BASE AXIOMS AND SUBBASE AXIOMS IN M-FUZZIFYING CONVEX SPACES

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ABSTRACT. Based on a completely distributive lattice M, base axioms and subbase axioms are introduced in M-fuzzifying convex spaces. It is shown that a mapping $\mathscr{B}$ (resp. φ) with the base axioms (resp. subbase axioms) can induce a unique M-fuzzifying convex structure with $\mathscr{B}$ (resp. φ) as its base (resp. subbase). As applications, it is proved that bases and subbases can be used to characterize CP mappings and CC mappings between M-fuzzifying convex spaces.

1. Introduction

Since Zadeh [19] introduced the concept of fuzzy sets, fuzzy set theory has been applied to various branches of mathematics, such as fuzzy control, fuzzy topology, fuzzy algebra and so on. In 1994, Rosa [8, 9] firstly generalized the notion of axiomatic convex structures [13] (convex structures, in short) to the fuzzy case, which is called an I-convex structure nowadays (I is the unit interval). Actually, convex structures exist in so many mathematical fields, such as linear spaces [13], lattices [13, 14], metric spaces and graphs [3, 12] and topological spaces [2, 5]. So the theory of convex structures deserved wide attention, especially in fuzzy set theory. In 2009, Maruyama [4] generalized I-convex structures to L-convex structures, where L is a completely distributive lattice. In the setting of L-convex structures, Pang and Shi [6] offered a categorical approach to establish the relations between L-convex structures and classical convex structures. Recently, Pang and Zhao [7] presented several characterizations of L-convex structures.

Each L-convex structure on a nonempty set X can be considered as a family of L-subsets of X satisfying several conditions. In a different way, Shi and Xiu [11] introduced a new approach to the fuzzifications of convex structures from a logical viewpoint. In this way, the new resulting concept is called M-fuzzifying convex structures. Different from L-convex structures, each M-fuzzifying convex structure on a nonempty set X is a mapping from $\mathbf{2}^X$ (the powerset of X) to M satisfying three conditions. In the situation of M-fuzzifying convex structures, Shi and Li [10] generalized the notion of restricted hull operators in classical convex spaces to M-fuzzifying restricted hull operators and used it to characterize M-fuzzifying convex

Received: October 2016; Revised: January 2017; Accepted: May 2017
Key words and phrases: M-fuzzifying convex structure, Base axiom, Subbase axiom, CP mapping, CC mapping.

structures. Wu and Bai [15] defined M-fuzzifying JHC convex structures and M-fuzzifying Peano interval spaces, and discussed their relations. Recently, Xiu and Shi [16] introduced the concept of M-fuzzifying interval operators and established its relations with M-fuzzifying convex structures from a categorical aspect.

Bases and subbases are two important concepts in the theory of convex structures, since they can be used to induce convex structures and to characterize properties of convex structures. In the fuzzy case, Shi and Xiu [11] proposed the concepts of bases and subbases in the framework of M-fuzzifying convex structures. Inspired by the axiomatic approach, we may ask that whether there are base axioms and subbase axioms to characterize M-fuzzifying convex structures as the case in fuzzy topological structures in [17, 18]. In this paper, we will consider base axioms and subbase axioms in the framework of M-fuzzifying convex structures and investigate their relations with M-fuzzifying convex structures.

2. **Preliminaries**

Throughout this paper, M denotes a completely distributive lattice. The smallest element and the largest element in M are denoted by $\perp$ and $\top$, respectively. The family of all subsets and all finite subsets of a nonempty set X are denoted by $\mathbf{2}^X$ and $\mathbf{2}^X_{fin}$, respectively. The binary relation $\prec$ in M is defined as follows: for $a, b \in M$, $a \prec b$ if and only if for every subset $D \subseteq M$, the relation $b \leq \bigvee D$ always implies the existence of $d \in D$ with $a \leq d$. A complete lattice is completely distributive if and only if $b = \bigvee\{a \in M : a \prec b\}$ for each $b \in L$ [1].

For a nonempty set X, let $\mathbf{2}^X$ denote the powerset of X. For $\{A_j\}_{j\in J} \subseteq \mathbf{2}^X$, we say $\{A_j\}_{j\in J}$ is an up-directed subset of $\mathbf{2}^X$ provided that for each $B, C \in \{A_j\}_{j\in J}$, there exists $D \in \{A_j\}_{j\in J}$ such that $B \subseteq D$ and $C \subseteq D$.

Let $f : X \longrightarrow Y$ be a mapping. Define $f^{\rightarrow} : \mathbf{2}^X \longrightarrow \mathbf{2}^Y$ by $f^{\rightarrow}(A) = \{f(x) : x \in A\}$ for each $A \in \mathbf{2}^X$ and $f^{\leftarrow} : \mathbf{2}^Y \longrightarrow \mathbf{2}^X$ by $f^{\leftarrow}(B) = \{x : f(x) \in B\}$ for each $B \in \mathbf{2}^Y$.

Definition 2.1. [11] A mapping $\mathscr{C} : \mathbf{2}^X \longrightarrow M$ is called an M-fuzzifying convex structure on X if it satisfies the following conditions:

(MYC1) $\mathscr{C}(\emptyset) = \mathscr{C}(X) = \top_M$;
(MYC2) If $\{A_i : i \in \Omega\} \subseteq \mathbf{2}^X$ is nonempty, then $\mathscr{C}(\bigcap_{i\in\Omega} A_i) \geq \bigwedge_{i\in\Omega} \mathscr{C}(A_i)$;
(MYC3) If $\{A_i : i \in \Omega\} \subseteq \mathbf{2}^X$ is nonempty and totally ordered by inclusion, then $\mathscr{C}(\bigcup_{i\in\Omega} A_i) \geq \bigwedge_{i\in\Omega} \mathscr{C}(A_i)$.

If $\mathscr{C}$ is an M-fuzzifying convex structure on X, then the pair $(X, \mathscr{C})$ is called an M-fuzzifying convex space.

If $\mathscr{C}$ satisfies (MYC1) and (MYC2), then $\mathscr{C}$ is called an M-fuzzifying closure system on X and the pair $(X, \mathscr{C})$ is called an M-fuzzifying closure space.

