CHARACTERIZATION OF REGULAR $\Gamma-$SEMIGROUPS THROUGH FUZZY IDEALS

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ABSTRACT. Notions of strongly regular, regular and left(right) regular $\Gamma-$semigroups are introduced. Equivalent conditions are obtained through fuzzy notion for a $\Gamma-$semigroup to be either strongly regular or regular or left regular.

1. **Introduction**

In 1965, Zadeh [9] introduced the concept of a fuzzy subset and studied its properties on the lines parallel to set theory. In 1967, Rosenfeld [5] defined the notion of a fuzzy subgroup and gave some of its properties. Rosenfeld's definition of a fuzzy group is a turning point for pure mathematicians. Since then, the study of fuzzy algebraic structure has been pursued in many directions such as groups, rings, modules, vector spaces and so on. In 1981, Das [1] explained the interrelationship between a fuzzy subgroup and its $t-$level subsets. Sen [7] introduced $\Gamma-$semigroups in 1981. Sen and Saha [6], [8] introduced $\Gamma-$semigroups different from the first definition of $\Gamma-$semigroups in the sense of Sen [7].

In this paper we introduce the concepts of strongly regularity, regularity and left(right) regularity in $\Gamma-$semigroups. We obtain equivalent conditions for a $\Gamma-$semigroup M to be either strongly regular or regular or left regular as well as conditions for a $\Gamma-$semigroup M to be either strongly regular or regular or left regular.

2. **Preliminary Notes**

We recall some definitions and results proposed by the pioneers in this field.

Definition 2.1. Let M and Γ be any two nonempty sets. M is called a $\Gamma-$*semigroup* if

(1) $M\Gamma M \subseteq M$ and $\Gamma M\Gamma \subseteq \Gamma$.
(2) $(axb)yc = a(xby)c = ax(byc)$ for all $a,\ b,\ c \in M$ and $x,\ y \in \Gamma$.

Notation 2.2. For subsets A, B of M, let $A\Gamma B = \{a\gamma b \,|\, a \in A,\ b \in B \text{ and } \gamma \in \Gamma\}$.

Definition 2.3. Let M be a $\Gamma-$semigroup and A a nonempty subset of M. A is called a *right* (resp. *left*) *ideal* of M if $A\,\Gamma\,M \subseteq A$ (resp. $M\,\Gamma\,A \subseteq A$).

Received: February 2006; Revised: October 2006; Accepted: December 2006

Key words and phrases: $\Gamma-$semigroup, Bi-ideal, Quasi-ideal, Regular, Strongly regular, Left(right) regular, Fuzzy (left, right)ideal, Fuzzy quasi-ideal, Fuzzy bi-ideal.

A is called an *ideal* of M if it is a right and a left ideal of M.

From now on, throughout this paper M will denote a $\Gamma-$semigroup unless otherwise specified.

Definition 2.4. A mapping $\mu : M \to [0,\ 1]$ is called a *fuzzy subset* of M.

A fuzzy subset $\mu : M \to [0, 1]$ is nonempty if μ is not the constant map which assumes the value 0. For any two fuzzy subsets λ and μ of M, $\lambda \subseteq \mu$ means that $\lambda(a) \leq \mu(a)$ for all $a \in M$. The characteristic function of M is denoted by $\mathbf{M}$ and of its subset A is denoted by f_A. If μ is a fuzzy subset of M, then the image of μ denoted by $Im(\mu) = \{\mu(m) \,|\, m \in M\}$ and $|Im(\mu)|$ denotes the cardinality of $Im(\mu)$. Hereafter we consider only nonempty fuzzy subsets of M.

Definition 2.5. Let μ be any fuzzy subset of M. For $t \in [0,\ 1]$, the set

$$\mu_t = \{x \in M \,|\, \mu(x) \geq t\}$$

is called a *level subset* of μ.

Definition 2.6. Let f and g be any two fuzzy subsets of M. Then $f \cap g$, $f \cup g$ and $f * g$ are fuzzy subsets of M defined by

$$(f \cap g)(x) = min\{f(x),\ g(x)\}$$

$$(f \cup g)(x) = max\{f(x),\ g(x)\}$$

$$(f * g)(z) = \begin{cases} \sup\limits_{z=x\gamma y} \{min\{f(x),\ g(y)\}\} \text{ if } z \text{ is expressed as } z = x\gamma y, \\ \qquad\qquad\qquad\qquad \text{where } x,\ y,\ z \in M \text{ and } \gamma \in \Gamma. \\ 0 \quad \text{otherwise.} \end{cases}$$

Note that if $f,\ g$ and h are fuzzy subsets of M, then

$$\begin{aligned}(f * (g * h))(z) &= \sup_{z=x\gamma y} \{\min\{f(x),\ (g * h)(y)\}\} \\ &= \sup_{z=x\gamma y} \{\min\{f(x),\ \sup_{y=p\gamma_1 q} \{\min\{g(p),\ h(q)\}\}\}\} \\ &= \sup_{z=x\gamma p\gamma_1 q} \{\min\{f(x),\ g(p),\ h(q)\}\} \\ &= (f * g * h)(z),\end{aligned}$$

where $x,\ y,\ z,\ p,\ q \in M$ and $\gamma_1,\ \gamma_2 \in \Gamma$.

Definition 2.7. A fuzzy subset μ of M is called a *fuzzy left* (resp. *right*) *ideal* of M if $\mu(m\,\gamma\,a) \geq \mu(a)$ (resp. $\mu(a\,\gamma\,m) \geq \mu(a)$) for all $m,\ a \in M$ and $\gamma \in \Gamma$.

If μ is both fuzzy left and right ideal of M, then μ is called a *fuzzy ideal* of M.

