4.5, this proof is trivial. □

Theorem 4.7. *Let E be a non-empty finite set.*

(i) *If $(E,\mathcal{I})$ is a fuzzifying matroid, then $\mathcal{I}_{\mathcal{B}_{\mathcal{I}}} = \mathcal{I}$.*

(ii) *If $\mathcal{B}: 2^E\to[0,1]$ is a mapping which satisfies* (FB1), (FB2) *and* (FB3), *then $\mathcal{B}_{\mathcal{I}_{\mathcal{B}}} = \mathcal{B}$.*

Proof. (i) Since $\mathcal{B}_{\mathcal{I}}\leq\mathcal{I}$, we get $\mathcal{I}_{\mathcal{B}_{\mathcal{I}}}(A) = \bigvee\{\mathcal{B}_{\mathcal{I}}(B) : A\subseteq B\}\leq\bigvee\{\mathcal{I}(B) : A\subseteq B\}\leq\mathcal{I}(A)$.

Conversely, by Lemma 4.1(iii), there exists a $B\in 2^E$ such that $B\supseteq A$ and $\mathcal{B}_{\mathcal{I}}(B) = \mathcal{I}(A)$. Then we have $\mathcal{I}_{\mathcal{B}_{\mathcal{I}}}(A)\geq\mathcal{I}(A)$. Thus $\mathcal{I}_{\mathcal{B}_{\mathcal{I}}} = \mathcal{I}$.

(ii)Let $A\in 2^E$. If $A = E$, then $\mathcal{B}_{\mathcal{I}_{\mathcal{B}}}(E) = \mathcal{I}_{\mathcal{B}}(E) = \mathcal{B}(E)$. Now let $A\neq E$.

If $\mathcal{B}(A) = 0$, then by the definition of $\mathcal{I}_{\mathcal{B}}$, there exist some $B\in 2^E$ such that $A\subset B$ and $\mathcal{I}_{\mathcal{B}}(A) = \mathcal{B}(B)$. Let B be a maximum set with this property. Then $\mathcal{B}(B) = \mathcal{I}_{\mathcal{B}}(B) = \mathcal{I}_{\mathcal{B}}(A)$. By the definition of fuzzifying base-map, we have $\mathcal{B}_{\mathcal{I}_{\mathcal{B}}}(A) = 0$.

If $\mathcal{B}(A)\neq 0$, then by (FB2), we get that $\mathcal{I}_{\mathcal{B}}(A) = \mathcal{B}(A)$ and $\mathcal{I}_{\mathcal{B}}(B) < \mathcal{B}(A)$ for an arbitrary $B\supset A$. Hence, $\mathcal{B}_{\mathcal{I}_{\mathcal{B}}}(A) = \mathcal{I}_{\mathcal{B}}(A) = \mathcal{B}(A)$. □

5. Circuit Axioms of Fuzzifying Matroids

In this section, we give the circuit axioms of fuzzifying matroids. Recall that $\mathcal{C}\subseteq 2^E$ is the family of circuits of a crisp matroid $(E,\mathcal{I})$ if and only if $\mathcal{C}$ satisfies the following conditions.

(C1) $\emptyset\notin\mathcal{C}$.

(C2) If $C_1, C_2\in\mathcal{C}$, then $C_1\subseteq C_2$ implies $C_1 = C_2$.

(C3) If $C_1, C_2\in\mathcal{C}$ with $C_1\neq C_2$ and $e\in C_1\cap C_2$, then there exists a $C\in\mathcal{C}$ such that $C\subseteq C_1\cup C_2 - e$.

In particular, if $(E,\mathcal{I})$ is a matroid, then $\mathcal{C}_{\mathcal{I}} = Min(Opp(\mathcal{I}))$ is the family of circuits and satisfies (C1), (C2) and (C3). Thus (C1), (C2), (C3) are called the circuit axioms of $(E,\mathcal{I})$. Now we generalize the circuit axioms to fuzzifying matroids.

