SOME FUNDAMENTAL RESULTS ON FUZZY CALCULUS

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Abstract. In this paper, we study fuzzy calculus in two main branches differential and integral. Some rules for finding limit and gH-derivative of gH-difference, constant multiple of two fuzzy-valued functions are obtained and we also present fuzzy chain rule for calculating gH-derivative of a composite function. Two techniques namely, Leibniz's rule and integration by parts are introduced for fuzzy integrals. Furthermore, we prove three essential theorems such as a fuzzy intermediate value theorem, fuzzy mean value theorem for integral and mean value theorem for gH-derivative. We derive a Bolzano's theorem, Rolle's theorem and some properties for gH-differentiable functions. To illustrate and explain these rules and theorems, we have provided several examples in details.

1. Introduction

The concept of fuzzy sets was originally introduced by Zadeh [36] as an extension of the classical notion of set. In parallel, the theory of fuzzy sets has developed with the theory of set-valued functions. The basic arithmetic structure of fuzzy numbers was later developed by Mizumoto and Tanaka [30], Dubois and Prade [19], in sense of the r-level sets. Also the concepts of integration and H-derivative of the fuzzy-valued functions were introduced by Aumann [8] and Hukuhara [21] respectively, and further similar concept can be found in literature [23, 27, 29, 31, 32]. The H-differentiability of the fuzzy-valued functions arise the restriction consisting in the increasing of the diameter of solutions level sets as time increases. For solving this shortcoming, strong and weak generalized differentiability are presented and studied by Bede and Gal in [10]. Here derivative exists and the solution of a fuzzy differential equation may have decreasing length of the support, but the uniqueness is lost. However, this disadvantage can be seen as an advantage since we can choose the singular points where the support of the solution changes its monotonicity. So we can obtain reversible solutions, stable and almost periodic solutions and asymptotic behavior of solutions to the fuzzy differential equations [10]. The approach followed in this paper, generalized Hukuhara differentiability was introduced in [35], which is based on a generalization of the H-difference between two intervals. See [34], for generalizations of difference and division as inverse operations of addition and multiplication in the context of real intervals and fuzzy numbers. On the other

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hand, for gH-differentiability, see [4, 5, 6, 11].
In order to rank fuzzy numbers, one fuzzy number needs to be evaluated and compared with the others, but this may not be easy. Since ehe fuzzy numbers are represented by possibility distributions, they can overlap with each other and it is difficult to determine clearly whether one fuzzy number is larger or smaller than another [28]. The method of ranking fuzzy numbers has been proposed firstly in [22] and it is developed in several papers (see e. g. [1, 2, 3, 13]). The ranking concept considered in this paper generalizes the notion of the metric space by setting the distance between two points to be a non-negative fuzzy number [24]. The fuzzy calculus is one of the principal branches of fuzzy math. The early preliminaries of fuzzy calculus are defined and extended by Dubois and Prade in [16, 17, 18]. In this way, Lakshmikantham and et al develop the necessary concepts of integral and differential calculus for fuzzy functions in [26]. Moreover Bede and et al in [9] proved the derivatives of the H-difference and the product of two fuzzy-valued function in the sense of strongly generalized differentiability. A significant number of these approaches have been introduced in the literature [7, 8, 12, 14, 19, 30, 33, 34]. In spite of recent progress in fuzzy calculus, there are many concepts that must be defined and proved for the fuzzy-valued functions. In this paper, we are going to explain and prove some basic concepts and rules in fuzzy calculus. To this aim, we present the sum, gH-difference, constant multiple and product rule for finding limits. We show that a gH-differentiable function is continuous too, and also the sum and gH-difference, the constant multiple and the product rules are provided for computing gH-differentiate functions without use definition of derivative, and the chain rule is introduced for $gH-$differentiability of fuzzy-valued functions. Since integration from every fuzzy-valued function is not commutable, here a technique is introduced by changing some integrals to easier integrals that called integration by parts. Moreover, we present a fuzzy Leibniz's rule that is for differentiating from integral. The main purpose of this paper is that presenting of important theorems in fuzzy arithmetic, such as a fuzzy intermediate value theorem, Bolzano's theorem, mean value theorem for integral and mean value theorem for gH-derivative and Rolles Theorem. Likewise, we have proved when a fuzzy function has a zero gH-derivative, it is a constant fuzzy function, and whenever gH-derivative of two fuzzy-valued functions is equal then the gH-deference of these functions is a constant.
This paper organized as follows:
In Section 2 we recall some basic concepts. Some rules for finding limits of fuzzy-valued functions are introduced in Section 3 a long with some necessary lemmas. The gH-differentiation rules are presented for gH-difference, constant multiples of fuzzy-valued functions, and fuzzy chain rule in Section 4. In Section 5, the fuzzy Leibniz's rule and integration by parts are given to evaluating gH-derivative of integral and changing integral to easier form, respectively. Some applied theorems and rules are defined and proved in Section 6 in details. Finally, conclusion and future research are drawn in Section 7.

2. Preliminaries

Let $\mathbb{R}_{\mathcal{F}}$ denote the set of fuzzy subsets of the real axis (i. e. $u : \mathbb{R} \to [0,1]$) satisfying the following properties:

(i). $\forall u \in \mathbb{R}_{\mathcal{F}}$, u is upper semi-continuous on $\mathbb{R}$,
(ii). $\forall u \in \mathbb{R}_{\mathcal{F}}$, u is fuzzy convex,
(iii). $\forall u \in \mathbb{R}_{\mathcal{F}}$, u is normal,
(iv). $cl\{x \in \mathbb{R} | u(x) > 0\}$ is compact, where cl denotes the closure of a subset.

Then $\mathbb{R}_{\mathcal{F}}$ is called the space of fuzzy numbers. Obviously, $\mathbb{R} \subset \mathbb{R}_{\mathcal{F}}$. Given $0 < r \leq 1$, we denote $[u]^r = \left\{x \in \mathbb{R} \middle| u(x) \geq r\right\}$, and $[u]^0 = cl\left\{x \in \mathbb{R} \middle| u(x) > 0\right\}$. Then from (i) to (iv), it follows that for each $r \in [0,\ 1]$, the r-level sets of $u \in \mathbb{R}_{\mathcal{F}}$ are nonempty closed intervals.

A triangular fuzzy number is defined as a fuzzy set in $\mathbb{R}_{\mathcal{F}}$, that is specified by an ordered triple $u = (a, b, c) \in \mathbb{R}^3$ with $a \leq b \leq c$ such that $\underline{u}(r) = a + (b - a)r$ and $\overline{u}(r) = c - (c - b)r$ are the endpoints of r-level sets for all $r \in [0, 1]$.

The Hausdorff distance between fuzzy numbers is given by $D : \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \longrightarrow \mathbb{R}^+ \cup \{0\}$ as $D(u, v) = \sup_{0 \leq r \leq 1} \max\{|\underline{u}(r) - \underline{v}(r)|, |\overline{u}(r) - \overline{v}(r)|\}$, where $[u]^r = [\underline{u}(r), \overline{u}(r)], [v]^r = [\underline{v}(r), \overline{v}(r)]$. The metric space $(\mathbb{R}_{\mathcal{F}}, D)$ is complete, separable and locally compact and the following properties of the metric D are well known:

$$D(u \oplus w, v \oplus w) = D(u, v),\ \forall u, v, w \in \mathbb{R}_{\mathcal{F}},$$
$$D(\lambda \odot u, \lambda \odot v) = |\lambda| D(u, v),\ \forall \lambda \in \mathbb{R},\ u, v \in \mathbb{R}_{\mathcal{F}},$$
$$D(u \oplus v, w \oplus z) \leq D(u, w) + D(v, z),\ \forall u, v, w, z \in \mathbb{R}_{\mathcal{F}}.$$

In this paper, for ranking concept, we will use a partial ordering is introduce in [24].