Lemma 2.8. *A fuzzy subset λ of M is a fuzzy left (right) ideal of M if and only if $\boldsymbol{M} * \lambda \subseteq \lambda$ $(\lambda * \boldsymbol{M} \subseteq \lambda)$.*

Proof. Assume that λ is a fuzzy left ideal of M and $z \in M$. Suppose that z is expressible as $z = x\gamma y$, where $x,\ y \in M$ and $\gamma \in \Gamma$. Then we have

$$\begin{aligned}(\mathbf{M} * \lambda)(z) &= \sup_{z=x\gamma y} \{\min\{\mathbf{M}(x),\ \lambda(y)\}\} \\ &= \sup_{z=x\gamma y} \{\lambda(y)\} \\ &\leq \lambda(x\gamma y) \text{ [since } \lambda(x\gamma y) \geq \lambda(y)] \\ &\leq \lambda(z)\end{aligned}$$

and we have $\mathbf{M} * \lambda \subseteq \lambda$. If z is not expressible as $z = x\gamma y$, then $(\mathbf{M} * \lambda)(z) = 0 \leq \lambda(z)$ holds. Thus $\mathbf{M} * \lambda \subseteq \lambda$ for any fuzzy left ideal λ of M.

Conversely, let us assume that $\mathbf{M} * \lambda \subseteq \lambda$ holds for any fuzzy subset λ of M. Let $x,\ y \in M$ and $\gamma \in \Gamma$. Then, we have

$$\begin{aligned}\lambda(x\gamma y) &\geq (\mathbf{M} * \lambda)(x\gamma y)\\ &= \sup_{x\gamma y = p\,\gamma_1 q} \{\min\{\mathbf{M}(p),\ \lambda(q)\}\};\ \ p,\ q \in M \text{ and } \gamma_1 \in \Gamma\\ &\geq \min\{\mathbf{M}(x),\ \lambda(y)\}\\ &= \lambda(y).\end{aligned}$$

This means that λ is a fuzzy left ideal of M. $\square$

Lemma 2.9. *Let μ be a fuzzy subset of M. μ is a fuzzy left (resp. right) ideal of M if and only if each level subset μ_t, $t \in Im(\mu)$, is a left (resp. right) ideal in M.*

Proof. Let μ be a fuzzy left ideal of M and $t \in Im(\mu)$. Let $x \in \mu_t$, $\alpha \in \Gamma$ and $u \in M$. Then $\mu(u\,\alpha\,x) \geq \mu(x) \geq t$, which implies that $u\,\alpha\,x \in \mu_t$. Hence μ_t is a left ideal of M.

Conversely, let us assume that μ_t is a left ideal in M, for every $t \in Im(\mu)$. Suppose there exist $\alpha \in \Gamma$ and $x,\ y \in M$ such that $\mu(y\alpha x) < \mu(x)$. Let $\mu(x) = t_1$. Then $x \in \mu_{t_1}$ and $y\,\alpha\,x \notin \mu_{t_1}$. This is a contradicts the assumption that μ_{t_1} is a left ideal in M. Thus μ is a fuzzy left ideal of M. $\square$

Theorem 2.10. *Let A be a left (resp. right) ideal in M. Then for any $t \in (0,\ 1]$, there exists a fuzzy left (resp. right) ideal μ of M such that $\mu_t = A$.*

Proof. Let A be a left ideal of M and $\mu : M \to [0,\ 1]$ be defined by

$$\mu(x) = \begin{cases} t & \text{if } x \in A\\ 0 & \text{otherwise,}\end{cases}$$

for all $x \in M$ and $t \in (0,\ 1]$. Clearly $\mu_t = A$. Now assume that $\mu(u\,\alpha\,x) < \mu(x)$, for some $x,\ u \in M$ and $\alpha \in \Gamma$. Since $|Im(\mu)| = 2$, we have $\mu(u\,\alpha\,x) = 0$ and $\mu(x) = t$, whence $u\,\alpha\,x \notin A$ and $x \in A$. This is impossible because A is a left ideal in M. Thus $\mu(u\,\gamma\,x) \geq \mu(x)$, for all $x,\ u \in M$ and $\gamma \in \Gamma$. Hence μ is a fuzzy left ideal of M. $\square$

Theorem 2.11. *Let A be a nonempty subset in M and μ be a fuzzy subset of M defined by*

$$\mu(x) = \begin{cases} s & \text{if } x \in A\\ t & \text{otherwise,}\end{cases}$$

for all $x \in M$ and $s, t \in [0,\ 1]$ with $s > t$. Then μ is a fuzzy left (resp. right) ideal of M if and only if A is a left (resp. right) ideal in M. Moreover

$$M_\mu = \{x \in M \mid \mu(x) = \sup_{y \in M} \mu(y)\} = A.$$

Proof. Let μ be a fuzzy left ideal of M and $x \in A$, $\alpha \in \Gamma$ and $u \in M$. Then $\mu(u\,\alpha\,x) \geq \mu(x) = s$ and so $u\,\alpha\,x \in A$. This shows that A is a left ideal in M.

Conversely, let us assume that A is a left ideal in M. Let $x,\ u \in M$ and $\alpha \in \Gamma$. If $x \in A$, then $u\,\alpha\,x \in A$. Thus $\mu(u\,\alpha\,x) = s = \mu(x)$. If $x \notin A$, then clearly $\mu(u\,\alpha\,x) \geq t = \mu(x)$. Hence μ is a fuzzy left ideal of M. Moreover

$$M_\mu = \{x \in M | \mu(x) = \sup_{y \in M} \mu(y)\} = \{x \in M \,|\, \mu(x) = s\} = A. \qquad \square$$

Corollary 2.12. *Let A be a subset in M. Then A is a left* (*resp. right*) *ideal in M if and only if f_A is a fuzzy left* (*resp. right*) *ideal of M.*

Theorem 2.13. *Let m be a fixed element in M. If μ is a fuzzy left* (*resp. right*) *ideal of M, then $\mu^m = \{\, x \in M \,|\, \mu(x) \geq \mu(m)\}$ is a left* (*resp. right*) *ideal in M.*

Proof. Let $x \in M$, $y \in \mu^m$ and $\gamma \in \Gamma$. Then $\mu(x\gamma y) \geq \mu(y) \geq \mu(m)$ and so $x\gamma y \in \mu^m$. Therefore μ^m is a left ideal in M. $\square$

Corollary 2.14. *Let μ be a fuzzy left* (*resp. right*) *ideal of M. Then the set M_μ is a left* (*resp. right*) *ideal in M.*

3. **Fuzzy Bi-ideal and Fuzzy Quasi-ideal of M**

Definition 3.1. A subset Q of M is called a *quasi-ideal* of M if $Q\Gamma M \cap M\Gamma Q \subseteq Q$.

