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INTEGRABILITY OF AN INTERVAL-VALUED MULTIFUNCTION WITH RESPECT TO AN INTERVAL-VALUED SET MULTIFUNCTION

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Abstract. Intervals are related to the representation of uncertainty. In this sense, we introduce an integral of Gould type for an interval-valued multifunction relative to an interval-valued set multifunction, with respect to Guo and Zhang order relation. Classical and specific properties of this new type of integral are established and several examples and applications from multicriteria decision making problems are provided.

1. Introduction

In the last decades, besides being an important mathematical theory, set-valued functions theory has become an important tool in several practical areas, especially in economic analysis, where it treats problems of individual demand, mean demand, competitive equilibrium, coalition production economies etc. For instance, applications of integration of set-valued functions in economy analysis have roots in Aumann's [3] research based on the classical Lebesgue integral.

Different types of integrals have been introduced and studied in order to generalize the Riemann integral. In this framework, a way of defining the integral is to use finite or infinite Riemann type sums as in, for instance, [4], [5], [6], [7], [8], [12], [17], [27], [41], [44]. An influential work in this direction was Gould's study [27], where the concept of an integral using finite sums for real functions with respect to finitely additive vector measures, is introduced. The Gould integral was generalized and studied in [19], [20], [25] (relative to submeasures), [38], [40] (relative to multimeasures), [21], [22], [42] (relative to multisubmeasures), [39] (relative to monotone set-valued set functions).

A special type of an interval-valued set multifunction was introduced by Sofian-Boca [42], with respect to the order relation of Guo and Zhang [28], in order to study a Gould type integral of a real function with respect to it. This type of set multifunction was also investigated by Gavriluţ in [24]. In [31] - [35] Jang studied interval valued Choquet integrals and pointed out their applications in multicriteria decision making problems. Although the procedure of the construction is analogue, our integral is a generalization of integral treated in [42], being define for an interval-valued set multifunction with respect to an other interval-valued set multifunction

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with respect to the order relation of Guo and Zhang [28]. Also, this integral is different from the one introduced by Jang [33] and it is an useful tool in modeling uncertainties, as we will further highlight.

On the other hand, fuzzy sets introduced by Zadeh [46] are used to deal with imprecision and uncertainty information. With the development of the fuzzy sets, many new theories came out, such as the interval valued fuzzy sets and the generalized theory of uncertainty. In fact, during the last decade, it has been suggested to use intervals in order to represent uncertainty in the area of decision theory and information theory, for example, calculation of economic uncertainty, theory of interval probability as a unifying concept for uncertainty. Thus, a special attention was recently paid to the study of interval-valued (set) multifunctions, since they are related to the representation of uncertainty, a necessity coming from economic uncertainty, fuzzy random variables, interval-probability, martingales of multivalued functions, interval-valued capacities (Bykzkan and Duan [9], Jang [31] - [35] (in multicriteria decision making problems), Li and Sheng [36], Weichselberger [45] and many others).

Interval valued functions are associated with the representation of uncertain functions. It has been suggested to use intervals in order to represent uncertainty, for instance, closed set-valued functions, interval-valued probabilities, fuzzy set-valued measures, economic uncertainties.

Interval valued measures are intensively used in interval valued intuitionistic fuzzy sets. Entropy, the well-known concept in physics, information theory and fuzzy set theory, describes the degree of uncertainties and fuzziness of the fuzzy sets. Entropy and similarity of intuitionistic fuzzy sets are very important in theory and applications in which the intuitionistic fuzzy sets are used to describe the imprecisions and uncertainties.

In this paper, motivated by the diversity of fields where interval-valued (set) multifunctions are applied, we introduce a new type of integral (modeling the expectation) of an interval-valued multifunction F relative to an interval-valued set multifunction μ, with respect to Guo and Zhang order relation [28]. So, the novelty of our paper is given by the fact that both F and μ are set-valued. Our integral, different from the one proposed by Jang in [31] - [35], provides a special, useful tool in modeling uncertainties. The paper is organized as follows: Section 1 is for introduction. After a section of basic concepts, in Section 3 we present various examples of order relations and results regarding monotone set multifunctions. In Section 4 we introduce the Gould integral of interval-valued multifunction relative to an interval-valued set multifunction. Various properties of the integral are established, concerning the behaviour with respect to the set multifunctions F or to μ. Section 5 presents results regarding important properties of this integral. So, we point out some of its continuity properties and we prove a result concerning the behaviour on atoms.

2. **Preliminaries**

If $n \in \mathbb{N}^* = \mathbb{N} \setminus \{0\}$, by $i = \overline{1,n}$ we mean $i \in \{1, ..., n\}$. Let be $\mathbb{R}_+ = [0, \infty)$, T a nonempty abstract set, $\mathcal{P}(T)$ the family of all subsets of T and $\mathcal{A}$ an arbitrary algebra of subsets of T.

Definition 2.1. A finite partition of T is a finite family of nonempty sets $P = \{A_i\}_{i=\overline{1,n}} \subset \mathcal{A}$ such that $A_i \cap A_j = \emptyset, i \neq j$ and $\bigcup\limits_{i=1}^{n} A_i = T$.