Definition 2.1. Let the partial ordering $\preceq$ in $\mathbb{R}_{\mathcal{F}}$ by

$$u \preceq v \text{ if and only if } \underline{u}(r) \leq \underline{v}(r) \text{ and } \overline{u}(r) \leq \overline{v}(r),\ \forall r \in [0, 1],$$

and the strict inequality $\prec$ in $\mathbb{R}_{\mathcal{F}}$ is defined by

$$u \prec v \text{ if and only if } \underline{u}(r) < \underline{v}(r) \text{ and } \overline{u}(r) < \overline{v}(r),\ \forall r \in [0, 1],$$

where $[u]^r = [\underline{u}(r), \overline{u}(r)], [v]^r = [\underline{v}(r), \overline{v}(r)]$.

Definition 2.2. [34] Let $u, v \in \mathbb{R}_{\mathcal{F}}$. If there exists $w \in \mathbb{R}_{\mathcal{F}}$ such that $u = v \oplus w$, then w is called the H-difference of u and v, and it is denoted by $u \ominus v$.

Definition 2.3. [11] Given two fuzzy numbers u , $v \in \mathbb{R}_{\mathcal{F}}$, the generalized Hukuhara difference (gH-difference for short) is the fuzzy number w, if it exists, such that

$$u \ominus_{gH} v = w \Longleftrightarrow \begin{cases} (i\ ) & u = v \oplus w\ , \\ or\ (ii) & v = u \oplus (-1)w\ . \end{cases}$$

It is easy to show that (i) and (ii) are both valid if and only if w is a crisp number.
In terms of r-levels we have $[u \ominus_{gH} v]^r = [\min\{\underline{u}(r) - \underline{v}(r), \overline{u}(r) - \overline{v}(r)\}, \max\{\underline{u}(r) - \underline{v}(r), \overline{u}(r) - \overline{v}(r)\}]$, and the conditions for the existence of $w = u \ominus_{gH} v \in \mathbb{R}_{\mathcal{F}}$ are

$$case(i) \begin{cases} \underline{w}(r) = \underline{u}(r) - \underline{v}(r) \text{ and } \overline{w}(r) = \overline{u}(r) - \overline{v}(r) \ \forall r \in [0,1]; \\ \text{with } \underline{w}(r) \text{ increasing, } \overline{w}(r) \text{ decreasing, } \underline{w}(r) \leq \overline{w}(r). \end{cases}$$

$$case(ii) \begin{cases} \underline{w}(r) = \overline{u}(r) - \overline{v}(r) \text{ and } \overline{w}(r) = \underline{u}(r) - \underline{v}(r) \ \forall r \in [0,1]; \\ \text{with } \underline{w}(r) \text{ increasing, } \overline{w}(r) \text{ decreasing, } \underline{w}(r) \leq \overline{w}(r). \end{cases}$$

If the gH-difference $u \ominus_{gH} v$ do not define a proper fuzzy number, the nested property can be used for r-levels and obtain a proper fuzzy number by

$$[u \ominus_g v]^r \quad := \quad cl\Bigg(\bigcup_{r_0 \geq r} ([u]^{r_0} \ominus_{gH} [v]^{r_0})\Bigg), \text{ for } r \in [0,1],$$

where $u \ominus_g v$ define generalized difference of two fuzzy number u , $v \in \mathbb{R}_{\mathcal{F}}$ [34], which has been extended and studied in [11].

Remark 2.4. Throughout this paper, we assume that if $u, v \in \mathbb{R}_{\mathcal{F}}$, then $u \ominus_{gH} v \in \mathbb{R}_{\mathcal{F}}$.

Proposition 2.5. [35] *For u , $v \in \mathbb{R}_{\mathcal{F}}$, we have*

$$D(u \ominus_{gH} v, 0) = D(u, v).$$

Proposition 2.6. [34] *Let u , $v \in \mathbb{R}_{\mathcal{F}}$. If $u \ominus_{gH} v$ exists, it is unique and has the following properties (0 denotes the crisp set $\{0\}$):*

(1) $u \ominus_{gH} u = 0$.
(2) *(a)*$(u \oplus v) \ominus_{gH} v = u$; *(b)*$u \ominus_{gH} (u - v) = v$.
(3) *If $u \ominus_{gH} v$ exists then also $(-v) \ominus_{gH} (-u)$ does and* $0 \ominus_{gH} (u \ominus_{gH} v) = (-v) \ominus_{gH} (-u)$.
(4) $u \ominus_{gH} v = v \ominus_{gH} u$ *if and only if $w = -w$ (in particular $w = 0$ if and only if $u = v$).*
(5) *If $v \ominus_{gH} u$ exists then either $u \oplus (v \ominus_{gH} u) = u$ or $v - (v \ominus_{gH} u) = u$ and if both equalities hold then $v \ominus_{gH} u$ is a crisp set.*

Definition 2.7. [34] Let u , $v \in \mathbb{R}_{\mathcal{F}}$ have r-levels $[u]^r = [\underline{u}(r), \overline{u}(r)]$, $[v]^r = [\underline{v}(r), \overline{v}(r)]$, with $0 \notin [v]^r \forall r \in [0,1]$. The gH-division $\div_{gH}$ is the operation that calculates the fuzzy number $w = u \div_{gH} v \in \mathbb{R}_{\mathcal{F}}$ defining by

$$u \div_{gH} v \quad = \quad w \Leftrightarrow \begin{cases} (i\ ) & u = v \odot w, \\ or\ (ii) & v = u \odot w^{-1}, \end{cases}$$

provided that w is a proper fuzzy number.

Proposition 2.8. [34] *Let u , $v \in \mathbb{R}_{\mathcal{F}}$ (here 1 is the same as $\{1\}$). We have:*

(1) *If $0 \notin [u]^r$ $\forall r$, then $u \div_{gH} u = 1$.*
(2) *If $0 \notin [v]^r$ $\forall r$, then $uv \div_{gH} v = u$.*

(3) *If* $0 \notin [v]^r$ $\forall r$*, then* $1 \div_{gH} v = v^{-1}$ *and* $1 \div_{gH} v^{-1} = v$*.*
(4) *If* $v \div_{gH} u$ *exists then either* $u(v \div_{gH} u) = v$ *or* $v(v \div_{gH} u)^{-1} = u$ *and both equalities hold if and only if* $v \div_{gH} u$ *is a crisp set.*

Note that a function $f : [a, b] \subseteq \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$ is called fuzzy-valued function. The r-level representation of the fuzzy-valued function f given by $f(x; r) = [\underline{f}(x; r), \overline{f}(x; r)]$, $x \in [a, b], r \in [0, 1]$.