Definition 3.2. A fuzzy subset λ of M is called a *fuzzy quasi-ideal* of M if
$(\lambda * \mathbf{M}) \cap (\mathbf{M} * \lambda) \subseteq \lambda$.

Definition 3.3. A subset B of M is called a *bi-ideal* of M if $B\Gamma M\Gamma B \subseteq B$.

Definition 3.4. A fuzzy subset λ of M is called a *fuzzy bi-ideal* of M if
$\lambda * \mathbf{M} * \lambda \subseteq \lambda$.

Lemma 3.5. *Every fuzzy left* (*right*) *ideal of M is a fuzzy quasi-ideal of M.*

Proof. Let λ be any fuzzy left ideal of M. Then we have
$(\lambda * \mathbf{M}) \cap (\mathbf{M} * \lambda) \subseteq \mathbf{M} * \lambda \subseteq \lambda$.
This means that λ is a fuzzy quasi-ideal of M. $\square$

Lemma 3.6. *Let λ and μ be any fuzzy right and left ideals of M, respectively. Then $\lambda \cap \mu$ is a fuzzy quasi-ideal of M.*

Proof. Since $((\lambda \cap \mu) * \mathbf{M}) \cap (\mathbf{M} * (\lambda \cap \mu)) \subseteq (\lambda * \mathbf{M}) \cap (\mathbf{M} * \mu) \subseteq \lambda \cap \mu$, hence $\lambda \cap \mu$ is a fuzzy quasi-ideal of M. $\square$

Lemma 3.7. *Any fuzzy quasi-ideal of M is a fuzzy bi-ideal of M.*

Proof. Let λ be any fuzzy quasi-ideal of M. Then we have
$\lambda * \mathbf{M} * \lambda \subseteq \lambda * \mathbf{M} * \mathbf{M} \subseteq \lambda * \mathbf{M}$
and
$\lambda * \mathbf{M} * \lambda \subseteq \mathbf{M} * \mathbf{M} * \lambda \subseteq \mathbf{M} * \lambda$.
Thus we have
$\lambda * \mathbf{M} * \lambda \subseteq (\lambda * \mathbf{M}) \cap (\mathbf{M} * \lambda) \subseteq \lambda$.
This implies that λ is a fuzzy bi-ideal of M. $\square$

Lemma 3.8. *Let λ and μ be any two fuzzy quasi-ideals of M. Then $\lambda * \mu$ is a fuzzy bi-ideal of M.*

Proof. By Lemma 3.7, μ is a fuzzy bi-ideal of M. Hence

$$\begin{aligned}(\lambda * \mu) * \mathbf{M} * (\lambda * \mu) &= \lambda * \{\mu * (\mathbf{M} * \lambda) * \mu\} \\ &\subseteq \lambda * \{\mu * (\mathbf{M} * \mathbf{M}) * \mu\} \\ &\subseteq \lambda * \{\mu * \mathbf{M} * \mu\} \\ &\subseteq \lambda * \mu.\end{aligned}$$

Thus $\lambda * \mu$ is a fuzzy bi-ideal of M. □

The following result is easily proved.

Lemma 3.9. *For any nonempty subsets A and B of M,*
(1) $f_A * f_B = f_{A\Gamma B}$,
(2) $f_A \cap f_B = f_{A\cap B}$.

We denote by $I(a)$ (resp. $R(a)$, $L(a)$, $Q(a)$ and $B(a)$) the ideal (resp. right ideal, left ideal, quasi-ideal and bi-ideal) of M generated by $a \in M$. We can easily prove that
$I(a) = a \cup M\Gamma a \cup a\Gamma M \cup M\Gamma a\Gamma M$,
$R(a) = a \cup a\Gamma M$,
$L(a) = a \cup M\Gamma a$,
$Q(a) = a \cup (a\Gamma M \cap M\Gamma a)$, and
$B(a) = a \cup a\Gamma M\Gamma a$
for all $a \in M$.

Lemma 3.10. *Let A be any nonempty subset of M. Then*
(1) *A is a quasi-ideal of M if and only if f_A is a fuzzy quasi-ideal of M.*
(2) *A is a bi-ideal of M if and only if f_A is a fuzzy bi-ideal of M.*

Proof. (1) Assume that A is a quasi-ideal of M. Then we have

$$\begin{aligned}(f_A * \mathbf{M}) \cap (\mathbf{M} * f_A) &= (f_A * f_M) \cap (f_M * f_A). \\ &= f_{(A\Gamma M)} \cap f_{(M\Gamma A)} \\ &= f_{(A\Gamma M)\cap(M\Gamma A)} \\ &\subseteq f_A.\end{aligned}$$

This means that f_A is a fuzzy quasi-ideal of M.
Conversely, let $x \in (A\Gamma M) \cap (M\Gamma A)$. Then we have

$$\begin{aligned}f_A(x) &\geq \{(f_A * \mathbf{M}) \cap (\mathbf{M} * f_A)\}(x) \\ &= \min\{(f_A * \mathbf{M})(x),\, (\mathbf{M} * f_A)(x)\} \\ &= \min\{f_{A\Gamma M}(x),\, f_{M\Gamma A}(x)\} \\ &= f_{A\Gamma M\cap M\Gamma A}(x) \\ &= 1.\end{aligned}$$

This implies that $x \in A$ and so $(A\Gamma M) \cap (M\Gamma A) \subseteq A$. In other words, A is a quasi-ideal of M.
(2) Assume that A is a bi-ideal of M. Then we have $f_A * \mathbf{M} * f_A \subseteq f_{A\Gamma\mathbf{M}\Gamma A} \subseteq f_A$ i.e. f_A is a fuzzy bi-ideal of M.
Conversely, let $x \in A\Gamma M\Gamma A$. Then we have
$f_A(x) \geq (f_A * \mathbf{M} * f_A)(x) = f_{A\Gamma M\Gamma A}(x) = 1$
and so $x \in A$. Thus $A\Gamma M\Gamma A \subseteq A$. Hence A is a bi-ideal of M. □

Theorem 3.11. *Let λ be a fuzzy subset of M. λ is a fuzzy bi-ideal of M if and only if λ_t is a bi-ideal of M, for every $t \in Im(\lambda)$.*