We denote by $\mathcal{P}$ the class of all partitions of T and by $\mathcal{P}_{\mathrm{A}}$, the class of all partitions of A, if $\mathrm{A} \in \mathcal{A}$ is fixed.

Definition 2.2. (i) If P, $P' \in \mathcal{P}$, P' is said to be finer than P (denoted by $P \leq P'$ or $P' \geq P$) if every set of P' is included in some set of P.

(ii) The common refinement of two finite partitions $P = \{A_i\}_{i=\overline{1,n}}$, $P' = \{B_j\}_{j=\overline{1,m}} \in \mathcal{P}$ is the partition $P \wedge P' = \{A_i \cap B_j\}_{\substack{i=\overline{1,n} \\ j=\overline{1,m}}}$.

Monotone set functions

In this subsection, letm be an arbitrary set function $m : \mathcal{A} \to \mathbb{R}_+$, with $m(\emptyset) = 0$.

Definition 2.3. [37] I. m is said to be:

(i) monotone or fuzzy if $m(A) \leq m(B)$, for every $A, B \in \mathcal{A}$, with $A \subseteq B$;

(ii) subadditive if $m(A \cup B) \leq m(A) + m(B)$, for every (disjoint) $A, B \in \mathcal{A}$;

iii) a submeasure (in the sense of Drewnowski [18]) if it is monotone and subadditive;

(iv) null-additive if $m(A \cup B) = m(A)$, for every $A, B \in \mathcal{A}$, with $m(B) = 0$;

(v) σ-subadditive if $m(A) \leq \sum\limits_{n=0}^{\infty} m(A_n)$, for every (pairwise disjoint) $(A_n)_{n\in\mathbb{N}} \subset \mathcal{A}$, with $A = \bigcup\limits_{n=0}^{\infty} A_n \in \mathcal{A}$;

(vi) finitely additive if $m(A \cup B) = m(A) + m(B)$, for every disjoint $A, B \in \mathcal{A}$;

(vii) σ-additive if $m(\bigcup\limits_{n=0}^{\infty} A_n) = \sum\limits_{n=0}^{\infty} m(A_n)$, for every pairwise disjoint $(A_n)_{n\in\mathbb{N}} \subset \mathcal{A}$;

II. A set $A \in \mathcal{A}$ is an atom with respect to m if $m(A) > 0$ and for every $B \in \mathcal{A}$, with $B \subset A$, we have either $m(B) = 0$ or $m(A \setminus B) = 0$.

III. m is said to be finitely purely atomic if $T = \bigcup\limits_{i=1}^{p} A_i$, where $A_i \in \mathcal{A}$, $i = \overline{1,p}$ are pairwise disjoint atoms of m.

Definition 2.4. (i) The variation $\overline{m}$ of m is the set function $\overline{m} : \mathcal{P}(T) \to [0, +\infty]$ defined by $\overline{m}(E) = \sup\{\sum\limits_{i=1}^{n} m(A_i)\}$, for every $E \in \mathcal{P}(T)$, where the supremum is extended over all finite families of pairwise disjoint sets $\{A_i\}_{i=1}^{n} \subset \mathcal{A}$, with $A_i \subseteq E$, for every $i = \overline{1,n}$.

(ii) m is said to be of finite variation on $\mathcal{A}$ if $\overline{m}(T) < \infty$.

Remark 2.5. I. If $E \in \mathcal{A}$, then in the definition of $\overline{m}$ one may consider the supremum over all finite partitions $\{A_i\}_{i=1}^{n} \in \mathcal{P}_E$.

II. $\overline{m}$ is monotone and super-additive on $\mathcal{P}(T)$, i.e., $\overline{m}(\bigcup\limits_{i\in I} A_i) \geq \sum\limits_{i\in I} \overline{m}(A_i)$, for every finite or countable $\{A_i\}_{i\in I} \in \mathcal{P}$.

III. If m is finitely additive, then $\overline{m}(A) = m(A)$, for every $A \in \mathcal{A}$.

IV. If m is subadditive ($\sigma-$subadditive, respectively) of finite variation, then $\overline{m}$ is finitely additive (σ-additive, respectively).

Example 2.6. Monotone measures are of a great importance in multicriteria decision making problems (see [35] for details): if $N = \{1, ..., n\}$, then for a monotone set function m defined on $\mathcal{P}(N)$, for every $S \in \mathcal{P}(N)$, $m(S)$ can be interpreted as the weight or the degree of importance of combination S of criteria, or better, its power to make the decision alone (without the remaining criteria). In addition to the usual weights on criteria taken separately, weights on any combination of criteria are also defined. Monotonicity means that adding a new element to a combination cannot decrease its importance.

3. Monotone Set Multifunctions

We denote by $\mathcal{P}_0(\mathbb{R}_+)$ the family of all nonempty subsets of $\mathbb{R}_+$ and by $\mathcal{P}_{kc}(\mathbb{R}_+)$ the family of nonempty, compact convex subsets of $\mathbb{R}_+$. Let "$\widetilde{\leq}$ " be an arbitrary order relation on $\mathcal{P}_0(\mathbb{R}_+)$. By "$\widetilde{\lneq}$" we mean "$\widetilde{\leq}$" and $''\neq''$.