Lemma 5.1. *Let $(E,\mathcal{I})$ be a fuzzifying matroid, and $\mathcal{C}_{\mathcal{I}}$ be the fuzzifying circuit-map. Then $\mathcal{C}_{\mathcal{I}}$ satisfies the following conditions.*

(i) *For $C\in 2^E$, $\mathcal{C}_{\mathcal{I}}(C)\neq 0$ iff $\mathcal{I}(A) > \mathcal{I}(C)$ for arbitrary $A\subset C$.*

(ii) *For arbitrary $C\in 2^E$, there exists a $C_0\subseteq C$ such that $\mathcal{C}_{\mathcal{I}}(C_0) = (\mathcal{I}(C))'$.*

Proof. (i) Let $\mathcal{C}_{\mathcal{I}}(C)\neq 0$ and $A\subset C_1$. We have $\mathcal{C}_{\mathcal{I}}(C) = (\mathcal{I}(C)'$. Then $(\mathcal{I}(A))' < (\mathcal{I}(C))'$ follows from $\bigvee_{B\subset C}(\mathcal{I}(B))'\not\geq(\mathcal{I}(C))'$. Hence $\mathcal{I}(A) > \mathcal{I}(C)$.

Conversely, if $\mathcal{I}(A) > \mathcal{I}(C)$ for arbitrary $A\subset C$, then we obtain $\bigvee_{B\subset C}(\mathcal{I}(B))' < (\mathcal{I}(C))'$. Thus $\mathcal{C}_{\mathcal{I}}(C) = (\mathcal{I}(C))' > (\mathcal{I}(A))'\geq(1)' = 0$.

(ii) For arbitrary $C \in 2^E$, it is clear that $\{A : A \subseteq C \text{ and } \mathcal{I}(A) = \mathcal{I}(C)\} \neq \emptyset$. Let $C_0 \in \{A : A \subseteq C \text{ and } \mathcal{I}(A) = \mathcal{I}(C)\}$ such that $|C_0|$ is minimal. Since $(E, \mathcal{I})$ is a fuzzifying matroid, we have $\mathcal{I}(B) > \mathcal{I}(C) = \mathcal{I}(C_0)$ for an arbitrary $B \subset C_0$. This implies $(\mathcal{I}(B))' < (\mathcal{I}(C_0))'$ and $\bigvee_{B \subset C_0} (\mathcal{I}(B))' < (\mathcal{I}(C_0))'$. Therefore $\mathcal{C}_\mathcal{I}(C_0) = (\mathcal{I}(C_0))' = (\mathcal{I}(C))'$. □

Remark 5.2. Analogous to Lemma 4.1, Lemma 5.1 is just valid for fuzzifying matroids. When $[0, 1]$ is replaced by a completely distributive lattice, we didn't prove it.

Theorem 5.3. *Let $(E, \mathcal{I})$ be a fuzzifying matroid, and $\mathcal{C}_\mathcal{I}$ be the fuzzifying circuit-map. Then $\mathcal{C}_\mathcal{I}$ satisfies the following conditions.*
(FC1) $\mathcal{C}_\mathcal{I}(\emptyset) = 0$.
(FC2) *Let $C_1, C_2 \in 2^E$ such that $\mathcal{C}_\mathcal{I}(C_i) \neq 0$ for $i = 1, 2$. If $C_1 \subset C_2$, then*

$$\mathcal{C}_\mathcal{I}(C_1) < \mathcal{C}_\mathcal{I}(C_2).$$

(FC3) *Let $C_1, C_2 \in 2^E$ such that $C_1 \cap C_2 \neq \emptyset$ and $C_1 \cap C_2 \neq C_i$ for $i = 1, 2$. Then*

$$\bigwedge_{e \in C_1 \cap C_2} \left(\bigvee_{C \subseteq C_1 \cup C_2 - e} \mathcal{C}_\mathcal{I}(C) \right) \geq \mathcal{C}_\mathcal{I}(C_1) \wedge \mathcal{C}_\mathcal{I}(C_2).$$

Proof. (FC1) Since $\mathcal{I}(\emptyset) = 1$, we get $\mathcal{C}_\mathcal{I}(\emptyset) = 0$.

