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SOME PROPERTIES OF UNCERTAIN INTEGRAL

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Abstract. Uncertainty theory is a mathematical methodology for dealing with non-determinate phenomena in nature. As we all know, uncertain process and uncertain integral are important contents of uncertainty theory, so it is necessary to have deep research. This paper presents the definition and discusses some properties of strong comonotonic uncertain process. Besides, some useful formulas of uncertain integral such as nonnegativity, monotonicity, intermediate results are studied.

1. **Introduction**

In the world, there are plenty of problems that are difficult to be described exactly. Randomness, fuzziness and uncertainty appear frequently in a system. To solve these problems, probability theory, fuzzy mathematics and rough set theory produced one after another. In 1933, Kolmogorov founded probability theory to settle random matters, which has been applied widely in many diverse subject areas. In order to describe fuzzy phenomena, the definition of fuzzy set was introduced by Zadeh [28] in 1965. Then Zadeh [29] presented possibility measure to deal with fuzzy events, Yager [21] and Dubois [2] have dedicated themselves in this study. However, possibility measure has no self duality, thus Liu and Liu [5] initiated a self duality credibility measure in 2002. As a generalization of randomness and fuzziness, the uncertainties of imprecise events can be considered by uncertainty theory which was given by Liu [6] for the first time and refined by Liu [10]. Thereafter, many researchers have conducted the thorough research and plenty of explorations were undertaken. In 2007, Liu [6] introduced the concepts of uncertain variable and uncertain distribution. In 2009, Gao [3] studied the properties of continuous uncertain measure. The convergence of uncertain sequences was discussed by You [23]. You and Yan [27] introduced the p-distance between two uncertain variables and the relationships among different convergences were discussed by You and Yan [24]. Prior to today, uncertainty theory has developed into an axiomatic system, which has been applied to uncertain programming ([8], [12, 13]), uncertain risk and uncertain reliability ([11], [17]), and etc.

In order to describe dynamic system with different uncertainties, stochastic process, fuzzy process, uncertain process (initiated by Liu [7]) were introduced, respectively. As we know, many practical matters can be abstracted as a differential equation. And to solve the equation, corresponding integral needs to be calculated.

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Classic integration and differentiation have been widely studied and used in many areas of natural and social sciences. The reader interested to stochastic differential and integral may consult [16, 19]. As for fuzzy integrals, there are Choquet fuzzy integral and Sugeno fuzzy integral (see [20], [18], [4]), which are all integrals with respect to variable, another type of fuzzy integral, an integral of a fuzzy process with respect to a fuzzy process was presented by Liu [7]. Then it was extended to the case of complex fuzzy integral (You and Wang [25]) and general fuzzy integral (You, Ma and Huo [26]). As a generalization of stochastic integral and fuzzy integral, uncertain integral was introduced by Liu [7]. Later, Liu [15] deduced the linear property of uncertain integral and presented the fundamental theorem of uncertain calculus, then the formulas of chain rule, change of variables, and integration by parts were derived. Applying uncertain integral, uncertain differential equation has been studied by many researches (see [7], [1], [9]).

Considering integration and differentiation play very important roles in the fundamental theory of mathematics, the main purpose of this paper is to study the properties of uncertain integral. The rest of this paper is organized as follows. Section 2 will recall some basic definitions and properties of uncertain process and uncertain integral. Then the concept of strong comonotonic uncertain process will be introduced in Section 3, furthermore, some properties of strong comonotonic uncertain process and uncertain integral will be discussed. In the last section, a brief conclusion is given.

2. **Preliminaries**

In this section, some basic knowledge which will be used in this paper are reviewed.

In order to measure uncertain event, uncertain measure and uncertain variable were defined.

Definition 2.1. (Liu [6]) Let $\mathcal{L}$ be a σ-algebra on a nonempty set Γ. A set function $\mathcal{M}$: $\Gamma \to [0,1]$ is called an uncertain measure if it satisfies the following axioms:

Axiom 1: (Normality Axiom) $\mathcal{M}\{\Gamma\} = 1$.