Proposition 2.2. [11] *Let $\mathscr{C} : \mathbf{2}^X \longrightarrow M$ be an M-fuzzifying closure system on X. That is, $\mathscr{C}$ satisfies* (MYC1) *and* (MYC2). *Then the following conditions are equivalent.*

(MYC3) *If $\{A_i : i \in \Omega\} \subseteq \mathbf{2}^X$ is nonempty and totally ordered by inclusion, then $\mathscr{C}(\bigcup_{i\in\Omega} A_i) \geq \bigwedge_{i\in\Omega} \mathscr{C}(A_i)$.*

(MYC3)$'$ *If* $\{A_i : i \in \Omega\} \subseteq \mathbf{2}^X$ *is up-directed by inclusion, then* $\mathscr{C}(\bigcup_{i\in\Omega}^{dir} A_i) \geq \bigwedge_{i\in\Omega} \mathscr{C}(A_i)$.

Definition 2.3. [11] A mapping $f : (X, \mathscr{C}_X) \longrightarrow (Y, \mathscr{C}_Y)$ between M-fuzzifying convex spaces is called a convexity-preserving mapping(a CP mapping) provided that $\mathscr{C}_X(f^{\leftarrow}(B)) \geq \mathscr{C}_Y(B)$ for each $B \in \mathbf{2}^Y$.

Definition 2.4. [11] A mapping $f : (X, \mathscr{C}_X) \longrightarrow (Y, \mathscr{C}_Y)$ between M-fuzzifying convex spaces is called a convex-to-convex mapping (a CC mapping) provided that $\mathscr{C}_X(A) \leq \mathscr{C}_Y(f^{\rightarrow}(A))$ for each $A \in \mathbf{2}^X$.

Definition 2.5. [11] Let $\varphi : \mathbf{2}^X \longrightarrow M$ be a mapping and define $\mathscr{C} : \mathbf{2}^X \longrightarrow M$ by

$$\forall A \in \mathbf{2}^X,\ \mathscr{C}(A) = \bigwedge\{\mathscr{D}(A) : \varphi \leq \mathscr{D} \in \mathfrak{H}\},$$

where $\mathfrak{H}$ denotes the family of all the M-fuzzifying convex structures on X. Then $\mathscr{C}$ is an M-fuzzifying convex structure on X and φ is called a subbase of the M-fuzzifying convex space $(X, \mathscr{C})$. In this case, we say that φ generates the M-fuzzifying convex structure $\mathscr{C}$.

Definition 2.6. [11] Let $(X, \mathscr{C})$ be an M-fuzzifying convex space and $\mathscr{B} : \mathbf{2}^X \longrightarrow M$ be a mapping with $\mathscr{B} \leq \mathscr{C}$. Then $\mathscr{B}$ is called a base of $(X, \mathscr{C})$ provided that

$$\forall A \in \mathbf{2}^X,\ \mathscr{C}(A) = \mathscr{B}^{(\sqcup)}(A),$$

where $\mathscr{B}^{(\sqcup)}(A) = \bigvee_{\bigcup_{\lambda\in\Lambda}^{dir} B_\lambda = A} \bigwedge_{\lambda\in\Lambda} \mathscr{B}(B_\lambda)$.

3. Base Axioms in M-fuzzifying Convex Spaces

In this section, we will provide an axiomatic approach to the concept of bases in M-fuzzifying convex spaces. Then we will establish its relations with M-fuzzifying convex spaces and use it to characterize CP mappings and CC mappings between M-fuzzifying convex spaces.

Theorem 3.1. *Let* $\mathscr{B} : \mathbf{2}^X \longrightarrow M$ *be a mapping satisfying*

(MYB1) $\bigvee_{\bigcup_{\lambda\in\Lambda}^{dir} B_\lambda = X} \bigwedge_{\lambda\in\Lambda} \mathscr{B}(B_\lambda) = \top$, $\mathscr{B}(\emptyset) = \top$;

(MYB2) *If* $\{A_i : i \in \Omega\} \subseteq \mathbf{2}^X$ *is nonempty, then*

$$\bigwedge_{i\in\Omega} \mathscr{B}(A_i) \leq \bigvee_{\bigcup_{\lambda\in\Lambda}^{dir} B_\lambda = \bigcap_{i\in\Omega} A_i} \bigwedge_{\lambda\in\Lambda} \mathscr{B}(B_\lambda);$$

(MYB3) *If* $\{A_i : i \in \Omega\} \subseteq \mathbf{2}^X$ *is up-directed by inclusion, then*

$$\bigwedge_{i\in\Omega} \bigvee_{\bigcup_{j\in J_i}^{dir} B_{i,j} = A_i} \bigwedge_{j\in J_i} \mathscr{B}(B_{i,j}) \leq \bigvee_{\bigcup_{\lambda\in\Lambda}^{dir} B_\lambda = \bigcup_{i\in\Omega}^{dir} A_i} \bigwedge_{\lambda\in\Lambda} \mathscr{B}(B_\lambda).$$

Then there is a unique M-fuzzifying convex structure on X with $\mathscr{B}$ as its base.

Proof. We prove this conclusion in two steps.

Step 1: Define $\mathscr{C} : \mathbf{2}^X \longrightarrow M$ by $\mathscr{C}(A) = \mathscr{B}^{(\sqcup)}(A)$ for each $A \in \mathbf{2}^X$. Next let us verify that $\mathscr{C}$ satisfies (MYC1), (MYC2) and (MYC3)$'$.

(MYC1) By (MYB1), we have

$$\mathscr{C}(\emptyset)=\mathscr{B}^{(\sqcup)}(\emptyset)\geq\mathscr{B}(\emptyset)=\top$$

and

$$\mathscr{C}(X)=\mathscr{B}^{(\sqcup)}(X)=\bigvee_{\bigcup^{dir}_{\lambda\in\Lambda}B_\lambda=X}\bigwedge_{\lambda\in\Lambda}\mathscr{B}(B_\lambda)=\top.$$

(MYC2) For $\{A_i : i\in\Omega\}\subseteq\mathbf{2}^X$, take each $a\in M$ such that

$$a\prec\bigwedge_{i\in\Omega}\mathscr{C}(A_i)=\bigwedge_{i\in\Omega}\mathscr{B}^{(\sqcup)}(A_i)=\bigwedge_{i\in\Omega}\bigvee_{\bigcup^{dir}_{j\in J_i}B_{i,j}=A_i}\bigwedge_{j\in J_i}\mathscr{B}(B_{i,j}).$$