Definition 2.9. [7] Let $f : [a,\ b] \to \mathbb{R}_{\mathcal{F}}$ be a fuzzy-valued function and $x_0 \in [a,\ b]$. If

$$\forall \epsilon > 0\ \exists \delta > 0\ \forall x \big(0 < |x - x_0| < \delta \Rightarrow D(f(x), L) < \epsilon\big),$$

then we say that $L \in \mathbb{R}_{\mathcal{F}}$ is limit of f in x_0, which is denoted by

$$\lim_{t \to x_0} f(x) = L.$$

Also the fuzzy-valued function f is said to be continuous if

$$\lim_{t \to x_0} f(x) = f(x_0),$$

and the function f is fuzzy continuous on $[a, b]$ if f is continuous at each $x_0 \in [a,\ b]$ such that the continuity is one-sided at end points a, b.

Definition 2.10. [11] Let $x_0 \in]a, b[$ and h be such that $x_0 + h \in]a, b[$, then the gH-derivative of a function $f :]a, b[\to \mathbb{R}_{\mathcal{F}}$ at x_0 is defined as

$$f'_{gH}(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) \ominus_{gH} f(x_0)}{h}. \tag{1}$$

If $f'_{gH}(x_0) \in \mathbb{R}_{\mathcal{F}}$, we say that f is generalized Hukuhara differentiable (gH-differentiable for short) at x_0.

Definition 2.11. [11] Let $f : [a, b] \to \mathbb{R}_{\mathcal{F}}$ and $x_0 \in]a, b[$, with $\underline{f}(x; r)$ and $\overline{f}(x; r)$ both differentiable at x_0. Also, we say that

- f is $(i) - gH$-differentiable at x_0 if

$$(i)\ (f)'_{gH}(x_0; r) = [(\underline{f})'(x_0; r), (\overline{f})'(x_0; r)],\ 0 \le r \le 1. \tag{2}$$

- f is $(ii) - gH$-differentiable at x_0 if

$$(ii)\ (f)'_{gH}(x_0; r) = [(\overline{f})'(x_0; r), (\underline{f})'(x_0; r)],\ 0 \le r \le 1. \tag{3}$$

Definition 2.12. [35] We say that a point $x_0 \in]a, b[$, is a switching point for the differentiability of f, if in any neighborhood V of x_0 there exist points $x_1 < x_0 < x_2$ such that

type(I): at x_1 (2) holds while (3) does not hold and at x_2 (3) holds and (2) does not hold, or

type(II): at x_1 (3) holds while (2) does not hold and at x_2 (2) holds and (3) does not hold.

Definition 2.13. [15] Let $f : [a,\ b] \to \mathbb{R}_{\mathcal{F}}$. We say that $f(t)$ is Fuzzy Riemann integrable to $J \in \mathbb{R}_F$ if for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any division $P = \{[u,\ v]; \xi\}$ with the norms $\Delta(P) < \delta$, we have

$$D\bigg(\sum_p^{*}(v-u)\odot f(\xi),\ J\bigg) < \varepsilon,$$

where $\sum_p^{*}$ denotes the fuzzy summation. We choose to write $J := \int_a^b f(x)\mathrm{d}x$. Further more,

$$\int_a^b f(x;r)\mathrm{d}x = [\int_a^b \underline{f}(x;r)\mathrm{d}x, \int_a^b \overline{f}(x;r)\mathrm{d}x].$$

Theorem 2.14. [23] *Let $f : [a,\ b] \to \mathbb{R}_{\mathcal{F}}$ be integrable and $c \in [a,\ b]$. Then*

$$\int_a^b f(x)dx \quad = \quad \int_a^c f(x)dx \oplus \int_c^b f(x)dx.$$

3. **Limits**

This section provides some theorems to calculate limits of fuzzy-valued functions. first, we show that the limit of the sum of two fuzzy-valued functions is the sum of their limits.

Theorem 3.1. *Let $f, g : \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$ be two fuzzy-valued functions. If $\lim_{x\to c} f(x) = L_1$ and $\lim_{x\to c} g(x) = L_2$, $L_1, L_2 \in \mathbb{R}_{\mathcal{F}}$ then*

$$\lim_{x\to c}[f(x) \oplus g(x)] = L_1 \oplus L_2.$$

Proof. Let $\epsilon > 0$. By assumption for $\frac{\epsilon}{2} > 0$, there exists $\delta_1 > 0$ such that

$$\forall x \in (c-\delta_1, c+\delta_1) \Rightarrow D(f(x), L_1) < \frac{\epsilon}{2}.$$

Also there is $\delta_2 > 0$, so that

$$\forall x \in (c-\delta_2, c+\delta_2) \Rightarrow D(g(x), L_2) < \frac{\epsilon}{2}.$$

Setting $\delta = \min\{\delta_1, \delta_2\}$, so for all $x \in (c-\delta, c+\delta)$, we find that

$$D(f(x) \oplus g(x), L_1 \oplus L_2) \le D(f(x), L_1) + D(g(x), L_2) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

and the proof is completed. □

Next, we prove that the limit of the difference of two fuzzy-valued functions is the difference of their limits.

Theorem 3.2. *Let $f, g : \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$ be two fuzzy-valued functions. If $\lim_{x\to c} f(x) = L_1$ and $\lim_{x\to c} g(x) = L_2$, $L_1, L_2 \in \mathbb{R}_{\mathcal{F}}$, then*

$$\lim_{x\to c}[f(x) \ominus_{gH} g(x)] = L_1 \ominus_{gH} L_2.$$

Proof. Define $h(x) = f(x) \ominus_{gH} g(x)$. According to Remark 2.4 and Definition 2.3, we have

$$f(x) \ominus_{gH} g(x) = h(x) \Longleftrightarrow \begin{cases} (i\ ) & f(x) = g(x) \oplus h(x)\ , \\ or\ (ii) & g(x) = f(x) \oplus (-1)h(x)\ . \end{cases} \tag{4}$$

First we consider the case (i) in (4), from assumptions we have $\lim_{x\to c} f(x) = L_1$. Thus limiting of both sides of the case (i) implies that

$$\lim_{x\to c}[g(x) \oplus h(x)] = L_1.$$

From Theorem 3.1, we conclude that

$$\lim_{x\to c} g(x) \oplus \lim_{x\to c} h(x) = L_1 \Rightarrow \lim_{x\to c} h(x) = L_1 \ominus L_2. \tag{5}$$

And for the case (ii) in (4), since $\lim_{x\to c} g(x) = L_2$, we have

$$\lim_{x\to c}[f(x) \oplus (-1)h(x)] = L_2.$$

By Theorem 3.1, find that

$$\lim_{x\to c} f(x) \oplus (-1) \lim_{x\to c} h(x) = L_2 \Rightarrow \lim_{x\to c} h(x) = (-1)L_2 \ominus (-1)L_1. \tag{6}$$

Then from the equations (5) and (6), the proof is obtained. □

Now we reveal that the limit of the product of the fuzzy-valued function and a positive real function is the product of their limits. To this, we need to have the following lemma and proposition.