Proof. Let λ be a fuzzy bi-ideal of M and $z \in M$. Let $t \in Im(\lambda)$. Suppose $z \in \lambda_t \Gamma M \Gamma \lambda_t$. Then there exist $x, y \in \lambda_t,\ m \in M$ and $\gamma_1, \gamma_2 \in \Gamma$ such that $z = x\gamma_1 m\gamma_2 y$. Then

$$\begin{aligned}(\lambda * \mathbf{M} * \lambda)(z) &= \sup_{z=a\alpha n\beta y} \{\min\{\lambda(a),\, \mathbf{M}(n),\, \lambda(b)\}\} \\ &\geq \min\{\lambda(x),\, \mathbf{M}(m),\, \lambda(y)\} \\ &\geq t.\end{aligned}$$

Since λ is a bi-ideal of M, $\lambda(z) \geq t$. Hence $z \in \lambda_t$. Thus λ_t is a bi-ideal of M.

Conversely, let us assume that $\lambda_t,\ t \in Im(\lambda)$ is a bi-ideal of M. Now $(\lambda * \mathbf{M} * \lambda)(p) = \sup\limits_{p=x\gamma_1 m\gamma_2 y} \{\min\{\lambda(x),\, \lambda(y)\}\}$. For $t_1 < t_2$ set $\lambda(x) = t_1$ and $\lambda(y) = t_2$. Then $x, y \in \lambda_{t_1}$ and $p = x\gamma_1 m\gamma_2 y \in \lambda_{t_1}\Gamma M \Gamma \lambda_{t_1} \subseteq \lambda_{t_1}$. Hence $\lambda(p) \geq t_1 \ \geq \min\{\lambda(x),\, \lambda(y)\}$. Since this is true for all $x,\ y$ with the property $p = x\gamma_1 m\gamma_2 y$, we have $\lambda(p) \geq \sup\limits_{p=x\gamma_1 m\gamma_2 y} \{\min\{\lambda(x),\, \lambda(y)\}\}$ and so $\lambda * \mathbf{M} * \lambda \subseteq \lambda$. Hence λ is a fuzzy bi-ideal of M. □

The following lemma can be proved similarly.

Lemma 3.12. *Let λ be a fuzzy subset of M. Then λ is a fuzzy quasi-ideal of M if and only if λ_t is a quasi-ideal in M, for all $t \in Im(\lambda)$.*

Lemma 3.13. *For a fuzzy bi-ideal λ of M, $\lambda(a\gamma_1 x\gamma_2 b) \geq \min\{\lambda(a),\ \lambda(b)\}$ holds for all $a,\ b,\ x \in M$ and $\gamma_1,\ \gamma_2 \in \Gamma$.*

Proof. Let $a,\ b,\ x \in M$ and $\gamma_1,\ \gamma_2 \in \Gamma$. Since λ is a fuzzy bi-ideal of M, $\lambda * \mathbf{M} * \lambda \subseteq \lambda$. Now

$$\begin{aligned}\lambda(a\,\gamma_1\,x\,\gamma_2\,b) &\geq (\lambda * \mathbf{M} * \lambda)(a\,\gamma_1\,x\,\gamma_2\,b) \\ &= \sup_{a\,\gamma_1\,x\,\gamma_2\,b=p\,\gamma_3\,y\,\gamma_4\,q} \min\{\{\lambda(p),\ \mathbf{M}(y),\ \lambda(q)\}\} \\ &\geq \min\{\lambda(a),\ \lambda(b)\}\}.\end{aligned}$$

This completes the proof. □

4. Regular Γ−semigroup M

Definition 4.1. M is said to be *regular* if for each element a in M, there exist $m \in M$ and $\gamma_1,\ \gamma_2 \in \Gamma$ such that $a = a\gamma_1 m\gamma_2 a$.

M is said to be *strongly regular* if for each element $a \in M$ and $\gamma_1,\ \gamma_2 \in \Gamma$, there exists $m \in M$ such that $a = x\,\gamma_1 m\gamma_2 y$, where $x \in R(a)$ and $y \in L(a)$.

Clearly M is regular if and only if $a \in a\,\Gamma\,a$, for all $a \in M$.

We recall ([3]), definitions of left (right) regularity.

Definition 4.2. M is said to be *left* (*right*) *regular* if for each element a in M, there exist $x \in M$ and $\gamma_1,\ \gamma_2 \in \Gamma$ such that $a = x\gamma_1 a\gamma_2 a$ $(a = a\gamma_1 a\gamma_2 x)$.

Example 4.3. Let M be the set of all integers of the form $4n + 1$ where n is an integer and Γ denotes the set of all integers of the form $4n + 3$. If $a\,\gamma\,b$ is $a+\gamma+b$, $\gamma\,a\,\mu$ is $\gamma+a+\mu$ (usual sum of the integers) for all $a,\ b \in M$ and $\gamma,\ \mu \in \Gamma$, then M is strongly regular. Set $a = 4n + 1,\ \gamma_1 = 4n_1 + 3$ and $\gamma_2 = 4n_2 + 3$. Let $x = a,\ y = a$ and $m = 4(-n - n_1 - n_2 - 2) + 1$. Then $a = a\gamma_1 m\gamma_2 a$. Thus M is strongly regular, regular and left (right) regular.

Example 4.4. Let $M = \{0, 1, 2, 3, 4, 5\}$ and $\Gamma = M$. If the composition of any two elements is taken as multiplication modulo six, then M is regular but not strongly regular. For $2 \in M$ and $3 \in \Gamma$ there is no $x \in R(2)$ and $y \in L(2)$ such that $2 = x3m3y$.

Example 4.5. In the above example if $\Gamma = \{0, 3\}$, then 2 is neither a regular nor a strongly regular element in M but 3 is a regular element in M.

We now find equivalent conditions for a $\Gamma-$semigroup M to be strongly regular, regular and left regular.