Then for each $i\in\Omega$, there exists $\{B_{i,j} : j\in J_i\}\subseteq\mathbf{2}^X$ such that $\bigcup^{dir}_{j\in J_i}B_{i,j}=A_i$ and for any $j\in J_i$, $\mathscr{B}(B_{i,j})\geq a$. By the completely distributive law, we have $\bigcap_{i\in\Omega}A_i=\bigcap_{i\in\Omega}\bigcup^{dir}_{j\in J_i}B_{i,j}=\bigcup_{f\in\prod_{i\in\Omega}J_i}\bigcap_{i\in\Omega}B_{i,f(i)}$. Obviously, $\mathscr{B}(B_{i,f(i)})\geq a$ for each $f\in\prod_{i\in\Omega}J_i$ and $i\in\Omega$. This implies that $\bigwedge_{f\in\prod_{i\in\Omega}J_i}\bigwedge_{i\in\Omega}\mathscr{B}(B_{i,f(i)})\geq a$. Moreover, it is trivial to check that $\{\bigcap_{i\in\Omega}B_{i,f(i)} : f\in\prod_{i\in\Omega}J_i\}$ is up-directed. Then it follows that $\bigcap_{i\in\Omega}A_i=\bigcup^{dir}_{f\in\prod_{i\in\Omega}J_i}\bigcap_{i\in\Omega}B_{i,f(i)}$. By (MYB2) and (MYB3),

$$\begin{aligned}
a &\leq \bigwedge_{f\in\prod_{i\in\Omega}J_i}\bigwedge_{i\in\Omega}\mathscr{B}(B_{i,f(i)})\\
&\leq \bigwedge_{f\in\prod_{i\in\Omega}J_i}\bigvee_{\bigcup^{dir}_{k\in K_f}C_{f,k}=\bigcap_{i\in\Omega}B_{i,f(i)}}\bigwedge_{k\in K_f}\mathscr{B}(C_{f,k})\\
&\leq \bigvee_{\bigcup^{dir}_{\lambda\in\Lambda}D_\lambda=\bigcup^{dir}_{f\in\prod_{i\in\Omega}J_i}\bigcap_{i\in\Omega}B_{i,f(i)}}\bigwedge_{\lambda\in\Lambda}\mathscr{B}(D_\lambda)\\
&= \bigvee_{\bigcup^{dir}_{\lambda\in\Lambda}D_\lambda=\bigcap_{i\in\Omega}A_i}\bigwedge_{\lambda\in\Lambda}\mathscr{B}(D_\lambda)\\
&= \mathscr{B}^{(\sqcup)}(\bigcap_{i\in\Omega}A_i)\\
&= \mathscr{C}(\bigcap_{i\in\Omega}A_i).
\end{aligned}$$

By the arbitrariness of a, we obtain $\bigwedge_{i\in\Omega}\mathscr{C}(A_i)\leq\mathscr{C}(\bigcap_{i\in\Omega}A_i)$.

(MYC3)$'$ For each up-directed set $\{A_i : i\in\Omega\}\subseteq\mathbf{2}^X$, by (MYB3) and the definition of $\mathscr{C}$, we have

$$\begin{aligned}
\bigwedge_{i\in\Omega}\mathscr{C}(A_i) &= \bigwedge_{i\in\Omega}\bigvee_{\bigcup^{dir}_{j\in J_i}B_{i,j}=A_i}\bigwedge_{j\in J_i}\mathscr{B}(B_{i,j})\\
&\leq \bigvee_{\bigcup^{dir}_{\lambda\in\Lambda}C_\lambda=\bigcup^{dir}_{i\in\Omega}A_i}\bigwedge_{\lambda\in\Lambda}\mathscr{B}(C_\lambda)\\
&= \mathscr{C}(\bigcup^{dir}_{i\in\Omega}A_i).
\end{aligned}$$

This shows $\mathscr{C}$ satisfies (MYC1)-(MYC3). Thus, $\mathscr{C}$ is an M-fuzzifying convex structure on X.

Step 2: Suppose that $\mathscr{C}^*$ is another M-fuzzifying convex structure on X with $\mathscr{B}$ as its base. By Definition 2.6, we know $\mathscr{C}^*=\mathscr{B}^{(\sqcup)}$. This means $\mathscr{C}^*=\mathscr{C}$. □

Conversely, we will show for a given M-fuzzifying convex structure, its base satisfies (MYB1)-(MYB3).

Theorem 3.2. *Let* $(X,\mathscr{C})$ *be an* M*-fuzzifying convex space with* $\mathscr{B}$ *as its base. Then* $\mathscr{B}$ *satisfies* (MYB1)-(MYB3).

Proof. By Definition 2.6, we know $\mathscr{C}=\mathscr{B}^{(\sqcup)}$. Now let us check that $\mathscr{B}$ satisfies (MYB1)-(MBY3).

(MYB1) By (MYC1), it follows that

$$\mathscr{B}(\emptyset)=\mathscr{B}^{(\sqcup)}(\emptyset)=\mathscr{C}(\emptyset)=\top$$

and

$$\bigvee_{\bigcup^{dir}_{\lambda\in\Lambda}B_\lambda=X}\ \bigwedge_{\lambda\in\Lambda}\mathscr{B}(B_\lambda)=\mathscr{B}^{(\sqcup)}(X)\geq\mathscr{C}(X)=\top.$$

(MYB2) For any nonempty set $\{A_i:i\in\Omega\}\subseteq\mathbf{2}^X$, since $\mathscr{B}\leq\mathscr{C}$, it follows that

$$\bigwedge_{i\in\Omega}\mathscr{B}(A_i)\leq\bigwedge_{i\in\Omega}\mathscr{C}(A_i)\leq\mathscr{C}(\bigcap_{i\in\Omega}A_i)=\mathscr{B}^{(\sqcup)}(\bigcap_{i\in\Omega}A_i)=\bigvee_{\bigcup^{dir}_{\lambda\in\Lambda}B_\lambda=\bigcap_{i\in\Omega}A_i}\ \bigwedge_{\lambda\in\Lambda}\mathscr{B}(B_\lambda).$$

(MYB3) For any up-directed set $\{A_i:i\in\Omega\}\subseteq\mathbf{2}^X$, we have

$$\begin{aligned}\bigwedge_{i\in\Omega}\ \bigvee_{\bigcup^{dir}_{j\in J_i}B_{i,j}=A_i}\ \bigwedge_{j\in J_i}\mathscr{B}(B_{i,j}) &\leq \bigwedge_{i\in\Omega}\ \bigvee_{\bigcup^{dir}_{j\in J_i}B_{i,j}=A_i}\ \bigwedge_{j\in J_i}\mathscr{C}(B_{i,j})\\ &\leq \bigwedge_{i\in\Omega}\ \bigvee_{\bigcup^{dir}_{j\in J_i}B_{i,j}=A_i}\mathscr{C}(\textstyle\bigcup^{dir}_{j\in J_i}B_{i,j})\\ &= \bigwedge_{i\in\Omega}\mathscr{C}(A_i)\\ &\leq \mathscr{C}(\bigcup^{dir}_{i\in\Omega}A_i)\\ &= \bigvee_{\bigcup^{dir}_{\lambda\in\Lambda}B_\lambda=\bigcup^{dir}_{i\in\Omega}A_i}\ \bigwedge_{\lambda\in\Lambda}\mathscr{B}(B_\lambda).\end{aligned}$$

□

By Theorems 3.1 and 3.2, we present the following definition.