Proposition 3.3. *For $u \in \mathbb{R}_{\mathcal{F}}$ and $\lambda_1, \lambda_2 \in \mathbb{R}$ such that $\lambda_1.\lambda_2 > 0$, we have*

$$D(\lambda_1 \odot u \ominus_{gH} \lambda_2 \odot u, 0) = |\lambda_1 - \lambda_2| D(u, 0).$$

Proof. See Proposition 2.2 in [20]. □

Lemma 3.4. *Let $f : \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$. If $\lim_{x\to c} f(x) = L$ show that there exists a number $\delta > 0$ such that*

$$0 < |x - c| < \delta \Rightarrow D(f(x), 0) < 1 + D(L, 0).$$

Proof. Taking $\epsilon = 1$, by Definition 2.9, we find a number $\delta > 0$ such that if $0 < |x - c| < \delta$, then $D\Big(f(x), L\Big) < 1$. Using proposition 2.5 and case(1) in 2.6, we have

$$\begin{aligned} D(f(x), 0) &= D\Big(f(x) \oplus L \ominus_{gH} L, 0\Big) = D\Big(f(x) \oplus L, L\Big) \\ &\leq D\Big(f(x), L\Big) + D(L, 0) < 1 + D(L, 0). \end{aligned}$$

□

Theorem 3.5. *Let $f : \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$ and $g : \mathbb{R} \to \mathbb{R}^+$. If $\lim_{x\to c} f(x) = L_1 \in \mathbb{R}_{\mathcal{F}}$ and $\lim_{x\to c} g(x) = L_2$, then*

$$\lim_{x\to c} \Big(f(x) \odot g(x)\Big) = L_1 \odot L_2.$$

Proof. Let $\epsilon > 0$ be given. Since $\lim_{x\to c} f(x) = L_1$, there exists $\delta_1 > 0$ such that $D(f(x), L_1) < \frac{\epsilon}{2L_2}$ if $0 < |x - c| < \delta_1$. Since $\lim_{x\to c} g(x) = L_2$, there exists $\delta_2 > 0$ such that $|g(x) - L_2| < \frac{\epsilon}{2(1+D(L_1,0))}$ if $0 < |x - c| < \delta_2$. By Lemma 3.4, there exists $\delta_3 > 0$ such that $D\Big(f(x), 0\Big) < 1 + D(L_1, 0)$ if $0 < |x - c| < \delta_3$. Let $\delta = \min\{\delta_1, \delta_2, \delta_3\}$. So by properties of D, propositions 2.5 and 3.3, for all x, when $0 < |x - c| < \delta$, we have

$$\begin{aligned} & D\Big(f(x) \odot g(x), L_1 \odot L_2\Big) = D\Big(f(x) \odot g(x) \oplus L_2 \odot f(x) \ominus_{gH} L_2 \odot f(x), L_1 \odot L_2\Big) \\ \leq\ & D\Big(f(x) \odot g(x) \ominus_{gH} L_2 \odot f(x), 0\Big) + D\Big(L_2 \odot f(x), L_1 \odot L_2\Big) \\ =\ & D\Big(f(x) \odot g(x) \ominus_{gH} L_2 \odot f(x), 0\Big) + D\Big(L_2 \odot f(x) \ominus_{gH} L_1 \odot L_2, 0\Big) \\ =\ & D\Big((g(x) - L_2) \odot f(x), 0\Big) + D\Big(L_2 \odot (f(x) \ominus_{gH} L_1), 0\Big) \\ =\ & |g(x) - L_2| D\Big(f(x), 0\Big) + |L_2| D\Big(f(x), L_1\Big) \\ \leq\ & \frac{\epsilon}{2(1 + D(L_1, 0))}(1 + D(L_1, 0)) + |L_2| \frac{\epsilon}{2|L_2|} \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

□

4. gH-derivatives

In this section, we prove that a gH-differentiable fuzzy-valued function is continuous. Also gH-derivatives from addition, gH-difference, constant multiplication, and product of the fuzzy-valued functions and fuzzy chain rule are obtained.

Theorem 4.1. *Let $f :]a, b[\to \mathbb{R}_{\mathcal{F}}$ be gH-differentiable at $c \in]a, b[$. Then f is fuzzy continuous at c.*

Proof. Using properties of distance D along with gH-differentiability of f and proposition 2.5 gets

$$\begin{aligned} \lim_{t\to c} D(f(t), f(c)) &= \lim_{t\to c} D(f(t) \ominus_{gH} f(c), 0) \\ &= \lim_{t\to c} D(\frac{f(t) \ominus_{gH} f(c)}{t - c}(t - c), 0) \\ &= \lim_{t\to c} \mid t - c \mid D(\frac{f(t) \ominus_{gH} f(c)}{t - c}, 0) \\ &= \lim_{t\to c} \mid t - c \mid \lim_{t\to c} D(\frac{f(t) \ominus_{gH} f(c)}{t - c}, 0) \\ &= 0 \cdot D(f'_{gH}(c), 0) = 0, \end{aligned}$$

which implies that f is continuous at $t = c$. □

Theorem 4.2. *Let $f, g : \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$ are $gH-$differentiable with same type of gH-differentiability. Then $f(x) \oplus g(x)$ is $gH-$differentiable and*

$$(f \oplus g)'_{gH}(x) = f'_{gH}(x) \oplus g'_{gH}(x)$$

Proof. See [11].

□

Theorem 4.3. *Let $f, g : \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$ be $gH-$differentiable with the same type of gH-differentiability. Then $f(x) \ominus_{gH} g(x)$ is gH-differentiable, and*

$$(f \ominus_{gH} g)'_{gH}(x) = f'_{gH}(x) \ominus_{gH} g'_{gH}(x).$$

Proof. Define $h(x) = f(x) \ominus_{gH} g(x)$. By considering Remark 2.4 and Definition 2.3, we have

$$f(x) \ominus_{gH} g(x) = h(x) \Longleftrightarrow \begin{cases} (i) & f(x) = g(x) \oplus h(x) \ , \\ or \ (ii) & g(x) = f(x) \oplus (-1)h(x) \ . \end{cases} \tag{7}$$

In the case (i) of (7), the Theorem 4.2 gets

$$f'_{gH}(x) = g'_{gH}(x) \oplus h'_{gH}(x) \Rightarrow h'_{gH}(x) = f'_{gH}(x) \ominus g'_{gH}(x). \tag{8}$$

And in the case (ii) of (7), similarly find that

$$g'_{gH}(x) = f'_{gH}(x) \oplus (-1)h'_{gH}(x) \Rightarrow h'_{gH}(x) = (-1)g'_{gH}(x) \ominus (-1)f'_{gH}(x) \tag{9}$$

Therefore the functions f'_{gH}, g'_{gH} and h'_{gH} satisfy in Definition 2.3, then from equations (8) and (9) we have $(f \ominus_{gH} g)'_{gH}(x) = f'_{gH}(x) \ominus_{gH} g'_{gH}(x)$. □

Theorem 4.4. *Let $f : \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$ be $gH-$differentiable and $c \in \mathbb{R}$. Then*

$$(c \odot f)'_{gH}(x) = c \odot f'_{gH}(x). \tag{10}$$

Proof. From assumption f is gH-differentiable so we find that

$$D\big(\frac{(c \odot f)(x+h) \ominus_{gH} (c \odot f)(x)}{h}, c \odot f'_{gH}(x)\big) =$$
$$D\big(\frac{c \odot f(x+h) \ominus_{gH} c \odot f(x)}{h}, c \odot f'_{gH}(x)\big) = |c| D\big(\frac{f(x+h) \ominus_{gH} f(x)}{h}, f'_{gH}(x)\big),$$

which leads to the conclusion $D\big(\frac{(c \odot f)(x+h) \ominus_{gH} (c \odot f)(x)}{h}, c \odot f'_{gH}(x)\big) \to 0$ as $h \to 0$, that is

$$(c \odot f)'_{gH}(x) = c \odot f'_{gH}(x).$$

□

Theorem 4.5. *Let $f : \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$ and $g : \mathbb{R} \to \mathbb{R}^+$ be two differentiable functions (f is gH-differentiable). Then*

$$(f \odot g)'_{gH}(x) = f'_{gH}(x) \odot g(x) \oplus f(x) \odot g'(x). \tag{11}$$

Proof. See Theorem 2.6. in [20]. □

Now we intend to extend chain rule in [7] for gH-derivative of the fuzzy-valued function.