Axiom 2: (Duality Axiom) $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$, for any event $\Lambda \subset \Gamma$.

Axiom 3: (Subadditivity Axiom) For every countable sequence of events $\Lambda_1, \Lambda_2, \cdots \subset \Gamma$, we have

$$\mathcal{M}\{\bigcup_{i=1}^{\infty} \Lambda_i\} \leq \sum_{i=1}^{\infty} \mathcal{M}\{\Lambda_i\}.$$

Definition 2.2. (Liu [6]) Let Γ be a nonempty set, $\mathcal{L}$ a σ-algebra over Γ, and $\mathcal{M}$ an uncertain measure. Then the triplet $(\Gamma, \mathcal{L}, \mathcal{M})$ is called an uncertainty space.

Besides, the product uncertain measure $\mathcal{M}$ on the product σ-algebra $\mathcal{L}$ is defined by the following product axiom.

Axiom 4: (Product Axiom, Liu [9]) Let $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ be uncertainty spaces, for $k = 1, 2, \cdots$. The product uncertain measure $\mathcal{M}$ is an uncertain measure satisfying

$$\mathcal{M}\{\prod_{k=1}^{\infty} \Lambda_k\} = \bigwedge_{k=1}^{\infty} \mathcal{M}_k\{\Lambda_k\},$$

where Λ_k are arbitrarily chosen events from $\mathcal{L}_k$, for $k = 1, 2, \cdots$, respectively.

Definition 2.3. (Liu [6]) An uncertain variable ξ is a measurable function from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers, i.e., for any Borel set B of real numbers, the set

$$\{\xi \in B\} = \{\gamma \in \Gamma | \xi(\gamma) \in B\}$$

is an event.

Definition 2.4. (Liu [6]) The uncertain distribution Φ of an uncertain variable ξ is defined by

$$\Phi(x) = \mathcal{M}\{\xi \le x\},$$

for any real number x.

Next, we recall some knowledge of uncertain process and Liu integral as follows.

Definition 2.5. (Liu [7]) Let T be an index set and let $(\Gamma, \mathcal{L}, \mathcal{M})$ be an uncertainty space. An uncertain process is a function $X_t(\gamma)$ from $T \times (\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers such that $\{X_t \in B\}$ is an event for any Borel set B of real numbers at each time t .

Definition 2.6. (Liu [14]) The uncertainty distribution $\Phi_t(x)$ of an uncertain process X_t is defined as

$$\Phi_t(x) = \mathcal{M}\{X_t \le x\}$$

for each time t and any number x.

Theorem 2.7. *(Liu* [15]*) Let X_t be an uncertain process with uncertainty distribution $\Phi_t(x)$, and let $f(x)$ be a measurable function. Then $f(X_t)$ is also an uncertain process.*

Furthermore, (i) if $f(x)$ is a strictly increasing function, then $f(X_t)$ has an uncertainty distribution

$$\Psi_t(x) = \Phi_t(f^{-1}(x));$$

(ii) if $f(x)$ is a strictly decreasing function and $\Phi_t(x)$ is continuous with respect to x, then $f(X_t)$ has an uncertainty distribution

$$\Psi_t(x) = 1 - \Phi_t(f^{-1}(x)).$$

Definition 2.8. (Liu [7]) Let X_t be an uncertain process. Then for each $\gamma \in \Gamma$, the function $X_t(\gamma)$ is called a sample path of X_t.

Definition 2.9. (Liu [9]) An uncertain process C_t is said to be a Liu process if

(i) $C_t = 0$ and almost all sample paths are Lipschitz continuous,

(ii) C_t has stationary and independent increments,

(iii) every increment $C_{t+s} - C_t$ is a normal uncertain variable with expected value 0 and variance t^2.