Definition 3.3. A mapping $\mathscr{B}:\mathbf{2}^X\longrightarrow M$ is called a base of some M-fuzzifying convex space provided that $\mathscr{B}$ satisfies (MYB1)-(MYB3).

Proposition 3.4. *Let* $\mathscr{C}$ *be an* M*-fuzzifying convex structure on* X*. Then* $\mathscr{C}$ *is a base of the* M*-fuzzifying convex space* $(X,\mathscr{C})$.

Proof. If $\mathscr{C}$ be an M-fuzzifying convex structure on X, then we first check that $\mathscr{C}=\mathscr{C}^{(\sqcup)}$. On one hand, $\mathscr{C}^{(\sqcup)}\geq\mathscr{C}$ holds obviously. On the other hand, for each $A\in\mathbf{2}^X$, it follows that

$$\mathscr{C}^{(\sqcup)}(A)=\bigvee_{\bigcup^{dir}_{\lambda\in\Lambda}B_\lambda=A}\ \bigwedge_{\lambda\in\Lambda}\mathscr{C}(B_\lambda)\leq\bigvee_{\bigcup^{dir}_{\lambda\in\Lambda}B_\lambda=A}\mathscr{C}(\bigcup^{dir}_{\lambda\in\Lambda}B_\lambda)=\mathscr{C}(A).$$

This proves that $\mathscr{C}=\mathscr{C}^{(\sqcup)}$. Next we need only show that $\mathscr{C}$ satisfies (MYB1)-(MYB3). (MYB1) and (MYB2) are straightforward.

(MYB3) For any up-directed set $\{A_i : i \in \Omega\} \subseteq \mathbf{2}^X$, we have

$$\begin{aligned}
\bigwedge_{i\in\Omega} \bigvee_{\bigcup^{dir}_{j\in J_i} B_{i,j}=A_i} \bigwedge_{j\in J_i} \mathscr{C}(B_{i,j}) &\leq \bigwedge_{i\in\Omega} \bigvee_{\bigcup^{dir}_{j\in J_i} B_{i,j}=A_i} \mathscr{C}(\bigcup^{dir}_{j\in J_i} B_{i,j}) \\
&= \bigwedge_{i\in\Omega} \mathscr{C}(A_i) \\
&\leq \mathscr{C}(\bigcup^{dir}_{i\in\Omega} A_i) \\
&= \mathscr{C}^{(\sqcup)}(\bigcup^{dir}_{i\in\Omega} A_i) \\
&= \bigvee_{\bigcup^{dir}_{\lambda\in\Lambda} B_\lambda = \bigcup^{dir}_{i\in\Omega} A_i} \bigwedge_{\lambda\in\Lambda} \mathscr{C}(B_\lambda).
\end{aligned}$$

□

Next we will show an M-fuzzifying closure system can be treated as a base for an M-fuzzifying convex space. For this, we first give the following lemma.

Lemma 3.5. *Suppose that* $A = \bigcup^{dir}_{i\in\Omega} A_i = \bigcup^{dir}_{i\in\Omega} \bigcup^{dir}_{j\in J_i} B_{i,j}$. *Define a mapping* $\sigma : \mathbf{2}^A_{fin} \longrightarrow \mathbf{2}^X$ *by*

$$\forall F \in \mathbf{2}^A_{fin},\ \sigma(F) = \bigcap \{B_{i,j} : F \subseteq B_{i,j}\}.$$

Then $\{\sigma(F) : F \in \mathbf{2}^A_{fin}\}$ *is up-directed and* $A = \bigcup\{\sigma(F) : F \in \mathbf{2}^A_{fin}\}$.

Proof. Firstly, we check the rationality of the definition of σ. For each $F \in \mathbf{2}^A_{fin}$, by the up-directness, there must be some $B_{i,j}$ such that $F \subseteq B_{i,j}$. This means $\{B_{i,j} : F \subseteq B_{i,j}\}$ is nonempty. Thus, the mapping σ is well defined.

Secondly, we show that $\mathfrak{D} = \{\sigma(F) : F \in \mathbf{2}^A_{fin}\}$ is up-directed. By the definition of σ, it is easy to see that σ is order-preserving. Take $\sigma(F_1), \sigma(F_2) \in \mathfrak{D}$. Since $\mathbf{2}^A_{fin}$ is directed, there exists $F_3 \in \mathbf{2}^A_{fin}$ such that $F_1 \subseteq F_3$ and $F_2 \subseteq F_3$. This implies that $\sigma(F_1) \subseteq \sigma(F_3)$ and $\sigma(F_2) \subseteq \sigma(F_3)$. Thus, $\mathfrak{D} = \{\sigma(F) : F \in \mathbf{2}^A_{fin}\}$ is up-directed.

Finally, we verify that $A = \bigcup\{\sigma(F) : F \in \mathbf{2}^A_{fin}\}$. By the definition of σ, it is easy to see that

$$\bigcup\{\sigma(F) : F \in \mathbf{2}^A_{fin}\} \subseteq \bigcup^{dir}_{i\in\Omega} \bigcup^{dir}_{j\in J_i} B_{i,j} = A.$$

Further, since $F \subseteq \sigma(F)$, it follows that

$$A = \bigcup\{F : F \in \mathbf{2}^A_{fin}\} \subseteq \bigcup\{\sigma(F) : F \in \mathbf{2}^A_{fin}\}.$$

Therefore, we obtain $A = \bigcup\{\sigma(F) : F \in \mathbf{2}^A_{fin}\}$. □

Proposition 3.6. *Let* $\mathscr{B} : \mathbf{2}^X \longrightarrow M$ *be an* M*-fuzzifying closure system on* X. *Then* $\mathscr{B}$ *is a base of some* M*-fuzzifying convex space.*

Proof. It suffices to show that $\mathscr{B}$ satisfies (MYB1)-(MYB3).