Theorem 4.6. *Let I be a closed interval in $\mathbb{R}$. If $g : I \to \phi \subseteq \mathbb{R}$ be differentiable at x, and $f : \phi \to \mathbb{R}_{\mathcal{F}}$ be gH-differentiable at $u = g(x)$ and g be strictly monotonic on I, then*

$$(f \circ g)'_{gH}(x) = f'_{gH}(g(x)) \odot g'(x) \ , \ \forall x \in I.$$

Proof. The gH-differentiability of f at $u = g(x)$ implies that

$$D\Big(\frac{f(u+k)\ominus_{gH} f(u)}{k}, f'_{gH}(u)\Big) \quad = \quad 0,$$

so

$$\frac{1}{|k|}D\Big(f(u+k)\ominus_{gH} f(u), k\odot f'_{gH}(u)\Big) \quad = \quad 0.$$

Setting $k = g(x+h) - g(x)$, thus we obtain

$$|\frac{h}{k}|D\Big(\frac{f(g(x+h))\ominus_{gH} f(g(x))}{h}, \frac{k}{h}\odot f'_{gH}(g(x))\Big) \quad = \quad 0. \tag{12}$$

Now by limiting both sides of (12), when $h \to 0$ gets

$$\frac{1}{|g'(x)|}D\Big((f\circ g)'_{gH}(x), f'_{gH}(g(x))\odot g'(x)\Big) \quad = \quad 0,$$

which proves the theorem. □

Example 4.7. Let $f(x) = (1.2, 2, 2.6)\odot\sqrt{x}$ and $g(x) = x^2+1$. Since the derivatives of f and g are

$$f'_{gH}(u) = \frac{(1.2, 2, 2.6)}{2\sqrt{u}}, \ g'(x) = 2x,$$

we find that

$$\begin{aligned}
(f\circ g)'_{gH}(x) &= f'_{gH}(g(x))\odot g'(x) \\
&= (1.2, 2, 2.6)\odot\frac{1}{2\sqrt{g(x)}}\odot g'(x) \\
&= (1.2, 2, 2.6)\odot\frac{1}{2\sqrt{x^2+1}}\odot(2x) = \frac{(1.2, 2, 2.6)\odot x}{\sqrt{x^2+1}}.
\end{aligned}$$

5. **Integral**

Theorem 5.1. *If f is gH-differentiable with no switching point in the interval* $[a, b]$ *then we have*

$$\int_a^b f'_{gH}(x)dx \quad = \quad f(b)\ominus_{gH} f(a).$$

Proof. See [11]. □

Now, we extend Theorem 5. 6 in [23, 7] for generalized Hukuhara derivative.

Theorem 5.2. *Let* $f : [a, b] \to \mathbb{R}_{\mathcal{F}}$ *be continuous fuzzy-valued function. Then*

$$F(t) = \int_a^t f(x)dx, \ t\in[a, b],$$

is gH-differentiable and $F'_{gH}(t) = f(t)$.

Proof. By assumption and Theorem 2.14, we can write

$$F(t+h) \quad = \quad \int_a^{t+h} f(x)\mathrm{d}x = \int_a^t f(x)\mathrm{d}x + \int_t^{t+h} f(x)\mathrm{d}x = F(t) + \int_t^{t+h} f(x)\mathrm{d}x.$$

Therefore by Remark 2.4 we get,

$$F(t+h) \ominus_{gH} F(t) \quad = \quad \int_t^{t+h} f(x)\mathrm{d}x.$$

Now using distance D, gives

$$\begin{aligned} D(\frac{F(t+h) \ominus_{gH} F(t)}{h}, f(t)) &= \frac{1}{|h|} D(\int_t^{t+h} f(x)\mathrm{d}x, h \odot f(t)) \\ &= \frac{1}{|h|} D(\int_t^{t+h} f(x)\mathrm{d}x, \big(\int_t^{t+h} \mathrm{d}x\big) \odot f(t)) \\ &= \frac{1}{|h|} D(\int_t^{t+h} f(x)\mathrm{d}x, \int_t^{t+h} f(t)\mathrm{d}x) \\ &\leq \frac{1}{|h|} \int_t^{t+h} D(f(x), f(t))\mathrm{d}x \\ &\leq \sup_{x\in[t,t+h]} D(f(x), f(t)) \Big(\frac{1}{|h|} \int_t^{t+h} \mathrm{d}x \Big) \\ &\leq \sup_{x\in[t,t+h]} D(f(x), f(t)). \end{aligned}$$

The right-hand side tends to zero as $h \to 0$. Hence $F'_{gH}(t) = f(t)$. □

Here we present a technique for computing integration of multiplying of two functions that is called integration by parts.

Theorem 5.3. *Let $f : [a,b] \to \mathbb{R}_{\mathcal{F}}$ and $g : [a,b] \to \mathbb{R}^+$ be two differentiable functions (f is gH-differentiable), then*

$$\int_a^b f'_{gH}(x) \odot g(x)dx = \big(f(b) \odot g(b)\big) \ominus_{gH} \big(f(a) \odot g(a)\big) \ominus_{gH} \int_a^b f(x) \odot g'(x)dx.$$

Proof. According to Theorem 4.5, we have

$$\Big(f(x) \odot g(x)\Big)'_{gH} = f'_{gH}(x) \odot g(x) \oplus f(x) \odot g'(x). \tag{13}$$

Integrating both sides of (13) respect to x and using Remark 2.4 and theorem 5.1 find that

$$f(b) \odot g(b) \ominus_{gH} f(a) \odot g(a) = \int_a^b f'_{gH}(x) \odot g(x)\mathrm{d}x \oplus \int_a^b f(x) \odot g'(x)\mathrm{d}x.$$

Therefore the required result is obtained. □

Theorem 5.4. *Let $f : [a,b] \to \mathbb{R}_{\mathcal{F}}$ and $g : [a,b] \to \mathbb{R}^+$ are two differentiable functions (f is gH-differentiable), then*

$$\int_a^x f'_{gH}(x) \odot g(x)dx = f(x) \odot g(x) \ominus_{gH} \int_a^x f(x) \odot g'(x)dx.$$

Proof. Similar to Theorem 5.3 by using Theorem 5.2 the proof is straightforward. □

Now we find the derivative of integral that is called fuzzy Leibniz's rule.