Definition 2.10. (Liu [9]) Let X_t be an uncertain process and let C_t be a Liu process. For any partition of closed interval $[a, b]$ with $a = t_1 < t_2 < \cdots < t_{k+1} = b$, the mesh is written as

$$\Delta = \max_{1 \le i \le k} | t_{i+1} - t_i | .$$

Then Liu integral of X_t with respect to C_t is defined as

$$\int_a^b X_t \mathrm{d}C_t = \lim_{\Delta\to 0}\sum_{i=1}^{k} X_{t_i}\cdot(C_{t_{i+1}} - C_{t_i}),$$

provided that the limit exists almost surely and is finite. In this case, the uncertain process X_t is said to be integrable.

Theorem 2.11. *(Liu [15]) If X_t is an integrable uncertain process on $[a,b]$, then it is integrable on each subinterval of $[a,b]$. Moreover, if $c\in[a,b]$, then*

$$\int_a^b X_t \mathrm{d}C_t = \int_a^c X_t \mathrm{d}C_t + \int_c^b X_t \mathrm{d}C_t.$$

3. The Properties of Uncertain Integral

Some concepts and theorems about Liu integral will be discussed in this section. First, we introduce the definition of strong comonotonic uncertain process.

Definition 3.1. Let X_t be a continuous and bounded uncertain process, C_t a Liu process. If X_t and C_t have the same monotonicity in $[a,b]$, and $X_b > X_a, C_b > C_a$, then we say X_t is a strong comonotonic uncertain process with respect to C_t in $[a,b]$, or X_t and C_t are said to be strong comonotonic in $[a,b]$, i.e.,

i)$(X_{t_1} - X_{t_2})(C_{t_1} - C_{t_2}) \geq 0$, for any $[t_1,t_2]\subset[a,b]$,

ii)$X_b > X_a, C_b > C_a$.

Next, some properties of strong comonotonic uncertain process are illustrated as follows.

Theorem 3.2. *If uncertain processes X_t, Y_t and Liu process C_t are strong comonotonic in $[a,b]$, then uncertain process X_t+Y_t and Liu process C_t are strong comonotonic in $[a,b]$.*

Proof. Since uncertain processes X_t, Y_t and Liu process C_t are strong comonotonic in $[a,b]$, we get

$$(X_{t_1} - X_{t_2})(C_{t_1} - C_{t_2}) \geq 0,$$
$$(Y_{t_1} - Y_{t_2})(C_{t_1} - C_{t_2}) \geq 0,$$

for any $[t_1,t_2]\subset[a,b]$, and $X_b > X_a, Y_b > Y_a, C_b > C_a$. Then

$$\begin{aligned}[(X_{t_1}+Y_{t_1}) - (X_{t_2}+Y_{t_2})](C_{t_1} - C_{t_2}) &= [(X_{t_1} - X_{t_2}) + (Y_{t_1} - Y_{t_2})](C_{t_1} - C_{t_2})\\ &= (X_{t_1} - X_{t_2})(C_{t_1} - C_{t_2})\\ &\quad + (Y_{t_1} - Y_{t_2})(C_{t_1} - C_{t_2}).\end{aligned}$$

Thus

$$[(X_{t_1}+Y_{t_1}) - (X_{t_2}+Y_{t_2})](C_{t_1} - C_{t_2}) \geq 0,$$

for any $[t_1,t_2]\subset[a,b]$ and $X_b+Y_b > X_a+Y_a, C_b > C_a$. Therefore, uncertain process X_t+Y_t and Liu process C_t are strong comonotonic in $[a,b]$. □

Theorem 3.3. *If uncertain process X_t and Liu process C_t are strong comonotonic in $[a,b]$, then for any real number $k>0$, uncertain process kX_t and Liu process C_t are strong comonotonic in $[a,b]$.*