Theorem 4.6. (1) *M is strongly regular if and only if $A \cap B = A\gamma B$, where A and B are respectively right and left ideals of M and $\gamma \in \Gamma$.*

(2) *M is regular if and only if $A \cap B = A\Gamma B$, where A and B are respectively right and left ideals of M .*

Proof. (1) Let M be strongly regular and $\gamma \in \Gamma$. Now $A\gamma B \subseteq A\Gamma M \subseteq A$ and $A\gamma B \subseteq M\Gamma B \subseteq B$ imply $A\gamma B \subseteq A\cap B$. Let $x \in A\cap B$. Since M is strongly regular, $x = y\gamma m\gamma z$ for some $m \in M$, $y \in R(x)$ and $z \in L(x)$. Since $y\gamma m \in R(x)\gamma m \subseteq A\gamma M \subseteq A$, $x = (y\gamma m)\gamma z \in A\gamma B$. Thus $A \cap B = A\gamma B$.

Conversely, let $x \in M$. Now $x \in R(x) \cap L(x) = R(x)\gamma L(x)$ for all $\gamma \in \Gamma$. Let $\gamma_1,\ \gamma_2 \in \Gamma$. Now $x \in (x \cup x\Gamma M)\gamma_1(x \cup M\Gamma x)$. Then

$x = x\gamma_1 x = x\gamma_1(x\gamma_1 x) = x\gamma_1 x\gamma_1 x$ or
$x = x\gamma_1 m_1\gamma_2 x$, for some $m_1 \in M$ and $\gamma_2 \in \Gamma$ or
$x = x\gamma_3 m_2\gamma_1 x$, for some $m_2 \in M$ and $\gamma_3 \in \Gamma$
$\quad = x\gamma_3 m_2\gamma_1(x\gamma_3 m_2\gamma_1 x) = (x\gamma_3 m_2)\gamma_1 x(\gamma_3 m_2\gamma_1)x$
$\quad = y\gamma_1 x\gamma_\alpha x$, for some $y = x\gamma_3 m_2 \in R(x)$ and $\gamma_\alpha = \gamma_3 m_2\gamma_1 \in \Gamma$ or
$x = x\gamma_4 m_3\gamma_1 m_4\gamma_5 x = (x\gamma_4 m_3)\gamma_1 m_4\gamma_5 x$, for some $m_3,\ m_4 \in M$ and $\gamma_4,\ \gamma_5 \in \Gamma$
$\quad = y\gamma_1 m_4\gamma_5 x$, for some $y = x\gamma_4 m_3 \in R(x)$, $m_4 \in M$ and $\gamma_5 \in \Gamma$.
Since $x \in R(x)$, we have $x = y\gamma_1 m\gamma x$, for some $m \in M$, $\gamma \in \Gamma$ and $y \in R(x)$.
Again $x \in (x \cup x\Gamma M)\gamma_2(x \cup M\Gamma x)$. Then
$x = x\gamma_2 x$ or
$x = x\gamma_2 m_1\gamma_3 x$, for some $m_1 \in M$ and $\gamma_3 \in \Gamma$
$\quad = x\gamma_2(m_1\gamma_3 x) = x\gamma_2 z$, for some $z = m_1\gamma_3 x \in L(x)$ or
$x = x\gamma_4 m_2\gamma_2 x$, for some $m_2 \in M$ and $\gamma_4 \in \Gamma$
$\quad = (x\gamma_4 m_2)\gamma_2 x = y\gamma_2 x$, for some $y = x\gamma_4 m_2 \in M$ or
$x = x\gamma_5 m_3\gamma_2 m_4\gamma_6 x$, for some $m_3,\ m_4 \in M$ and $\gamma_5,\ \gamma_6 \in \Gamma$
$\quad = (x\gamma_5 m_3)\gamma_2(m_4\gamma_6 x) = y_\alpha\gamma_2 z_1$, for some $y_\alpha \in M$ and $z_1 = m_4\gamma_6 x \in L(x)$.
Since $x \in L(x)$, we have $x = y_1\gamma_2 z$, for some $y_1 \in M$ and $z \in L(x)$.

Now $x = y\gamma_1 m\gamma x = y\gamma_1(m\gamma y_1)\gamma_2 z = y\gamma_1 m_1\gamma_2 z$. Thus M is strongly regular.

(2) Let A and B be right and left ideals of M respectively. Now $A\Gamma B \subseteq A\Gamma M \subseteq A$, $A\Gamma B \subseteq M\Gamma B \subseteq B$ and hence $A\Gamma B \subseteq A \cap B$. Let $x \in A \cap B$. Then $x \in A$ and $x \in B$. Since M is regular, there exist $m \in M$ and $\gamma_1,\ \gamma_2 \in \Gamma$ such that $x = x\gamma_1 m\gamma_2 x = x(\gamma_1 m\gamma_2)x \in A\Gamma B$. Thus $A\Gamma B \supseteq A \cap B$. Hence $A\Gamma B = A \cap B$.

Conversely, let us assume that $A\Gamma B = A \cap B$ for every right ideal A and every left ideal B and $x \in M$. Then $R(x)\Gamma L(x) = R(x) \cap L(x)$, where $L(x),\ R(x)$ are respectively left and right ideals generated by x. Thus

$x \in R(x)\Gamma L(x) = x\Gamma x \cup x\Gamma M\Gamma x \cup x\Gamma M\Gamma M\Gamma x \subseteq x\Gamma M\Gamma x$.

Hence M is regular. □

Lemma 4.7. *M is left regular if and only if $I \cap B \subseteq I\Gamma B$ for every ideal I and for every bi-ideal B of M.*

Proof. Let M be left regular and $a \in M$. Suppose $a \in I \cap B$. Since M is left regular, $a = (x\gamma_1 a)\gamma_2 a \in I\Gamma B$. Thus $I \cap B \subseteq I\Gamma B$.