Theorem 5.5. *Let $f : \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$ be a continuous fuzzy-valued function and $u, v : \mathbb{R} \to \mathbb{R}$ be two differentiable functions. Then*

$$\left(\int_{u(x)}^{v(x)} f(t)\, dt \right)'_{gH} = f(v(x)) \odot \frac{dv}{dx} \ominus_{gH} f(u(x)) \odot \frac{du}{dx}.$$

Proof. According to Theorem 5.2, there exists a function $F : \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$ such that $F'_{gH}(x) = f(x)$, so by Remark 2.4 and Theorem 5.1, we have

$$\int_{u(x)}^{v(x)} f(t)\mathrm{d}t = F(v(x)) \ominus_{gH} F(u(x)). \tag{14}$$

Differentiating both sides of (14) and using Theorem 4.6 implies that

$$\begin{aligned} \left(\int_{u(x)}^{v(x)} f(t)\mathrm{d}t \right)'_{gH} &= \Big(F(v(x)) \Big)'_{gH} \ominus_{gH} \Big(F(u(x)) \Big)'_{gH} \\ &= f(v(x)) \odot v'(x) \ominus_{gH} f(u(x)) \odot u'(x). \end{aligned}$$

□

Example 5.6. Find gH-derivative of $f(x) = \int_{\sin x}^{\sqrt{x}} \big((0.4, 1, 1.6) \odot e^t \oplus (-1.3, -1, -0.3)\big)\mathrm{d}t$. Using fuzzy Leibniz's rule gives

$$\begin{aligned} f'_{gH}(x) &= \big((0.4, 1, 1.6) \odot e^{\sqrt{x}} \oplus (-1.3, -1, -0.3)\big) \odot \frac{1}{2\sqrt{x}} \\ &\quad \ominus_{gH} \big((0.4, 1, 1.6) \odot e^{\sin x} \oplus (-1.3, -1, -0.3)\big) \odot \cos x \\ &= \frac{(0.4, 1, 1.6)}{2\sqrt{x}} \odot e^{\sqrt{x}} \ominus_{gH} (0.4, 1, 1.6) \odot e^{\sin x} \cos x \\ &\quad \oplus \frac{(-1.3, -1, -0.3)}{2\sqrt{x}} \ominus_{gH} (-1.3, -1, -0.3) \odot \cos x. \end{aligned}$$

6. Some Main Theorems in Fuzzy Calculus

In this section, fuzzy intermediate value theorems, mean value theorem for integral and mean value theorem for gH-derivative are presented. These theorems have some important consequences like Bolzano's theorem and Rolle's theorem which are provided in details. for this purpose, we need some properties of ranking which have proven.

Proposition 6.1. *If $u \preceq v$ then $-v \preceq -u$.*

Proof. Using Definition 2.1 gives

$$\underline{u}(r) \le \underline{v}(r),\ \overline{u}(r) \le \overline{v}(r),\ \forall r \in [0, 1].$$

Therefore we have

$$-\underline{v}(r) \le -\underline{u}(r),\ -\overline{v}(r) \le -\overline{u}(r),\ \forall r \in [0, 1],$$

i. e. $-v \preceq -u$. □

Proposition 6.2. *If $u \preceq v$ and $v \preceq u$ then $u = v$.*

Proof. By Definition 2.1 and assumption, we have

$$\underline{u}(r) \leq \underline{v}(r),\ \overline{u}(r) \leq \overline{v}(r),\ \forall r \in [0,1],$$

also

$$\underline{v}(r) \leq \underline{u}(r),\ \overline{v}(r) \leq \overline{u}(r),\ \forall r \in [0,1].$$

So for all $r \in [0,1]$, we find that $\underline{u}(r) = \underline{v}(r)$, $\overline{u}(r) = \overline{v}(r)$, which proves the theorem. □

Proposition 6.3. *Let $f, g : [a,b] \to \mathbb{R}_{\mathcal{F}}$ be two continuous fuzzy-valued function. If $f(x) \preceq g(x)$ for all $x \in [a,b]$ then $\int_a^b f(x)dx \preceq \int_a^b g(x)dx$.*

Proof. Using Definition 2.1, we get

$$\underline{f}(x,r) \leq \underline{g}(x,r),\ \overline{f}(x,r) \leq \overline{g}(x,r),\ \forall r \in [0,1],\ \forall x \in [a,b].$$

By monotonicity of the integral we have

$$\int_a^b \underline{f}(x,r)\mathrm{d}x \leq \int_a^b \underline{g}(x,r)\mathrm{d}x,\ \int_a^b \overline{f}(x,r)\mathrm{d}x \leq \int_a^b \overline{g}(x,r)\mathrm{d}x,\ \forall r \in [0,1].$$

So the required result is obtained. □

6.1. **Fuzzy Intermediate Value Theorem.** Now the intermediate value theorem states that if a and b are any two points in an interval on which f is continuous, then f takes every value between $f(a)$ and $f(b)$.

Theorem 6.4. *If $f(x)$ is continuous fuzzy-valued function on $[a,b]$, and there exists a fuzzy number γ such that $f(a) \preceq \gamma \preceq f(b)$, then there exists at least $c \in [a,b]$, such that $f(c) = \gamma$.*

Proof. Define

$$\mathcal{S} = \{x | x \in [a,b], f(x) \preceq \gamma\}.$$

Since $\mathcal{S}$ is a nonempty set and is bounded from above by b, $c = \sup \mathcal{S}$ exists as a real number.
First suppose that $f(c) \succ \gamma$. By Definition 2.1, for every $r \in [0,1]$ we have $\underline{f}(c;r) > \underline{\gamma}(r), \overline{f}(c;r) > \overline{\gamma}(r)$.
Moreover, since f is continuous we have

$$\forall \epsilon > 0,\ \exists \delta > 0,\ \forall x\big(\mid x - c \mid < \delta \Rightarrow D(f(x), f(c)) < \epsilon\big). \tag{15}$$

For any fixed $r \in [0,1]$, let $\epsilon = \min\{\underline{f}(c;r) - \underline{\gamma}(r), \overline{f}(c;r) - \overline{\gamma}(r)\}$, so from Definition of distance D find that

$$\mid \underline{f}(x;r) - \underline{f}(c;r) \mid < \epsilon \Rightarrow \underline{f}(x;r) > \underline{f}(c;r) - \epsilon \geq \underline{\gamma}(r), \tag{16}$$

and

$$\mid \overline{f}(x;r) - \overline{f}(c;r) \mid < \epsilon \Rightarrow \overline{f}(x;r) > \overline{f}(c;r) - \epsilon \geq \overline{\gamma}(r). \tag{17}$$

From (16) and (17), for all $x \in (c-\delta, c+\delta)$, we deduce $\underline{f}(x;r) > \underline{\gamma}(r)$, $\overline{f}(x;r) > \overline{\gamma}(r)$. This requires that $c-\delta$ be an upper bounded for $\mathcal{S}$, which is a contradiction, because no point in the interval $(c-\delta, c]$ for $\underline{f}(x;r) > \underline{\gamma}(r)$, $\overline{f}(x;r) > \overline{\gamma}(r)$, can be belonged to $\mathcal{S}$ and c is considered as the supremum for $\mathcal{S}$. We then conclude that $\underline{f}(c;r) \leq \underline{\gamma}(r)$, $\overline{f}(c;r) \leq \overline{\gamma}(r)$ for every $r \in [0,1]$. So, according to Definition 2.1, $f(c) \preceq \gamma$.

Now suppose that $f(c) \prec \gamma$. Again, by continuity and properties of D, for every $r \in [0,1]$ in (15) we have

$$| \underline{f}(x;r) - \underline{f}(c;r) | < \epsilon \Rightarrow \underline{f}(x;r) < \underline{f}(c;r) + \epsilon \leq \underline{\gamma}(r), \tag{18}$$

and

$$| \overline{f}(x;r) - \overline{f}(c;r) | < \epsilon \Rightarrow \overline{f}(x;r) < \overline{f}(c;r) + \epsilon \leq \overline{\gamma}(r). \tag{19}$$

From (18) and (19), for all $x \in (c-\delta, c+\delta)$, $\underline{f}(x;r) < \underline{\gamma}(r)$, $\overline{f}(x;r) < \overline{\gamma}(r)$, thus $c + \frac{\delta}{2} \in \mathcal{S}$, which is a contradiction. Hence, for every $r \in [0,1]$, $\underline{f}(c;r) \geq \underline{\gamma}(r)$, $\overline{f}(c;r) \geq \overline{\gamma}(r)$, and it means that $f(c) \succeq \gamma$. Therefore, by Proposition 6.2 gets $f(c) = \gamma$. □

As a consequence of fuzzy intermediate value theorem, fuzzy Bolzano's theorem is introduced to prove that a continuous fuzzy-valued function has at least one zero on any interval which the sign of a function changes.