(FC2) Since $\mathcal{C}_\mathcal{I}(C_i) \neq 0$, we have $\mathcal{C}_\mathcal{I}(C_i) = (\mathcal{I}(C_i))'$ for $i = 1, 2$. By Lemma 5.1, we know $\mathcal{I}(C_1) > \mathcal{I}(C_2)$ since $C_1 \subset C_2$, which implies $(\mathcal{I}(C_1))' < (\mathcal{I}(C_2))'$. That is to say, $\mathcal{C}_\mathcal{I}(C_1) < \mathcal{C}_\mathcal{I}(C_2)$.

(FC3) If $\mathcal{C}_\mathcal{I}(C_1) = 0$ or $\mathcal{C}_\mathcal{I}(C_2) = 0$, then it is easy to verify this inequality. Now we suppose that $\mathcal{C}_\mathcal{I}(C_i) \neq 0$ for $i = 1, 2$ and let $e \in C_1 \cap C_2$.

We denote $b_i = \mathcal{C}_\mathcal{I}(C_i)$ and $a_i = (b_i)' = \mathcal{I}(C_i)$ for $i = 1, 2$. Without loss of generality, suppose that $b_1 \geq b_2 > 0$. Then $a_1 \leq a_2 < 1$. By Lemma 2.2, we get $(E, \mathcal{I}_{(a_2)})$ is a crisp matroid and $C_2 \notin \mathcal{I}_{(a_2)}$. Since $b_2 > 0$, we have $\mathcal{I}(A) > a_2$ for arbitrary $A \subset C_2$, which implies $A \in \mathcal{I}_{(a_2)}$. Thus C_2 is a crisp circuit of $(E, \mathcal{I}_{(a_2)})$.

According to the definition of the circuits of crisp matroids, there exists a $C_3 \in 2^E$ such that $e \in C_3 \subseteq C_1$ and C_3 is a circuit of $\mathcal{I}_{(a_2)}$. So $C_3 \neq C_2$ since $C_1 \cap C_2 \neq C_i$ for $i = 1, 2$. Thus there exists a $C_4 \in 2^E$ such that C_4 is a crisp circuit of $\mathcal{I}_{(a_2)}$ and $C_4 \subseteq C_3 \cup C_2 - e \subseteq C_1 \cup C_2 - e$, which implies $\mathcal{I}(C_4) \leq a_2$. By Lemma 5.1, there exists a $C \subseteq C_4 \subseteq C_1 \cup C_2 - e$ such that $\mathcal{C}_\mathcal{I}(C) = (\mathcal{I}(C_4))' \geq (a_2)' = b_2 = \mathcal{C}_\mathcal{I}(C_1) \wedge \mathcal{C}_\mathcal{I}(C_2)$. Hence

$$\bigwedge_{e \in C_1 \cap C_2} \left(\bigvee_{C \subseteq C_1 \cup C_2 - e} \mathcal{C}_\mathcal{I}(C) \right) \geq \mathcal{C}_\mathcal{I}(C_1) \wedge \mathcal{C}_\mathcal{I}(C_2).$$

□

Remark 5.4. Let $(E, \mathcal{I})$ be a crisp matroid. Then it is easy to see (FC1) is equivalent to (C1). (FC2) is equivalent to the following condition: if $C_1, C_2 \in 2^E$, $\mathcal{C}_\mathcal{I}(C_i) = 1$ for $i = 1, 2$, then $C_1 \not\subset C_2$ since $\mathcal{C}_\mathcal{I}(C_1) \nless \mathcal{C}_\mathcal{I}(C_2)$. This is consist of

(C2). Finally, it is easy to verify that (FC3) is equivalent to (C3). Thus (FC1), (FC2) and (FC3) are the semantic extension for the circuit axioms of crisp matroids.