Proof. Since uncertain process X_t and Liu process C_t are strong comonotonic in $[a,b]$, we have

$$(X_{t_1}-X_{t_2})(C_{t_1}-C_{t_2})\geq 0,$$

for any $[t_1,t_2]\subset[a,b]$, and $X_b>X_a, C_b>C_a$. Considering

$$(kX_{t_1}-kX_{t_2})(C_{t_1}-C_{t_2})=k(X_{t_1}-X_{t_2})(C_{t_1}-C_{t_2}),$$

then for any $[t_1,t_2]\subset[a,b]$, and $k>0$, we have $(kX_{t_1}-kX_{t_2})(C_{t_1}-C_{t_2})\geq 0$, and $kX_b>kX_a, C_b>C_a$. Thus uncertain process kX_t and Liu process C_t are strong comonotonic in $[a,b]$, for any $k>0$. □

Remark 3.4. If uncertain process X_t, Y_t and Liu process C_t are strong comonotonic in $[a,b]$, then for any real numbers $k_1>0, k_2>0$, uncertain process $k_1X_t+k_2Y_t$ and Liu process C_t are strong comonotonic in $[a,b]$.

Theorem 3.5. *Let uncertain process X_t and Liu process C_t be strong comonotonic in $[a,b]$ and let $f(x)$ be an increasing function. Then $f(X_t)$ is also an uncertain process. Furthermore, $f(X_t)$ and Liu process C_t are strong comonotonic in $[a,b]$.*

Proof. By Theorem 2.7, $f(X_t)$ is an uncertain process. By the property of compound function, $f(X_t)$ and Liu process C_t are strong comonotonic in $[a,b]$. □

Considering the importance of Liu integral, next, we will discuss some useful properties of Liu integral.

Theorem 3.6. *(Nonnegativity) Let Liu integrable uncertain process X_t and Liu process C_t be strong comonotonic in $[a,b]$. If $X_t\geq 0$, then $\int_a^b X_t\mathrm{d}C_t\geq 0$.*

Proof. Since Liu integral $\int_a^b X_t\mathrm{d}C_t$ exists, whether closed interval $[a,b]$ is even partitioned or not, the results are the same. Let $a=t_1<t_2<\cdots<t_{k+1}=b$, and

$$|C_{t_2}-C_{t_1}|=|C_{t_3}-C_{t_2}|=\cdots=|C_{t_{n+1}}-C_{t_n}|.$$

Denote $\Delta=\max\limits_{1\leq i\leq n}|t_{i+1}-t_i|, \Delta C_t=C_{t_{i+1}}-C_{t_i}$. By the definition of Liu integral,

$$\begin{aligned}\int_a^b X_t\mathrm{d}C_t &= \lim_{\Delta\to 0}\sum_{i=1}^{n}X_{t_i}\cdot(C_{t_{i+1}}-C_{t_i})\\ &= \lim_{\Delta\to 0,\Delta C_t\geq 0}\sum_{i=1}^{m}X_{t_i}\cdot(C_{t_{i+1}}-C_{t_i})+\lim_{\Delta\to 0,\Delta C_t\leq 0}\sum_{j=1}^{k}X_{t_j}\cdot(C_{t_{j+1}}-C_{t_j})\\ &= \lim_{\Delta\to 0,\Delta C_t\geq 0}\sum_{i=1}^{m}X_{t_i}\Delta C_t+\lim_{\Delta\to 0,\Delta C_t\leq 0}\sum_{j=1}^{k}X_{t_j}\Delta C_t,\end{aligned}$$

where $m+k=n$. Since X_t and C_t are strong comonotonic, we obviously have $X_b>X_a$, for each X_{t_j}, there exists X_{t_i} corresponding to X_{t_j}. Thus

$$\lim_{\Delta\to 0,\Delta C_t\geq 0}\sum_{i=1}^{m}X_{t_i}\Delta C_t+\lim_{\Delta\to 0,\Delta C_t\leq 0}\sum_{j=1}^{k}X_{t_j}\Delta C_t=\lim_{\Delta\to 0,\Delta C_t\geq 0}\sum_{i_j=1}^{l}X_{i_j}\Delta C_t,$$

where $\Delta C_t>0, l=m-k$. Since $X_t\geq 0$, $X_{i_j}\geq 0$. The theorem is verified. □