Conversely, let $a \in M$. Then $a \in I(a) \cap B(a) \subseteq I(a)\Gamma B(a)$. Thus

$$\begin{aligned}
&a \in (a \cup a\Gamma M \cup M\Gamma a \cup M\Gamma a\Gamma M)\Gamma(a \cup a\Gamma M\Gamma a)\\
&= (a\Gamma a) \cup (a\Gamma a\Gamma M\Gamma a) \cup (a\Gamma M\Gamma a) \cup (a\Gamma M\Gamma a\Gamma M\Gamma a) \cup (M\Gamma a\Gamma a) \cup (M\Gamma a\Gamma a\Gamma M\Gamma a)\\
&\cup (M\Gamma a\Gamma M\Gamma a) \cup (M\Gamma a\Gamma M\Gamma a\Gamma M\Gamma a)\\
&\subseteq a\Gamma a \cup a\Gamma a\Gamma a \cup M\Gamma a\Gamma a.
\end{aligned}$$

We have three cases: 1) $a \in a\Gamma a$, 2) $a \in a\Gamma a\Gamma a$ and 3) $a \in M\Gamma a\Gamma a$. For all these cases, we have $a = x\gamma_1 a\gamma_2 a$ for some $x \in M$ and $\gamma_1,\ \gamma_2 \in \Gamma$. Thus M is left regular. □

Now we characterize regularity, strongly regularity and left regularity of M through fuzzy concepts.

Lemma 4.8. *The following conditions are equivalent:*

(1) *M is left regular.*

(2) *$\lambda \cap \mu \subseteq \lambda * \mu$, for every fuzzy ideal λ and fuzzy bi-ideal μ of M.*

Proof. (1) $\Rightarrow$ (2) Let M be left regular and $a \in M$. Then, there exist $x \in M$ and $\gamma_1,\ \gamma_2 \in \Gamma$ such that $a = x\gamma_1 a\gamma_2 a$. Consider

$$\begin{aligned}
(\lambda * \mu)(a) &= \sup_{a = x\gamma y}\{\min\{\lambda(x),\ \mu(y)\}\}\\
&\geq \min\{\lambda(x\gamma_1 a),\ \mu(a)\}\\
&\geq \min\{\lambda(a),\ \mu(a)\}\\
&= (\lambda \cap \mu)(a).
\end{aligned}$$

Thus $\lambda \cap \mu \subseteq \lambda * \mu$.

(2) $\Rightarrow$ (1) Let I be any ideal and B be any bi-ideal of M. Let $x \in I \cap B$. This implies that $1 = \{f_I \cap f_B\}(x) \leq f_{I\Gamma B}(x)$ and hence $x \in I\Gamma B$. Thus $I \cap B \subseteq I\Gamma B$. By Lemma 4.7, M is left regular. □

Lemma 4.9. *M is regular if and only if $Q\Gamma M\Gamma Q = Q$, for every quasi-ideal Q of M.*

Proof. Let M be regular and Q be a quasi-ideal of M. Then $Q\Gamma M\Gamma Q \subseteq Q\Gamma M\Gamma M \subseteq Q\Gamma M$ and $Q\Gamma M\Gamma Q \subseteq M\Gamma M\Gamma Q \subseteq M\Gamma Q$. Since Q is a quasi-ideal of M, $Q\Gamma M\Gamma Q \subseteq Q\Gamma M \cap M\Gamma Q \subseteq Q$. Next, let $a \in Q$. Since M is regular, there exist $m \in M$ and $\gamma_1,\ \gamma_2 \in \Gamma$ such that $a = a\gamma_1 m\gamma_2 a$. Hence $a = a\gamma_1 m\gamma_2 a \in Q\Gamma M\Gamma Q$. Thus $Q \subseteq Q\Gamma M\Gamma Q$. Hence $Q = Q\Gamma M\Gamma Q$.

Conversely, let us assume $Q = Q\Gamma M\Gamma Q$ is true for every quasi-ideal Q of M. Let $a \in M$. Then $Q(a) = Q(a)\Gamma M\Gamma Q(a)$, where $Q(a)$ is the quasi-ideal generated by a. Hence

$a \in Q(a)\Gamma M\Gamma Q(a) = \{a \cup ((a\Gamma M) \cap (M\Gamma a))\}\Gamma M\Gamma\{a \cup ((a\Gamma M) \cap (M\Gamma a))\}$.

Thus M is regular. □

Lemma 4.10. *M is regular if and only if $Q\Gamma A\Gamma Q = A \cap Q$, for every quasi-ideal Q and every ideal A of M.*

Proof. Let M be regular, Q be a quasi-ideal and A be an ideal of M. Then $Q\Gamma A\Gamma Q \subseteq Q\Gamma M\Gamma M \subseteq Q\Gamma M$ and $Q\Gamma A\Gamma Q \subseteq M\Gamma M\Gamma Q \subseteq M\Gamma Q$. Since Q is a quasi-ideal of M, $Q\Gamma A\Gamma Q \subseteq Q\Gamma M \cap M\Gamma Q \subseteq Q$. Since A is an ideal of M, $Q\Gamma A\Gamma Q \subseteq M\Gamma A\Gamma M \subseteq A\Gamma M \subseteq A$. Thus $Q\Gamma A\Gamma Q \subseteq A \cap Q$.

Next, let $a \in A \cap Q$. Then $a \in A$ and $a \in Q$. Since M is regular, there exist $\gamma_1,\ \gamma_2 \in \Gamma$ and $m \in M$ such that $a = a\gamma_1 m\gamma_2 a = a\gamma_1 m\gamma_2(a\gamma_1 m\gamma_2 a) = a\gamma a\gamma a$, where $\gamma = \gamma_1 m\gamma_2 \in \Gamma$. Hence $a = a\gamma a\gamma a \in Q\Gamma A\Gamma Q$. Thus $A \cap Q \subseteq Q\Gamma A\Gamma Q$. Hence $Q\Gamma A\Gamma Q = A \cap Q$.

Conversely, let us assume $Q\Gamma A\Gamma Q = A \cap Q$ is true for every quasi-ideal Q and every ideal A of M. Let $a \in M$. Then $I(a) \cap Q(a) = Q(a)\,\Gamma\, I(a)\,\Gamma\, Q(a)$, where $Q(a)$ is the quasi-ideal generated by a and $I(a)$ is the ideal generated by a. Hence

$$\begin{aligned} a &\in Q(a)\Gamma I(a)\Gamma Q(a) \\ &= (a \cup (a\Gamma M \cap M\Gamma a))\Gamma(a \cup (a\Gamma M) \cup (M\Gamma a) \cup (M\Gamma a\Gamma M))\Gamma(a \cup (a\Gamma M \cap M\Gamma a)) \\ &\subseteq a\Gamma M\Gamma a. \end{aligned}$$

Thus M is regular. □

Theorem 4.11. *M is regular if and only if $\lambda * \mu = \lambda \cap \mu$, for every fuzzy right ideal λ and every fuzzy left ideal μ of M.*

Proof. Let M be regular. Let λ be a fuzzy right ideal, μ be a fuzzy left ideal of M and $z \in M$.

$$\begin{aligned} (\lambda * \mu)(z) &= \sup_{z = x\gamma y}\{\min\{\lambda(x),\ \mu(y)\}\} \\ &\leq \sup_{z = x\gamma y}\{\min\{\lambda(x\gamma y),\ \mu(x\gamma y)\}\} \\ &= (\lambda \cap \mu)(z) \end{aligned}$$

Thus $\lambda * \mu \subseteq \lambda \cap \mu$. Next, let $y \in M$. Since M is regular, there exist $\gamma_1,\ \gamma_2 \in \Gamma$ and $x \in M$ such that $y = y\gamma_1 x\gamma_2 y$.