Theorem 5.5. *Let $\mathcal{C}: 2^E \to [0,1]$ be a mapping which satisfies* (FC1), (FC2) *and* (FC3). *We define a mapping $\mathcal{I}_{\mathcal{C}}: 2^E \to [0,1]$ such that*

$$\mathcal{I}_{\mathcal{C}}(A) = \left(\bigvee\{\mathcal{C}(C) : C \subseteq A\}\right)'.$$

Then $(E, \mathcal{I}_{\mathcal{C}})$ is a fuzzifying matroid.

Proof. According to Lemma 2.2, it suffices to prove $(E, (\mathcal{I}_{\mathcal{C}})_{(b)})$ is a crisp matroid for arbitrary $b \in [0,1)$.

By the definition of $\mathcal{I}_{\mathcal{C}}$, it is easy to verify that $A_1 \subseteq A_2$ implies $\mathcal{I}_{\mathcal{C}}(A_1) \geq \mathcal{I}_{\mathcal{C}}(A_2)$. Hence, $(\mathcal{I}_{\mathcal{C}})_{(b)} = Opp(Upp(Min(Opp((\mathcal{I}_{\mathcal{C}})_{(b)}))))$. Thus by the relation between crisp matroids and circuits, it just needs to prove $Min(Opp((\mathcal{I}_{\mathcal{C}})_{(b)}))$ satisfies (C1), (C2) and (C3). By the definition of $\mathcal{I}_{\mathcal{C}}$, we have

$$\begin{aligned}
&A \in Min(Opp((\mathcal{I}_{\mathcal{C}})_{(b)}))\\
\Leftrightarrow &A \in Opp((\mathcal{I}_{\mathcal{C}})_{(b)}) \text{ and } B \notin Opp((\mathcal{I}_{\mathcal{C}})_{(b)}) \text{ for arbotrary } B \subset A\\
\Leftrightarrow &A \notin (\mathcal{I}_{\mathcal{C}})_{(b)} \text{ and } B \in (\mathcal{I}_{\mathcal{C}})_{(b)} \text{ for arbotrary } B \subset A\\
\Leftrightarrow &\bigvee\{\mathcal{C}(C) : C \subseteq A\} \geq b' \text{ and } \bigvee\{\mathcal{C}(C) : C \subseteq B\} < b' \text{ for arbotrary } B \subset A\\
\Leftrightarrow &\mathcal{C}(A) \geq b' \text{ and } \mathcal{C}(B) < b' \text{ for arbotrary } B \subset A\\
\Leftrightarrow &A \in Min(\mathcal{C}_{[b']}).
\end{aligned}$$

Since $b \in [0,1)$ if and only if $b' \in (0,1]$, thus it suffices to prove that $Min(\mathcal{C}_{[a]})$ satisfies (C1), (C2) and (C3) for an arbitrary $a \in (0,1]$.

(C1) By (FC1), $\emptyset \notin \mathcal{C}_{[a]}$. Thus $\emptyset \notin Min(\mathcal{C}_{[a]})$.

(C2) If $C_1, C_2 \in Min(\mathcal{C}_{[a]})$ and $C_1 \subseteq C_2$, then it is clear that $C_1 = C_2$.

(C3) Let $C_1, C_2 \in Min(\mathcal{C}_{[a]})$, $C_1 \neq C_2$ and $e \in C_1 \cap C_2$. By (C2), we get $C_1 \cap C_2 \neq C_i$ for $i = 1, 2$. Thus according to (FC3), there exists a $C_3 \subseteq C_1 \cup C_2 - e$ such that $\mathcal{C}(C_3) \geq \mathcal{C}(C_1) \wedge \mathcal{C}(C_2) \geq a$. Hence $C_3 \in \mathcal{C}_{[a]}$ and there exists a $C \subseteq C_3$ such that $C \in Min(\mathcal{C}_{[a]})$. □