Theorem 3.7. *(Monotonicity) Let X_t, Y_t be Liu integrable uncertain processes, and let uncertain process $Y_t - X_t$ and Liu process C_t be strong comonotonic in $[a, b]$. If $Y_t \geq X_t$, then*

$$\int_a^b Y_t \mathrm{d}C_t \geq \int_a^b X_t \mathrm{d}C_t.$$

Proof. Since $Y_t \geq X_t$, we have $Y_t - X_t \geq 0$. It follows from Theorem 3.6 that

$$\int_a^b Y_t \mathrm{d}C_t - \int_a^b X_t \mathrm{d}C_t = \int_a^b (Y_t - X_t) \mathrm{d}C_t \geq 0.$$

That is

$$\int_a^b Y_t \mathrm{d}C_t \geq \int_a^b X_t \mathrm{d}C_t.$$

The theorem is proved. □

Theorem 3.8. *Let Liu integrable uncertain process X_t and Liu process C_t be strong comonotonic in $[a, b]$. If there exist real numbers m, M such that $m \leq X_t \leq M$, then*

$$m(C_b - C_a) \leq \int_a^b X_t \mathrm{d}C_t \leq M(C_b - C_a).$$

Proof. Since $m \leq X_t \leq M$, it follows from Theorem 3.5 that

$$\int_a^b m \mathrm{d}C_t \leq \int_a^b X_t \mathrm{d}C_t \leq \int_a^b M \mathrm{d}C_t.$$

Furthermore,

$$\int_a^b m \mathrm{d}C_t = m(C_b - C_a), \int_a^b M \mathrm{d}C_t = M(C_b - C_a).$$

Then the theorem is verified. □

Theorem 3.9. *If uncertain processes Y_t, X_t and Liu process C_t are strong comonotonic in $[a, b]$, then*

$$\int_a^b (Y_t \vee X_t) \mathrm{d}C_t \geq \int_a^b Y_t \mathrm{d}C_t \vee \int_a^b X_t \mathrm{d}C_t.$$

Proof. Since $Y_t \vee X_t \geq X_t, Y_t \vee X_t \geq Y_t$, by Theorem 3.7

$$\int_a^b (Y_t \vee X_t) \mathrm{d}C_t \geq \int_a^b Y_t \mathrm{d}C_t,$$

$$\int_a^b (Y_t \vee X_t) \mathrm{d}C_t \geq \int_a^b X_t \mathrm{d}C_t.$$

Then

$$\int_a^b (Y_t \vee X_t) \mathrm{d}C_t \geq \int_a^b Y_t \mathrm{d}C_t \vee \int_a^b X_t \mathrm{d}C_t.$$

□

Corollary 3.10. *If uncertain processes Y_t, X_t and Liu process C_t are strong comonotonic in $[a, b]$, then*

$$\int_a^b (Y_t \wedge X_t)\mathrm{d}C_t \le \int_a^b Y_t\mathrm{d}C_t \wedge \int_a^b X_t\mathrm{d}C_t.$$

Proof. Since $Y_t \wedge X_t \le X_t, Y_t \wedge X_t \le Y_t$, by Theorem 3.7

$$\int_a^b (Y_t \wedge X_t)\mathrm{d}C_t \le \int_a^b Y_t\mathrm{d}C_t,$$

$$\int_a^b (Y_t \wedge X_t)\mathrm{d}C_t \le \int_a^b X_t\mathrm{d}C_t.$$

Then

$$\int_a^b (Y_t \wedge X_t)\mathrm{d}C_t \le \int_a^b Y_t\mathrm{d}C_t \wedge \int_a^b X_t\mathrm{d}C_t.$$