Theorem 6.5. *Let the fuzzy-valued function is continuous on $[a,b]$ such that $f(a) \prec 0$ and $f(b) \succ 0$ or vice versa. Then f has at least one zero on (a,b).*

Proof. The proof derives from Theorem 6.4, taking $\gamma = 0$. □

Please note that the Bolzano's theorem does not indicate the value or values of zeros of functions, it only confirms their existence.

Example 6.6. Consider continuous fuzzy-valued function $f(x) = (2.3, 3, 3.9)(x-1)$ on an interval $[0,2]$ which is represented by $[(2.3+0.7r), (3.9-0.9r)](x-1)$. So we have

$$\begin{aligned} f(0;r) &= [-3.9+0.9r, -2.3-0.7r],\ 0 \leq r \leq 1, \\ f(2;r) &= [2.3+0.7r, 3.9-0.9r],\ 0 \leq r \leq 1. \end{aligned}$$

By Definition 2.1 $f(0) \prec 0$, $f(2) \succ 0$, hence according to Theorem 6.5, the fuzzy-valued function f has at least one zero on $[0,2]$. See Figure 1.

6.2. **Fuzzy Mean Value Theorem for Integrals.** The fuzzy mean value theorem for integrals is an applied theorem in numerical analysis. One of the main results of this theorem is that a continuous fuzzy-valued function has at least one point where the function equals the average value of the function.

Theorem 6.7. *Let $f : [a,b] \to \mathbb{R}_{\mathcal{F}}$ be a continuous fuzzy-valued function such that $f(a) \preccurlyeq f(x) \preccurlyeq f(b)$ for all $x \in [a,b]$, $g : [a,b] \to \mathbb{R}$ is an integrable real function on the interval $]a,b[$, then there exists a number $c \in]a,b[$ such that*

$$\int_a^b f(x)g(x)\,dx = f(c)\int_a^b g(x)\,dx.$$

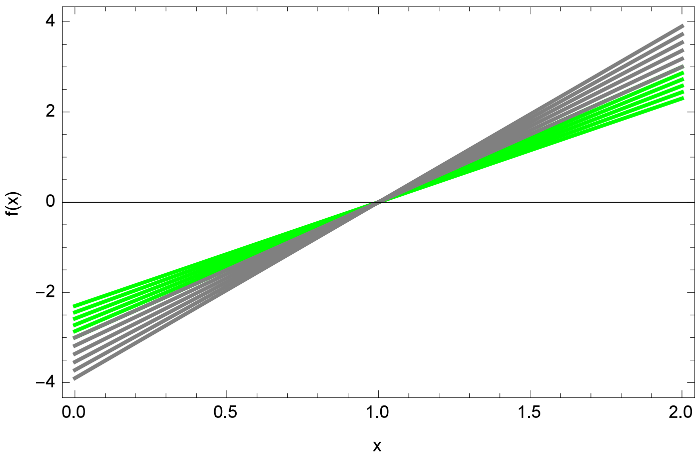

FIGURE 1. The Level Sets of the Function in Example 6.6

Proof. First we suppose that $g(x) > 0$. Then by Proposition 6.3 and assumption, we have

$$\int_a^b f(a)g(x)\mathrm{d}x \preccurlyeq \int_a^b f(x)g(x)\mathrm{d}x \preccurlyeq \int_a^b f(b)g(x)\mathrm{d}x.$$

Therefore,

$$f(a) \preccurlyeq \frac{\int_a^b f(x)g(x)\mathrm{d}x}{\int_a^b g(x)\mathrm{d}x} \preccurlyeq f(b).$$

By fuzzy intermediate value theorem, there exiets at least one element $c \in [a, b]$, such that

$$f(c) = \frac{\int_a^b f(x)g(x)\mathrm{d}x}{\int_a^b g(x)\mathrm{d}x}.$$

The proof of $g(x) \leq 0$ by considering proposition 6.1 is similar to the case when $g(x) > 0$, which proves the theorem. □

Corollary 6.8. *Let $f : [a, b] \to \mathbb{R}_{\mathcal{F}}$ be a continuous fuzzy-valued function such that $f(a) \preccurlyeq f(x) \preccurlyeq f(b)$ for all $x \in [a, b]$, then there exists a number $c \in]a, b[$ such that*

$$\int_a^b f(x)\,dx = f(c)(b - a).$$

Proof. It is an immediate consequence of Theorem 6.7. □

Example 6.9. Consider $f : [0, 1] \to \mathbb{R}_{\mathcal{F}}$ defined by $f(x) = (0.2, 0.35, 0.75)e^x + (0, 0.5, 1)$. So according to Corollary 6.8, there exists $c \in]1, 2[$ such that

$$f(c) \quad = \quad \int_0^1 f(x)\mathrm{d}x.$$

Therefore, we have

$$(0.2, 0.35, 0.75)e^c + (0, 0.5, 1) \quad = \quad (0.2, 0.35, 0.75)(e-1) + (0, 0.5, 1).$$

By Proposition 2.6 and 2.7 conclude that c is equal to 0.541325.

Theorem 6.10. *Let f be a fuzzy continuous gH-derivative on $]a, b[$ with no switching points. Let for all $x \in]a, b[$, $f'_{gH}(a) \preccurlyeq f'_{gH}(x) \preccurlyeq f'_{gH}(b)$ and $\varphi : [a, b] \to \mathbb{R}$ be an integrable real function on the interval $]a, b[$, then there exists at least a number $c \in]a, b[$ such that*

$$\int_a^b f'_{gH}(x)\varphi(x)\,dx = f'_{gH}(c)\int_a^b \varphi(x)\,dx.$$

Proof. The proof is immediate by substituting $f'_{gH}(x)$ instead of $f(x)$ in Theorem 6.7. □

6.3. **Fuzzy Mean Value Theorem for gH-Derivative.** In what follows, we present an essential theorem in fuzzy Calculus that is called a fuzzy mean value theorem for gH-derivative. Also some corollaries are derived of mean value theorem for gH-derivative of fuzzy-valued functions.