Corollary 5.6. *Let $\mathcal{C}: 2^E \to [0,1]$ be a mapping which satisfies* (FC1), (FC2) *and* (FC3) *and $a \in (0,1]$. Then $(\mathcal{I}_{\mathcal{C}})_{(a')} = \mathcal{I}_{Min(\mathcal{C}_{[a]})}$.*

Proof. According to the proof of Theorem 5.5, this proof is trivial. □

Theorem 5.7. *Let E be a non-empty finite set.*

(i) *If $(E, \mathcal{I})$ is a fuzzifying matroid, then $\mathcal{I}_{\mathcal{C}_{\mathcal{I}}} = \mathcal{I}$.*

(ii) *If $\mathcal{C}: 2^E \to [0,1]$ is a mapping which satisfies* (FC1), (FC2) *and* (FC3), *then $\mathcal{C}_{\mathcal{I}_{\mathcal{C}}} = \mathcal{C}$.*

Proof. (i) By th definition of $\mathcal{C}_{\mathcal{I}}$, we have $\mathcal{C}_{\mathcal{I}}(A) \leq (\mathcal{I}(A))'$ for an arbitrary $A \in 2^E$. Thus $\mathcal{I}_{\mathcal{C}_{\mathcal{I}}}(A) = (\bigvee\{\mathcal{C}_{\mathcal{I}}(C) : C \subseteq A\})' = \bigwedge\{(\mathcal{C}_{\mathcal{I}}(C))' : C \subseteq A\} \geq \mathcal{I}(A)$.

Conversely, by Lemma 5.1, there exists a $C \in 2^E$ such that $C \subseteq A$ and $\mathcal{C}_{\mathcal{I}}(C) = (\mathcal{I}(A))'$. Thus $\mathcal{I}_{\mathcal{C}_{\mathcal{I}}}(A) \leq (\mathcal{C}_{\mathcal{I}}(C))' = \mathcal{I}(A)$.

(ii) Let $A \in 2^E$. If $A = \emptyset$, then $\mathcal{C}_{\mathcal{I}_{\mathcal{C}}}(\emptyset) = 0 = \mathcal{C}(\emptyset)$. Now let $A \neq \emptyset$.

If $\mathcal{C}(A) = 0$, then there exists some $C \in 2^E$ such that $C \subset A$ and $\mathcal{I}_{\mathcal{C}}(A) = (\mathcal{C}(C))'$. Let C be the minimum set with this property. By (FC2), we have $\mathcal{I}_{\mathcal{C}}(C) = (\mathcal{C}(C))'$. So $\mathcal{C}_{\mathcal{I}_{\mathcal{C}}}(A) = 0 = \mathcal{C}(A)$.

If $\mathcal{C}(A) \neq 0$, then by (FC2), we have that $\mathcal{C}(B) < \mathcal{C}(A)$ for arbitrary $B \subset A$. Thus $\mathcal{I}_{\mathcal{C}}(A) = (\bigvee\{\mathcal{C}(C) : C \subseteq A\})' = \bigwedge\{(\mathcal{C}(C))' : C \subseteq A\} = (\mathcal{C}(A))'$ and $\mathcal{I}_{\mathcal{C}}(B) = \bigwedge\{(\mathcal{C}(C))' : C \subseteq B\} > (\mathcal{C}(A))'$. So by Lemma 5.1, we get $\mathcal{C}_{\mathcal{I}_{\mathcal{C}}}(A) = (\mathcal{I}_{\mathcal{C}}(A))' = \mathcal{C}(A)$. □

6. Conclusions

Firstly, some properties about fuzzifying base-maps of fuzzifying matroids are discussed in this paper. By these properties, the fuzzifying basis axioms of fuzzifying matroids are given. In the second part, fuzzifying circuit axioms of fuzzifying matroids are introduced. It is proved that there is a one-to-one correspondence relation between fuzzifying matroids and mappings which satisfy the basis axioms or circuit axioms.

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