□

Theorem 3.11. *If uncertain process X_t and Liu process C_t are strong comonotonic in $[a, b]$, and $X_t \ge 0$, then $\int_a^c X_t\mathrm{d}C_t \ge \int_a^b X_t\mathrm{d}C_t$, where $a < b < c$.*

Proof. It follows from Theorem 2.11 that

$$\int_a^c X_t\mathrm{d}C_t = \int_a^b X_t\mathrm{d}C_t + \int_b^c X_t\mathrm{d}C_t.$$

Since uncertain process X_t and Liu process C_t are strong comonotonic, $X_t \ge 0$, we have

$$\int_b^c X_t\mathrm{d}C_t \ge 0.$$

Then

$$\int_a^c X_t\mathrm{d}C_t - \int_a^b X_t\mathrm{d}C_t = \int_b^c X_t\mathrm{d}C_t \ge 0.$$

The theorem is proved. □

Definition 3.12. If X_t is an uncertain process, then $|X_t|$ is a measurable function from $T \times (\Gamma, \mathcal{L}, \mathcal{M})$ to the set of nonnegative real numbers, where T denotes the index set of time.

Theorem 3.13. *If uncertain process X_t and Liu process C_t are strong comonotonic in $[a, b]$, Liu integral $\int_a^b X_t\mathrm{d}C_t$ and $\int_a^b |X_t|\mathrm{d}C_t$ exist, then*

$$\left|\int_a^b X_t\mathrm{d}C_t\right| \le \int_a^b |X_t|\mathrm{d}C_t.$$

Proof. Since Liu integral $\int_a^b X_t\mathrm{d}C_t$ exists, whether closed interval $[a, b]$ is even partitioned or not, the results are the same. Let $a = t_1 < t_2 < \cdots < t_{k+1} = b$, and

$$|C_{t_2} - C_{t_1}| = |C_{t_3} - C_{t_2}| = \cdots = |C_{t_{n+1}} - C_{t_n}|.$$

Denote $\Delta = \max_{1\le i\le n} |t_{i+1} - t_i|, \Delta C_t = C_{t_{i+1}} - C_{t_i}$. By the definition of Liu integral,

$$\int_a^b X_t \mathrm{d}C_t = \lim_{\Delta\to 0}\sum_{i=1}^{n} X_{t_i}\cdot (C_{t_{i+1}} - C_{t_i})$$
$$= \lim_{\Delta\to 0, \Delta C_t\ge 0}\sum_{i=1}^{m} X_{t_i}\Delta C_t + \lim_{\Delta\to 0, \Delta C_t\le 0}\sum_{j=1}^{k} X_{t_j}\Delta C_t,$$

where $m+k=n$. Since X_t and C_t are strong comonotonic, we obviously have $X_b > X_a$, then for each X_{t_j}, there exists X_{t_i} corresponding to X_{t_j}. Thus

$$\lim_{\Delta\to 0, \Delta C_t\ge 0}\sum_{i=1}^{m} X_{t_i}\Delta C_t + \lim_{\Delta\to 0, \Delta C_t\le 0}\sum_{j=1}^{k} X_{t_j}\Delta C_t = \lim_{\Delta\to 0, \Delta C_t\ge 0}\sum_{i_j=1}^{l} X_{i_j}\Delta C_t,$$

where $\Delta C_t > 0, l = m-k$. Then

$$|\int_a^b X_t \mathrm{d}C_t| = |\lim_{\Delta\to 0, \Delta C_t\ge 0}\sum_{i_j=1}^{l} X_{i_j}\Delta C_t|,$$

$$\int_a^b |X_t| \mathrm{d}C_t = \lim_{\Delta\to 0, \Delta C_t\ge 0}\sum_{i_j=1}^{l} |X_{i_j}|\Delta C_t,$$

where $\Delta C_t > 0, l = m-k$. Since

$$|\lim_{\Delta\to 0, \Delta C_t\ge 0}\sum_{i_j=1}^{l} X_{i_j}\Delta C_t| \le \lim_{\Delta\to 0, \Delta C_t\ge 0}\sum_{i_j=1}^{l} |X_{i_j}|\Delta C_t,$$