(MYB1) Since $\mathscr{B}$ satisfies (MYC1), we have $\mathscr{B}(\emptyset) = \top$ and

$$\bigvee_{\bigcup_{\lambda\in\Lambda}^{dir} B_\lambda = X} \bigwedge_{\lambda\in\Lambda} \mathscr{B}(B_\lambda) \geq \mathscr{B}^{(\sqcup)}(X) = \top.$$

(MYB2) For each nonempty set $\{A_i : i \in \Omega\} \subseteq \mathbf{2}^X$, by (MYC2), we have

$$\bigwedge_{i\in\Omega} \mathscr{B}(A_i) \leq \mathscr{B}(\bigcap_{i\in\Omega} A_i) \leq \mathscr{B}^{(\sqcup)}(\bigcap_{i\in\Omega} A_i) = \bigvee_{\bigcup_{\lambda\in\Lambda}^{dir} B_\lambda = \bigcap_{i\in\Omega} A_i} \bigwedge_{\lambda\in\Lambda} \mathscr{B}(B_\lambda).$$

(MYB3) For each up-directed set $\{A_i : i \in \Omega\} \subseteq \mathbf{2}^X$, let $a \in M$ such that

$$a \prec \bigwedge_{i\in\Omega} \bigvee_{\bigcup_{j\in J_i}^{dir} B_{i,j} = A_i} \bigwedge_{j\in J_i} \mathscr{B}(B_{i,j}).$$

Then for any $i \in \Omega$, there exists $\{B_{i,j} : j \in J_i\} \subseteq \mathbf{2}^X$ such that $\bigcup_{j\in J_i}^{dir} B_{i,j} = A_i$ and for each $j \in J_i$, $\mathscr{B}(B_{i,j}) \geq a$. Let $A = \bigcup_{i\in\Omega}^{dir} A_i = \bigcup_{i\in\Omega}^{dir} \bigcup_{j\in J_i}^{dir} B_{i,j}$. Define a mapping $\sigma : \mathbf{2}^A_{fin} \longrightarrow \mathbf{2}^X$ as follows:

$$\forall F \in \mathbf{2}^A_{fin},\ \sigma(F) = \bigcap\{B_{i,j} : F \subseteq B_{i,j}\}.$$

By Lemma 3.5, we know $\{\sigma(F) : F \in \mathbf{2}^A_{fin}\}$ is up-directed and $A = \bigcup\{\sigma(F) : F \in \mathbf{2}^A_{fin}\}$. For each $F \in \mathbf{2}^A_{fin}$, by (MYC2), it follows that

$$a \leq \bigwedge_{F\subseteq B_{i,j}} \mathscr{B}(B_{i,j}) \leq \mathscr{B}(\bigcap_{F\subseteq B_{i,j}} B_{i,j}) = \mathscr{B}(\sigma(F)).$$

This implies that

$$\bigvee_{\bigcup_{\lambda\in\Lambda}^{dir} B_\lambda = \bigcup_{i\in\Omega}^{dir} A_i} \bigwedge_{\lambda\in\Lambda} \mathscr{B}(B_\lambda) \geq a.$$

By the arbitrariness of a, we obtain

$$\bigwedge_{i\in\Omega} \bigvee_{\bigcup_{j\in J_i}^{dir} B_{i,j} = A_i} \bigwedge_{j\in J_i} \mathscr{B}(B_{i,j}) \leq \bigvee_{\bigcup_{\lambda\in\Lambda}^{dir} B_\lambda = \bigcup_{i\in\Omega}^{dir} A_i} \bigwedge_{\lambda\in\Lambda} \mathscr{B}(B_\lambda),$$

as desired. □

Next we will aim to characterize Cp mappings and CC mappings between M-fuzzifying convex spaces by using bases of M-fuzzifying convex spaces.

Proposition 3.7. *Let $f : (X, \mathscr{C}_X) \longrightarrow (Y, \mathscr{C}_Y)$ be a mapping between M-fuzzifying convex spaces and $\mathscr{B}_Y$ be a base of $(Y, \mathscr{C}_Y)$. Then $f : (X, \mathscr{C}_X) \longrightarrow (Y, \mathscr{C}_Y)$ is a CP mapping if and only if $\mathscr{C}_X(f^{\leftarrow}(B)) \geq \mathscr{B}_Y(B)$ for each $B \in \mathbf{2}^Y$.*

Proof. Necessity. Suppose that $f : (X, \mathscr{C}_X) \longrightarrow (Y, \mathscr{C}_Y)$ is a CP mapping. Since $\mathscr{C}_Y \geq \mathscr{B}_Y$, it follows that for each $B \in \mathbf{2}^Y$,

$$\mathscr{C}_X(f^{\leftarrow}(B)) \geq \mathscr{C}_Y(B) \geq \mathscr{B}_Y(B).$$

Sufficiency. Since $\mathscr{B}_Y$ is a base of $(Y, \mathscr{C}_Y)$, we have for each $B \in \mathbf{2}^Y$,

$$\mathscr{C}_Y(B) = \bigvee_{\bigcup_{\lambda\in\Lambda}^{dir} B_\lambda = B} \bigwedge_{\lambda\in\Lambda} \mathscr{B}_Y(B_\lambda).$$

Then it follows that

$$\begin{aligned} \mathscr{C}_Y(B) &= \bigvee_{\bigcup_{\lambda\in\Lambda}^{dir} B_\lambda = B} \bigwedge_{\lambda\in\Lambda} \mathscr{B}_Y(B_\lambda) \\ &\leq \bigvee_{\bigcup_{\lambda\in\Lambda}^{dir} B_\lambda = B} \bigwedge_{\lambda\in\Lambda} \mathscr{C}_X(f^{\leftarrow}(B_\lambda)) \\ &= \bigvee_{\bigcup_{\lambda\in\Lambda}^{dir} B_\lambda = B} \mathscr{C}_X(\bigcup_{\lambda\in\Lambda}^{dir} f^{\leftarrow}(B_\lambda)) \\ &= \bigvee_{\bigcup_{\lambda\in\Lambda}^{dir} B_\lambda = B} \mathscr{C}_X(f^{\leftarrow}(\bigcup_{\lambda\in\Lambda}^{dir} B_\lambda)) \\ &= \mathscr{C}_X(f^{\leftarrow}(B)). \end{aligned}$$

This shows that $f : (X, \mathscr{C}_X) \longrightarrow (Y, \mathscr{C}_Y)$ is a CP mapping. □

Proposition 3.8. *Let $f : (X, \mathscr{C}_X) \longrightarrow (Y, \mathscr{C}_Y)$ be a mapping between M-fuzzifying convex spaces and $\mathscr{B}_X$ be a base of $(X, \mathscr{C}_X)$. Then $f : (X, \mathscr{C}_X) \longrightarrow (Y, \mathscr{C}_Y)$ is a CC mapping if and only if $\mathscr{B}_X(A) \leq \mathscr{C}_Y(f^{\rightarrow}(A))$ for each $A \in \mathbf{2}^X$.*