Theorem 6.11. *Let $f : [a, b] \to \mathbb{R}_{\mathcal{F}}$ be continuous fuzzy-valued function. If f has continuous gH-derivative on $]a, b[$, then there exists at least, $c \in]a, b[$ such that*

$$f'_{gH}(c) \quad = \quad \frac{f(b) \ominus_{gH} f(a)}{b-a}.$$

Proof. From assumption and Theorem 6.10, we have

$$\int_a^b f'_{gH}(x)\mathrm{d}x \quad = \quad f'_{gH}(c)(b-a).$$

Also by Theorem 5.1, find that

$$\int_a^b f'_{gH}(x)\mathrm{d}x \quad = \quad f(b) \ominus_{gH} f(a).$$

Then

$$f'_{gH}(c)(b-a) \quad = \quad f(b) \ominus_{gH} f(a).$$

So the theorem is proved. □

Example 6.12. Consider $f : (0, 2) \to \mathbb{R}_{\mathcal{F}}$ defined by $f(x) = (0.8, 1.1, 1.5)x^2$. So according to Theorem 6.11, there exists $c \in (0, 2)$ such that

$$f'_{gH}(c) \quad = \quad \frac{f(2) \ominus_{gH} f(0)}{2}.$$

Therefore, we have

$$(0.8, 1.1, 1.5)2c \quad = \quad \frac{(0.8, 1.1, 1.5)(4) \ominus_{gH} (0.8, 1.1, 1.5)(0)}{2}.$$

Then by Propositions 2.6 and 2.7 we find that $c = 1$. In Figure 2, we illustrate the result of Theorem 6.11.

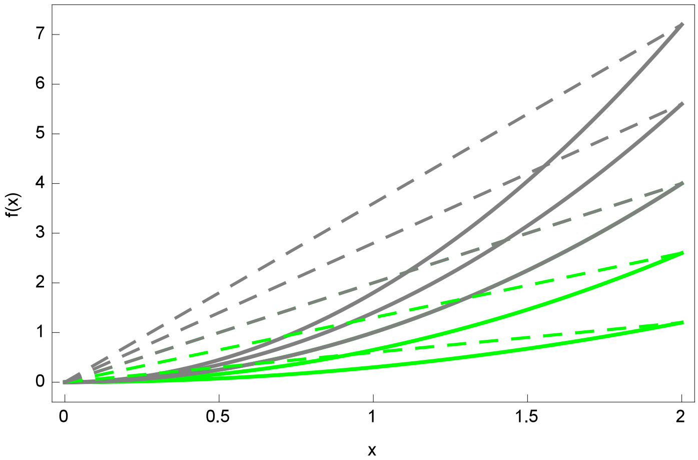

FIGURE 2. The Level Sets of the Function in Example 6.12

Now we introduce fuzzy Rolle's theorem.

Theorem 6.13. *Let f be continuous fuzzy-valued function on a interval $[a, b]$, and gH-differentiable on $]a, b[$. If $f(a) = f(b)$ then there exist at least a number $c \in]a, b[$ at which*

$$f'_{gH}(c) \quad = \quad 0.$$

Proof. It is a consequence of Theorem 6.11. □

Example 6.14. Let $f : [-\pi, \pi] \to \mathbb{R}_{\mathcal{F}}$ defined by $f(x) = (0.2, 1, 1.7) \sin x$ that graphed in Figure 3. The function f is continuous on $[-\pi, \pi]$ and is differentiable at every point of $(-\pi, \pi)$. Since $f(-\pi) = f(\pi) = 0$, the fuzzy Rolle's theorem implies that f'_{gH} must be zero at least once in the open interval $(-\pi, \pi)$. In fact, $f'_{gH}(x) = (0.2, 1, 1.7) \cos x$ is zero, once at $x = \frac{-\pi}{2}$ and again at $x = \frac{\pi}{2}$ in the interval $(-\pi, \pi)$.

Corollary 6.15. *Let $f : [a, b] \to \mathbb{R}_{\mathcal{F}}$ be gH-differentiable function and $f'_{gH}(x) = 0$ on]a, b[, then $f(x)$ is constant on $]a, b[$.*

Proof. Take $x_1, x_2 \in]a, b[$, by assumption $f(x)$ is continuous and differentiable on $[x_1, x_2] \subset]a, b[$, then Theorem 6.11 implies that

$$f(x_1) \ominus_{gH} f(x_2) = (x_1 - x_2) f'_{gH}(c),$$

so we conclude that

$$f(x_1) \ominus_{gH} f(x_2) \quad = \quad 0 \Rightarrow \; f(x_1) = f(x_2).$$

□

Corollary 6.16. *Let $f, g : [a, b] \to \mathbb{R}_{\mathcal{F}}$ be two gH-differentiable functions on $]a, b[$. If $f'_{gH}(x) = g'_{gH}(x)$ for all $x \in]a, b[$, then in $f(x) = g(x) \oplus c$, where c is some constant.*

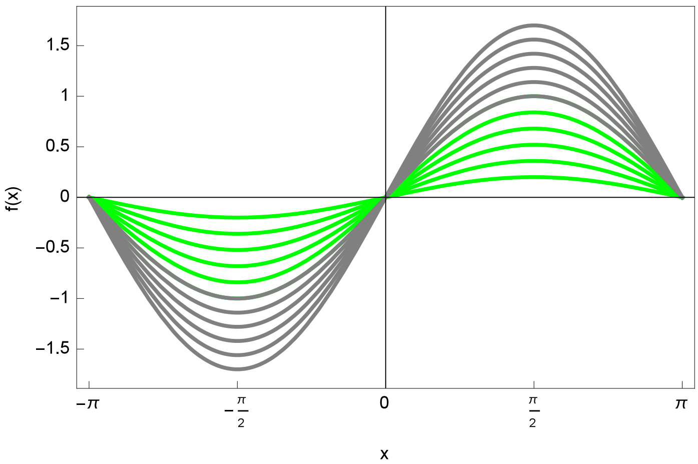

FIGURE 3. The Level Sets of the Function in Example 6.14

Proof. Define $h(x) = f(x) \ominus_{gH} g(x)$, by Theorem 4.3 the gH-derivative of $h(x)$ is

$$h'_{gH}(x) \quad = \quad f'_{gH} \ominus_{gH} g'_{gH},$$

Also by assumption $f'_{gH}(x) = g'_{gH}(x)$ for all $x \in]a,b[$, then $h'_{gH}(x) = 0$ on this interval. Hence by Corollary 6.15 we have that $h(x)$ is a constant, and this implies that $f(x) = g(x) \oplus c$ for all $x \in]a,b[$. □

7. Conclusion

In this paper, we suggested the sum, gH-difference, constant multiple and product rules for finding limits and gH-derivative of the fuzzy-valued functions. Moreover, the chain rule was introduced for $gH-$differentiability and we showed that a gH-differentiable function is a fuzzy continuous function. The integration by parts was obtained for computing some integrals by changing to easier integrals. Also, we derived the fuzzy Leibniz's rule for obtaining gH-derivative of the integrals. Furthermore, some fundamental theorems and rules were presented for fuzzy calculus, including intermediate value theorem, Bolzano's theorem, mean value theorem for integral, mean value theorem for gH-derivative and Rolle's theorem. Hence, with these theorems and rules, we had shown that the continuous fuzzy-valued functions take all the value in between. This property tells us that if f is continuous, then any interval on which f changes sign must contain a zero of the function. Also, fuzzy mean value theorem for integral was introduced for producing of a crisp function and a continuous fuzzy-valued function which it can be very useful in the fuzzy numerical analysis. One of the most important theorems in fuzzy calculus is the mean value theorem for gH-derivative. We proved this theorem for the fuzzy-valued functions under generalized Hukuhara differentiability and Rolle's theorem was concluded from it. Moreover, two properties were provided for gH-differentiability as a consequence of mean value theorem for gH-derivative. Some examples were given to illustrate the application of these assertions. For future research, we will present some numerical methods based on these results for solving fuzzy nonlinear equations.

Notice 1: The authors declare that they have no conflict of interest.

Notice 2: This article does not contain any studies with human participants or animals performed by any of the authors.

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