Now $(\lambda*\mu)(y) = \sup\limits_{y = a\gamma b}\{\min\{\lambda(a),\ \mu(b)\}\} \geq \min\{\lambda(y),\ \mu(y)\}$, since $y = y(\gamma_1 x\gamma_2)y$ and $\Gamma M\Gamma \subseteq \Gamma$, $\gamma_1 x\gamma_2 \in \Gamma$. So $(\lambda * \mu)(y) \geq (\lambda \cap \mu)(y)$ and hence $\lambda * \mu \supseteq \lambda \cap \mu$. Thus $\lambda * \mu = \lambda \cap \mu$.

Conversely, suppose that $\lambda * \mu = \lambda \cap \mu$ whenever λ is a fuzzy right ideal and μ is a fuzzy left ideal of M. Let A be a right ideal and B be a left ideal of M. Clearly, f_A is a fuzzy right ideal and f_B is a fuzzy left ideal of M. Therefore, we get $f_A \cap f_B = f_A * f_B = f_{A\Gamma B}$, by Lemma 3.9. Thus $A \cap B = A\Gamma B$. This is true for all right ideals A and left ideals B. Hence by Lemma 4.6, M is regular. □

Theorem 4.12. *The following conditions are equivalent:*
(1) *M is regular.*
(2) *$\lambda = \lambda * \boldsymbol{M} * \lambda$ for every fuzzy bi-ideal λ of M.*
(3) *$\lambda = \lambda * \boldsymbol{M} * \lambda$ for every fuzzy quasi-ideal λ of M.*

Proof. First assume that (1) holds. Let λ be any fuzzy bi-ideal of M and $a \in M$. Since M is regular, there exist $\gamma_1,\ \gamma_2 \in \Gamma$ and $m \in M$ such that $a = a\gamma_1 m\gamma_2 a$. Then we have

$$\begin{aligned}(\lambda * \mathbf{M} * \lambda)(a) &= \sup_{a=x\gamma_3 m_1\gamma_4 y}\{\min\{\lambda(x),\ (\mathbf{M} * \lambda)(m_1\gamma_4 y)\}\}\\ &\geq \min\{\lambda(a),\ (\mathbf{M} * \lambda)(m\gamma_2 a)\},\ [\text{since } a = a\gamma_1(m\gamma_2 a)]\\ &= \min\{\lambda(a),\ \sup_{m\gamma_2 a=p\gamma_5 q}\{\min\{\mathbf{M}(p),\ \lambda(q)\}\}\}\\ &\geq \{\lambda(a),\ \lambda(a)\}\\ &= \lambda(a)\end{aligned}$$

and so $\lambda * \mathbf{M} * \lambda \supseteq \lambda$. Since λ is a fuzzy bi-ideal of M, the converse inclusion holds. Thus we have $\lambda = \lambda * \mathbf{M} * \lambda$ and (1) implies (2).

Since any fuzzy quasi-ideal of M is a fuzzy bi-ideal of M, by Lemma 3.7, (2) implies (3).

Assume (3) holds. Let Q be any quasi-ideal in M and let $a \in Q$. Then we have $f_{Q\Gamma M\Gamma Q}(a) = f_Q * f_M * f_Q(a) = f_Q(a) = 1$. This implies that $a \in Q\Gamma M\Gamma Q$. Thus $Q \subseteq Q\Gamma M\Gamma Q$. On the other hand, since Q is a quasi-ideal of M, $Q\Gamma M \cap M\Gamma Q \subseteq Q$, we have $Q\Gamma M\Gamma Q = Q$. It follows from Lemma 4.9 that M is regular and (3) implies (1). □

Theorem 4.13. *The following conditions are equivalent:*
(1) *M is regular.*
(2) *$\lambda \cap \mu = \mu * \lambda * \mu$ for every fuzzy ideal λ and every fuzzy bi-ideal μ of M.*
(3) *$\lambda \cap \mu = \mu * \lambda * \mu$ for every fuzzy ideal λ and every fuzzy quasi-ideal μ of M.*

Proof. First assume that (1) holds. Let λ and μ respectively be a fuzzy ideal and a fuzzy bi-ideal of M. Then $\mu * \lambda * \mu \subseteq \mu * M * \mu \subseteq \mu$. Now we have $\mu * \lambda * \mu \subseteq M * \lambda * M \subseteq \lambda$. Thus $\mu * \lambda * \mu \subseteq \lambda \cap \mu$. In order to see that the converse inclusion holds, let $a \in M$. Since M is regular, there exist $\gamma_1,\ \gamma_2 \in \Gamma$ and $m \in M$ such that $a = a\gamma_1 m\gamma_2 a = a\gamma_1 m\gamma_2(a\gamma_1 m\gamma_2 a) = a\gamma a\gamma a$, where $\gamma = \gamma_1 m\gamma_2 \in \Gamma$. Since λ is a fuzzy ideal of M, $\lambda(a\gamma a\gamma a) \geq \lambda(a)$. Then we have

$$\begin{aligned}(\mu * \lambda * \mu)(a) &= \sup_{a=x\gamma_1 y}\{\min\{\mu(x),\ (\lambda * \mu)(y)\}\}\\ &\geq \{\mu(a),\ (\lambda * \mu)(a\gamma a)\}\\ &= \min\{\mu(a),\ \sup_{a\gamma a=x_1\gamma_2 y_1}\{\min\{\lambda(x_1),\ \mu(y_1)\}\}\}\end{aligned}$$

$$\geq \min\{\mu(a),\ \{\lambda(a),\ \mu(a)\}\}$$
$$= (\mu \cap \lambda)(a),$$

and hence $\mu * \lambda * \mu \supseteq \lambda \cap \mu$. Thus we obtain that $\lambda \cap \mu = \mu * \lambda * \mu$ and so (1) implies (2).