Proof. Necessity. Suppose that $f : (X, \mathscr{C}_X) \longrightarrow (Y, \mathscr{C}_Y)$ is a CC mapping. Since $\mathscr{B}_X \leq \mathscr{C}_X$, it follows that for each $A \in \mathbf{2}^X$,

$$\mathscr{B}_X(A) \leq \mathscr{C}_X(A) \leq \mathscr{C}_Y(f^{\rightarrow}(A)).$$

Sufficiency. Since $\mathscr{B}_X$ is a base of $(X, \mathscr{C}_X)$, we have for each $A \in \mathbf{2}^X$,

$$\mathscr{C}_X(A) = \bigvee_{\bigcup_{\lambda\in\Lambda}^{dir} A_\lambda = A} \bigwedge_{\lambda\in\Lambda} \mathscr{B}_X(A_\lambda).$$

Then it follows that

$$\begin{aligned} \mathscr{C}_X(A) &= \bigvee_{\bigcup_{\lambda\in\Lambda}^{dir} A_\lambda = A} \bigwedge_{\lambda\in\Lambda} \mathscr{B}_X(A_\lambda) \\ &\leq \bigvee_{\bigcup_{\lambda\in\Lambda}^{dir} A_\lambda = A} \bigwedge_{\lambda\in\Lambda} \mathscr{C}_Y(f^{\rightarrow}(A_\lambda)) \\ &= \bigvee_{\bigcup_{\lambda\in\Lambda}^{dir} A_\lambda = A} \mathscr{C}_Y(\bigcup_{\lambda\in\Lambda}^{dir} f^{\rightarrow}(A_\lambda)) \\ &= \bigvee_{\bigcup_{\lambda\in\Lambda}^{dir} A_\lambda = A} \mathscr{C}_Y(f^{\rightarrow}(\bigcup_{\lambda\in\Lambda}^{dir} A_\lambda)) \\ &= \mathscr{C}_Y(f^{\rightarrow}(A)). \end{aligned}$$

This shows that $f:(X,\mathscr{C}_X)\longrightarrow(Y,\mathscr{C}_Y)$ is a CC mapping. □

4. Subbase Axioms in M-fuzzifying Convex Spaces

In Definition 2.5, it is easy to see each mapping φ without any limitations can be treated as a subbase of some M-fuzzifying convex spaces. This seems to be abnormal. In this section, we will redefine subbases of an M-fuzzifying convex space and provided an axiomatic approach to the definition of subbases in M-fuzzifying convex spaces. Then we will investigate the relations between axiomatic subbases and M-fuzzifying convex spaces. Moreover, we will present a characterization of CP mappings between M-fuzzifying convex spaces by means of axiomatic subbases.

Definition 4.1. Let $(X,\mathscr{C})$ be an M-fuzzifying convex space and $\varphi:2^X\longrightarrow M$ be a mapping. Then φ is called a subbase of $\mathscr{C}$ provided that $\mathscr{B}:2^X\longrightarrow M$ defined by

$$\forall B\in 2^X,\ \mathscr{B}(B)=\bigvee_{\bigcap_{i\in\Omega}B_i=B}\bigwedge_{i\in\Omega}\varphi(B_i)$$

is a base of $\mathscr{C}$.

Remark 4.2. Definition 4.1 implies Definition 2.5. Suppose that φ is a subbase of $(X,\mathscr{C})$ in the sense of Definition 4.1, then $\mathscr{B}$ defined in Definition 4.1 is a base of $\mathscr{C}$. Now let us show that φ is a subbase of $(X,\mathscr{C})$ in the sense of Definition 2.5. Take each M-fuzzifying convex space $(X,\mathscr{D})$ with $\varphi\leq\mathscr{D}$. Then for each $B\in 2^X$, it follows that

$$\mathscr{B}(B)=\bigvee_{\bigcap_{i\in\Omega}B_i=B}\bigwedge_{i\in\Omega}\varphi(B_i)\leq\bigvee_{\bigcap_{i\in\Omega}B_i=B}\bigwedge_{i\in\Omega}\mathscr{D}(B_i)\leq\bigvee_{\bigcap_{i\in\Omega}B_i=B}\mathscr{D}(\bigcap_{i\in\Omega}B_i)=\mathscr{D}(B).$$

and further for each $A\in 2^X$,

$$\mathscr{C}(A)=\bigvee_{\bigcup^{dir}_{j\in J}A_j=A}\bigwedge_{j\in J}\mathscr{B}(A_j)\leq\bigvee_{\bigcup^{dir}_{j\in J}A_j=A}\bigwedge_{j\in J}\mathscr{D}(A_j)\leq\bigvee_{\bigcup^{dir}_{j\in J}A_j=A}\mathscr{D}(\bigcup^{dir}_{j\in J}A_j)=\mathscr{D}(A).$$

By the arbitrariness of $\mathscr{D}$, we know φ is a subbase of $(X,\mathscr{C})$ in the sense of Definition 2.5.

In the sequel, we will adopt Definition 4.1 as the definition of subbase of M-fuzzifying convex spaces.

Theorem 4.3. *Let $\varphi:\mathbf{2}^X\longrightarrow M$ be a mapping satisfying*

(MYSB1) $\bigvee_{\bigcap_{i\in\Omega}A_i=\emptyset}\bigwedge_{i\in\Omega}\varphi(A_i)=\top$,

(MYSB2) $\bigvee_{\bigcup^{dir}_{j\in J}A_j=X}\bigwedge_{j\in J}\bigvee_{\bigcap_{i\in I_j}A_{j,i}=A_j}\bigwedge_{i\in I_j}\varphi(A_{j,i})=\top$.

Then there is a unique M-fuzzifying convex structure on X with φ as its subbase.

Proof. Define $\mathscr{B}:\mathbf{2}^X\longrightarrow M$ as follows:

$$\forall B\in\mathbf{2}^X,\ \mathscr{B}(B)=\bigvee_{\bigcap_{i\in\Omega}B_i=B}\bigwedge_{i\in\Omega}\varphi(B_i).$$

Then it suffices to show that $\mathscr{B}$ satisfies (MYB1)-(MYB3).

(MYB1) By (MYSB1) and (MYSB2), it is straightforward.