we have

$$|\int_a^b X_t \mathrm{d}C_t| \le \int_a^b |X_t| \mathrm{d}C_t.$$

□

Theorem 3.14. *Let Liu integral* $\int_a^b X_t \mathrm{d}C_t$ *exist. If* $X_t \ge 0$*, and* $\int_a^b X_t \mathrm{d}C_t > 0$*, then there exists an interval* $[\alpha,\beta] \subset [a,b]$*, such that* $X_t > 0$*, for any* $t \in [\alpha,\beta]$*.*

Proof. If $X_t = 0$ for each $t \in [a,b]$, then by the definition of Liu integral, $\int_a^b X_t \mathrm{d}C_t = 0$. There is a contradiction between $\int_a^b X_t \mathrm{d}C_t = 0$ and $\int_a^b X_t \mathrm{d}C_t > 0$. Thus there exists an interval $[\alpha,\beta] \subset [a,b]$, such that $X_t > 0$, for any $t \in [\alpha,\beta]$. □

Theorem 3.15. *Let Liu integrable uncertain process* X_t *and Liu process* C_t *be strong comonotonic in* $[a,b]$*. Assume* $X_t \ge 0$*, and* $X_t > 0$*, for each* $t \in [\alpha,\beta] \subset [a,b]$*. If* $\int_{t_1}^{t_2} X_t \mathrm{d}C_t = 0$*, for fixed* $t_1 \in [\alpha,\beta], t_2 \in [a,b]$*, then* $t_1 = t_2$*.*

Proof. Let $t_1 \ne t_2$. Without loss of generality, assume that $t_1 < t_2$. Since $X_t \ge 0$, it follows from Theorem 3.6 that

$$\int_a^b X_t \mathrm{d}C_t \ge 0.$$

i) If $t_2 \in [\alpha, \beta]$, then for each $t \in [t_1, t_2] \subset [\alpha, \beta] \subset [a, b]$, we have $X_t > 0$. By the definition of Liu integral, we get $\int_{t_1}^{t_2} X_t \mathrm{d}C_t > 0$. There is a contradiction between $\int_{t_1}^{t_2} X_t \mathrm{d}C_t > 0$ and $\int_{t_1}^{t_2} X_t \mathrm{d}C_t = 0$. Therefore, $t_1 = t_2$.

ii) If $t_2 \notin [\alpha, \beta]$, then $t_1 < \beta < t_2$. By Theorem 2.11

$$\int_{t_1}^{t_2} X_t \mathrm{d}C_t = \int_{t_1}^{\beta} X_t \mathrm{d}C_t + \int_{\beta}^{t_2} X_t \mathrm{d}C_t.$$

For each $t \in [\beta, t_2] \subset [a, b]$, we have $\int_{\beta}^{t_2} X_t \mathrm{d}C_t \geq 0$, and for each $t \in [t_1, \beta] \subset [\alpha, \beta] \subset [a, b]$, we have $X_t > 0$. By the definition of Liu integral, $\int_{t_1}^{\beta} X_t \mathrm{d}C_t > 0$. Thus $\int_{t_1}^{t_2} X_t \mathrm{d}C_t > 0$, there is a contradiction between $\int_{t_1}^{t_2} X_t \mathrm{d}C_t > 0$ and $\int_{t_1}^{t_2} X_t \mathrm{d}C_t = 0$. Therefore, $t_1 = t_2$.

In case of $t_2 < t_1$, we can obtain the results in a similar proof. □

4. **Conclusions**

In this paper, the definition of strong comonotonic uncertain process was presented. Based on strong comonotonic uncertain process, some theorems about mathematical operations such as addition, and scalar multiplication were deduced. Furthermore, some useful properties of Liu integral were presented. These results broaden the research field of uncertain calculus and promote the further developments of uncertainty theory.

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