It is clear that (2) implies (3).

Assume that (3) holds. Let A and Q be any ideal and any quasi-ideal of M, respectively. Let $x \in A \cap Q$. Then $x \in A$ and $x \in Q$ and $Q \cap A \subseteq Q\Gamma A\Gamma Q$. The converse inequality is easily obtained and so $f_{Q\Gamma A\Gamma Q}(x) = (f_Q * f_A * f_Q)(x) = (f_A \cap f_Q)(x) = \min\{f_A(x),\ f_Q(x)\} = 1$. Thus $x \in Q\Gamma A\Gamma Q$ and so $Q\Gamma A\Gamma Q = A \cap Q$. It follows from Lemma 4.10, that M is regular and (3) implies (1). □

Theorem 4.14. *The following conditions are equivalent:*

(1) *M is regular.*
(2) *$\lambda \cap \mu \subseteq \lambda * \mu$ for every fuzzy right ideal λ and every fuzzy bi-ideal μ of M.*
(3) *$\lambda \cap \mu \subseteq \lambda * \mu$ for every fuzzy right ideal λ and every fuzzy quasi-ideal μ of M.*
(4) *$\mu \cap \nu \subseteq \mu * \nu$ for every fuzzy left ideal ν and every fuzzy bi-ideal μ of M.*
(5) *$\mu \cap \nu \subseteq \mu * \nu$ for every fuzzy left ideal ν and every fuzzy quasi-ideal μ of M.*
(6) *$\lambda \cap \mu \cap \nu \subseteq \lambda * \mu * \nu$ for every fuzzy right ideal λ, every fuzzy left ideal ν and every fuzzy bi-ideal μ of M.*
(7) *$\lambda \cap \mu \cap \nu \subseteq \lambda * \mu * \nu$ for every fuzzy right ideal λ, every fuzzy left ideal ν and every fuzzy quasi-ideal μ of M.*

Proof. First assume that (1) holds. Let λ and μ respectively be a fuzzy right ideal and a fuzzy bi-ideal of M, and suppose $a \in M$. Since M is regular, there exist $\gamma_1,\ \gamma_2 \in \Gamma$ and $m \in M$ such that $a = a\gamma_1 m\gamma_2 a = a\gamma_1 m\gamma_2(a\gamma_1 m\gamma_2 a) = a\gamma a\gamma a$, where $\gamma = \gamma_1 m\gamma_2 \in \Gamma$. Then we have

$$\begin{aligned}(\lambda * \mu)(a) &= \sup_{a = x\gamma y}\{\min\{\lambda(x),\ \mu(y)\}\}\\ &\geq \min\{\lambda(a\gamma a),\ \mu(a)\}\\ &\geq \min\{\lambda(a),\ \mu(a)\}\\ &= (\lambda \cap \mu)(a)\end{aligned}$$

and hence $\lambda \cap \mu \subseteq \lambda * \mu$. Thus (1) implies (2).

It can be seen in a similar way that (1) implies (4). Clearly (2) implies (3) and (4) implies (5).

Assume that (3) holds. Let λ be a fuzzy right ideal and μ be a fuzzy left ideal of M. Since every fuzzy left ideal is a fuzzy quasi-ideal of M, by (3) we have $\lambda \cap \mu \subseteq \lambda * \mu$. Again since λ is a fuzzy right ideal and μ is a fuzzy left ideal of M, we have $\lambda * \mu \subseteq \lambda \cap \mu$. Thus $\lambda * \mu = \lambda \cap \mu$. By Theorem 4.11, M is regular. Hence we obtain (3) implies (1).

Similarly, we have (5) implies (1).

Assume that (1) holds. Let λ, ν and μ respectively be a fuzzy right ideal, a fuzzy left ideal and a fuzzy bi-ideal of M, and suppose $a \in M$. Since M is regular, there exist $\gamma_1,\ \gamma_2 \in \Gamma$ and $m \in M$ such that $a = a\gamma_1 m\gamma_2 a = a\gamma_1 m\gamma_2(a\gamma_1 m\gamma_2 a) = a\gamma a\gamma a$ where $\gamma = \gamma_1 m\gamma_2 \in \Gamma$. Then we have

$$
\begin{aligned}
(\lambda * \mu * \nu)(a) &= \sup_{a = x\gamma_1 y} \{\min\{\lambda(x),\, (\mu * \nu)(y)\}\} \\
&\geq \min\{\lambda(a),\, (\mu * \nu)(a\gamma a)\} \\
&\geq \min\{\lambda(a),\, \sup_{a\gamma a = p\gamma_2 q} \{\min\{\mu(p),\, \nu(q)\}\}\} \\
&\geq \min\{\lambda(a),\, \min\{\mu(a),\, \nu(a)\}\} \\
&\geq \min\{\lambda(a),\, \mu(a),\, \nu(a)\} \\
&= (\lambda \cap \mu \cap \nu)(a)
\end{aligned}
$$

and hence $\lambda \cap \mu \cap \nu \subseteq \lambda * \mu * \nu$. So (1) implies (6).

It is clear that (6) implies (7).

Finally, assume that (7) holds. Let λ and ν respectively be a fuzzy right ideal and left ideals of M. Since $\mathbf{M}$ itself is a fuzzy quasi-ideal of M, we have $\lambda \cap \nu = \lambda \cap \mathbf{M} \cap \nu \subseteq \lambda * \mathbf{M} * \nu \subseteq \lambda * \nu$, by Lemma 2.8. Clearly $\lambda * \nu \subseteq \lambda \cap \nu$. Hence $\lambda * \nu = \lambda \cap \nu$. It follows from Theorem 4.11 that M is regular and (7) implies (1). □

Acknowledgments. Authors would like to express their sincere thanks to the referees for the valuable suggestions

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