(MYB2) For each nonempty $\{B_i : i \in \Omega\} \subseteq \mathbf{2}^X$, let a be any element in M with the property of

$$a \prec \bigwedge_{i\in\Omega} \mathcal{B}(B_i) = \bigwedge_{i\in\Omega} \bigvee_{\bigcap_{j\in J_i} G_{i,j}=B_i} \bigwedge_{j\in J_i} \varphi(G_{i,j}).$$

Then for each $i \in \Omega$, there exists a set $\{G_{i,j} : j \in J_i\} \subseteq \mathbf{2}^X$ such that $\bigcap\{G_{i,j} : j \in J_i\} = B_i$ and for each $j \in J_i$, $\varphi(G_{i,j}) \geq a$. Then it follows that

$$\bigcap_{i\in\Omega} B_i = \bigcap_{i\in\Omega} \bigcap_{j\in J_i} G_{i,j} = \bigcap\{G_{i,j} : i \in \Omega, j \in J_i\}.$$

Let $\{G_{i,j} : i \in \Omega, j \in J_i\} = \{B_\lambda : \lambda \in \Lambda\}$. Then $\bigcap_{i\in\Omega} B_i = \bigcap_{\lambda\in\Lambda} B_\lambda$ and $\varphi(B_\lambda) \geq a$ for each $\lambda \in \Lambda$. This implies that

$$\begin{aligned} & \bigvee_{\bigcup^{dir}_{\lambda\in\Lambda} D_\lambda=\bigcap_{i\in\Omega} B_i} \bigwedge_{\lambda\in\Lambda} \mathscr{B}(D_\lambda) \\ \geq\ & \mathscr{B}(\bigcap_{i\in\Omega} B_i) = \bigvee_{\bigcap_{\lambda\in\Lambda} C_\lambda=\bigcap_{i\in\Omega} B_i} \bigwedge_{\lambda\in\Lambda} \varphi(C_\lambda) \\ \geq\ & \bigwedge_{\lambda\in\Lambda} \varphi(B_\lambda) \geq a. \end{aligned}$$

By the arbitrariness of a, we obtain $\bigwedge_{i\in\Omega} \mathscr{B}(A_i) \leq \bigvee_{\bigcup^{dir}_{\lambda\in\Lambda} B_\lambda=\bigcap_{i\in\Omega} B_i} \bigwedge_{\lambda\in\Lambda} \mathscr{B}(B_\lambda)$.

(MYB3) For any up-directed set $\{A_i : i \in \Omega\} \subseteq \mathbf{2}^X$, we need to prove that

$$\bigwedge_{i\in\Omega} \bigvee_{\bigcup^{dir}_{j\in J_i} B_{i,j}=A_i} \bigwedge_{j\in J_i} \mathscr{B}(B_{i,j}) \leq \bigvee_{\bigcup^{dir}_{\lambda\in\Lambda} B_\lambda=\bigcup^{dir}_{i\in\Omega} A_i} \bigwedge_{\lambda\in\Lambda} \mathscr{B}(B_\lambda).$$

That is,

$$\begin{aligned} & \bigwedge_{i\in\Omega} \bigvee_{\bigcup^{dir}_{j\in J_i} B_{i,j}=A_i} \bigwedge_{j\in J_i} \bigvee_{\bigcap_{k\in K_{i,j}} G_{i,j,k}=B_{i,j}} \bigwedge_{k\in K_{i,j}} \varphi(G_{i,j,k}) \\ \leq\ & \bigvee_{\bigcup^{dir}_{\lambda\in\Lambda} B_\lambda=\bigcup^{dir}_{i\in\Omega} A_i} \bigwedge_{\lambda\in\Lambda} \bigvee_{\bigcap_{\gamma\in\Gamma_\lambda} D_{\lambda,\gamma}=B_\lambda} \bigwedge_{\gamma\in\Gamma_\lambda} \varphi(D_{\lambda,\gamma}). \end{aligned}$$

Let a be any element in M with the property of

$$a \prec \bigwedge_{i\in\Omega} \bigvee_{\bigcup^{dir}_{j\in J_i} B_{i,j}=A_i} \bigwedge_{j\in J_i} \bigvee_{\bigcap_{k\in K_{i,j}} G_{i,j,k}=B_{i,j}} \bigwedge_{k\in K_{i,j}} \varphi(G_{i,j,k}).$$

Then for each $i \in \Omega$, there exists an up-directed set $\{B_{i,j} : j \in J_i\}$ with $\bigcup^{dir}_{j\in J_i} B_{i,j} = A_i$ and for each $j \in J_i$, there exist a set $\{G_{i,j,k} : k \in K_{i,j}\}$ with $\bigcap_{k\in K_{i,j}} G_{i,j,k} = B_{i,j}$ such that $\varphi(G_{i,j,k}) \geq a$ for each $k \in K_{i,j}$. Let

$$A = \bigcup^{dir}_{i\in\Omega} A_i = \bigcup^{dir}_{i\in\Omega} \bigcup^{dir}_{j\in J_i} B_{i,j} = \bigcup^{dir}_{i\in\Omega} \bigcup^{dir}_{j\in J_i} \bigcap_{k\in K_{i,j}} G_{i,j,k}.$$

Then define a mapping $\sigma : \mathbf{2}^A_{fin} \longrightarrow \mathbf{2}^X$ as follows:

$$\forall F \in \mathbf{2}^A_{fin}, \quad \sigma(F) = \bigcap\{B_{i,j} : F \subseteq B_{i,j}\}.$$

By Lemma 3.5, we know $\{\sigma(F) : F \in \mathbf{2}^A_{fin}\}$ is up-directed and $A = \bigcup\{\sigma(F) : F \in \mathbf{2}^A_{fin}\}$. Moreover, take each $F \in \mathbf{2}^A_{fin}$. It follows that

$$\sigma(F) = \bigcap_{F\subseteq B_{i,j}} B_{i,j} = \bigcap_{F\subseteq B_{i,j}} \bigcap_{k\in K_{i,j}} G_{i,j,k} = \bigcap_{F\subseteq B_{i,j}, k\in K_{i,j}} G_{i,j,k}.$$

Since $\bigcup^{dir}_{F\in\mathbf{2}^A_{fin}} \sigma(F) = \bigcup^{dir}_{i\in\Omega} A_i$, we have

$$\begin{aligned}
& \bigvee_{\bigcup^{dir}_{\lambda\in\Lambda} B_\lambda = \bigcup^{dir}_{i\in\Omega} A_i} \bigwedge\nolimits_{\lambda\in\Lambda} \mathscr{B}(B_\lambda) \\
= & \bigvee_{\bigcup^{dir}_{\lambda\in\Lambda} B_\lambda = \bigcup^{dir}_{i\in\Omega} A_i} \bigwedge\nolimits_{\lambda\in\Lambda} \bigvee_{\bigcap_{\gamma\in\Gamma_\lambda} D_{\lambda,\gamma} = B_\lambda} \bigwedge_{\gamma\in\Gamma_\lambda} \varphi(D_{\lambda,\gamma}) \\
\geq & \bigwedge_{F\in\mathbf{2}^A_{fin}} \bigvee_{\bigcap_{F\subseteq B_{i,j}, k\in K_{i,j}} G_{i,j,k} = \sigma(F)} \bigwedge_{k\in K_{i,j}} \varphi(G_{i,j,k}) \\
\geq & \ a.
\end{aligned}$$

By the arbitrariness of a, we obtain

$$\bigwedge_{i\in\Omega} \bigvee_{\bigcup^{dir}_{j\in J_i} B_{i,j} = A_i} \bigwedge_{j\in J_i} \mathscr{B}(B_{i,j}) \leq \bigvee_{\bigcup^{dir}_{\lambda\in\Lambda} B_\lambda = \bigcup^{dir}_{i\in\Omega} A_i} \bigwedge_{\lambda\in\Lambda} \mathscr{B}(B_\lambda).$$

By Theorem 3.1, there is a unique M-fuzzifying convex structure $\mathscr{C} = \mathscr{B}^{(\sqcup)}$ with $\mathscr{B}$ as its base. That is, there is a unique M-fuzzifying convex structure $\mathscr{C}$ with φ as its subbase, as desired. □

Theorem 4.4. *Let $(X, \mathscr{C})$ be an M-fuzzifying convex space with φ as its subbase. Then φ satisfies* (MYSB1) *and* (MYSB2).

Proof. By Definition 4.1, it is straightforward and the proof is omitted. □

By Theorems 4.3 and 4.4, we present the axiomatic definition of subbases of M-fuzzifying convex spaces.

Definition 4.5. A mapping $\varphi : \mathbf{2}^X \longrightarrow M$ is called a subbase of some M-fuzzifying convex space provided that φ satisfies (MYSB1) and (MYSB2).

The following result is obvious and the proof is omitted.

Proposition 4.6. (1) *If $(X, \mathscr{C})$ is an M-fuzzifying convex space, then $\mathscr{C}$ is a subbase of $(X, \mathscr{C})$.*

(2) *If $(X, \mathscr{C})$ is an M-fuzzifying convex space with $\mathscr{B}$ as its base, then $\mathscr{B}$ is a subbase of $(X, \mathscr{C})$.*

Next we will characterize CP mappings between M-fuzzifying convex spaces by means of subbases of M-fuzzifying convex spaces.

Proposition 4.7. *Let $f : (X, \mathscr{C}_X) \longrightarrow (Y, \mathscr{C}_Y)$ be a mapping between M-fuzzifying convex spaces and φ_Y be a subbase of $(Y, \mathscr{C}_Y)$. Then $f : (X, \mathscr{C}_X) \longrightarrow (Y, \mathscr{C}_Y)$ is a CP mapping if and only if $\mathscr{C}_X(f^{\leftarrow}(B)) \geq \varphi_Y(B)$ for each $B \in \mathbf{2}^Y$.*

Proof. Since φ_Y is a subbase of $(Y, \mathscr{C}_Y)$, we know $\mathscr{B}_Y : \mathbf{2}^Y \longrightarrow M$ defined by

$$\forall B \in \mathbf{2}^Y,\ \mathscr{B}_Y(B) = \bigvee_{\bigcap_{i\in\Omega} B_i = B} \bigwedge_{i\in\Omega} \varphi_Y(B_i)$$

is a base of $\mathscr{C}_Y$. Next let us show the necessity and sufficiency.

Necessity. It follows from the definition of $\mathscr{B}_Y$ that $\mathscr{C}_Y(B) \geq \mathscr{B}_Y(B) \geq \varphi_Y(B)$ for each $B \in \mathbf{2}^Y$. Further, since $f : (X, \mathscr{C}_X) \longrightarrow (Y, \mathscr{C}_Y)$ is a CP mapping, it follows that for each $B \in \mathbf{2}^Y$, $\mathscr{C}_X(f^{\leftarrow}(B)) \geq \mathscr{C}_Y(B)$. This implies that $\mathscr{C}_X(f^{\leftarrow}(B)) \geq \varphi_Y(B)$ for each $B \in \mathbf{2}^Y$.

Sufficiency. Take each $B \in \mathbf{2}^Y$. Then

$$\begin{aligned}
\mathscr{B}_Y(B) &= \bigvee_{\bigcap_{i\in\Omega} B_i = B} \bigwedge_{i\in\Omega} \varphi_Y(B_i) \\
&\leq \bigvee_{\bigcap_{i\in\Omega} B_i = B} \bigwedge_{i\in\Omega} \mathscr{C}_X(f^{\leftarrow}(B_i)) \\
&= \bigvee_{\bigcap_{i\in\Omega} B_i = B} \mathscr{C}_X(\bigcap_{i\in\Omega} f^{\leftarrow}(B_i)) \\
&= \bigvee_{\bigcap_{i\in\Omega} B_i = B} \mathscr{C}_X(f^{\leftarrow}(\bigcap_{i\in\Omega} B_i)) \\
&= \mathscr{C}_X(f^{\leftarrow}(B)).
\end{aligned}$$

Since $\mathscr{B}_Y$ is a base of $(Y, \mathscr{C}_Y)$, it follows from Proposition 3.7 that $f : (X, \mathscr{C}_X) \longrightarrow (Y, \mathscr{C}_Y)$ is a CP mapping. □

5. **Conclusions**

In this paper, we provided an axiomatic approach to bases and subbases in M-fuzzifying convex spaces. Concretely, we gave the axiomatic conditions to define bases and subbases of M-fuzzifying convex spaces. From the axiomatic bases and subbases, we can induce a unique M-fuzzifying convex structure. Also, we show that bases can be used to characterize both CP mappings and CC mappings, and subbases can be used to characterize CP mappings.

In the theory of classical convex structures, join spaces and product spaces are both defined by means of subbases. Following the subbases in this paper, we will consider join spaces and product spaces in the framework of M-fuzzifying convex spaces in the future.

Acknowledgements. The authors would like to express their sincere thanks to the anonymous reviewers and the area editor for their careful reading and constructive comments. This work is supported by the National Natural Science Foundation of China (11701122), Beijing Institute of Technology Research Fund Program for Young Scholars and the Scientific Research Foundation of CUIT (KYTZ201631,CRF201611,17ZB0093).

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