Example 3.1. [28] (the "standard" partial order relation "$\preceq$" on $\mathcal{P}_0(\mathbb{R}_+)$, which extends the usual order on $\mathcal{P}_{kc}(\mathbb{R}_+)$) : If $A, B \in \mathcal{P}_0(\mathbb{R}_+)$, then $A \preceq B$ if the following two conditions hold:
(i) for every $x \in A$, there exists $y_x \in B$ so that $x \leq y_x$;
(ii) for every $y \in B$, there exists $x_y \in A$ so that $x_y \leq y$.

In general, there is no implication between the order relation "$\preceq$" and the inclusion one. However, on the family $\{[0, a]; 0 \leq a < \infty\}$ they coincide. By convention, $\{0\} = [0, 0]$.

If $[a, b], [c, d] \in \mathcal{P}_{kc}(\mathbb{R}_+)$, the following operations are considered:

I. $[a, b] + [c, d] = [a + b, c + d]$;
II. $\alpha \cdot [a, b] = [\alpha a, \alpha b], \alpha \geq 0$;
III. $[a, b] \cdot [c, d] = [a \cdot c, b \cdot d]$;
IV. $[a, b] \wedge [c, d] = [\min\{a, c\}, \min\{b, d\}]$;
V. $[a, b] \vee [c, d] = [\max\{a, c\}, \max\{b, d\}]$;
VI. $[a, b] \subseteq [c, d]$ if and only if $c \leq a \leq b \leq d$;
VII. $[a, b] \preceq [c, d]$ if and only if $a \leq c$ and $b \leq d$.

Example 3.2. [10] Let $A, B : [0, 1]^2 \to [0, 1]$ be two aggregation functions [26] such that $A(a, b) = A(c, d)$ and $B(a, b) = B(c, d)$ can happen only if $(a, b) = (c, d)$. Define a relation "$\preceq_{A,B}$" on $\{[a, b]; 0 \leq a \leq b \leq 1\}$ by $[a, b] \preceq_{A,B} [c, d]$ whenever $A(a, b) < A(c, d)$ or $A(a, b) = A(c, d)$ and $B(a, b) \leq B(c, d)$. Then $\preceq_{A,B}$ is a linear order on $\{[a, b]; 0 \leq a \leq b \leq 1\}$ with the minimal element $\{0\} = [0, 0]$, and the maximal element $\{1\} = [1, 1]$.

Remark 3.3. I) The linear order "$\preceq_{A,B}$" refines the above standard partial order "$\preceq$" on intervals from Example 3.1: $[a,\ b] \preceq [c, d]$ whenever $a \leq c$ and $b \leq d$, so $[a,\ b] \preceq [c, d]$ implies $[a,\ b] \preceq_{A,B} [c, d]$.

II. By taking $A(a,b)=a$, $B(a,b)=b$ and $A(a,b)=b$, $B(a,b)=a$, respectively, in the previous example, the following lexicographical order relations can be recovered: $[a,b]\preceq_1[c,d]$ whenever $a<c$ or $a=c$ and $b\le d$ and $[a,b]\preceq_2[c,d]$ whenever $b<d$ or $b=d$ and $a\le c$, respectively.

Example 3.4. In [10], an algorithm that makes use of entropies and the interval valued set function concepts is proposed to determine which is the best alternative between a set of them following some criteria that are provided by one or several experts. Suppose that a set of alternatives $\{A_1,...,A_n\}$ and a set of criteria $\{x_1,...,x_k\}$ are given. Then one can write the following multicriteria decision making (MCDM) matrix:

$$\begin{array}{c} \\ A_1 \\ A_2 \\ \\ A_n \end{array}\begin{array}{c} \begin{array}{ccc} x_1 & \dots & x_k \end{array} \\ \begin{pmatrix} ([\underline{\mu_{A_1}}(x_1),\overline{\mu_{A_1}}(x_1)] & \dots & [\underline{\mu_{A_1}}(x_k),\overline{\mu_{A_1}}(x_k)] \\ ([\underline{\mu_{A_2}}(x_1),\overline{\mu_{A_2}}(x_1)] & \dots & [\underline{\mu_{A_2}}(x_k),\overline{\mu_{A_2}}(x_k)] \\ \dots & \dots & \\ ([\underline{\mu_{A_n}}(x_1),\overline{\mu_{A_n}}(x_1)] & \dots & [\underline{\mu_{A_n}}(x_k),\overline{\mu_{A_n}}(x_k)] \end{pmatrix}\end{array}$$

where $[\underline{\mu_{A_i}}(x_j),\overline{\mu_{A_i}}(x_j)]$ denotes the degree to which alternative A_i satisfies criterion x_j. We assume that this satisfaction is expressed in an interval-valued way. Observe that in this way we can understand each alternative as an interval-valued fuzzy set over the referential set of criteria and in such a way that each of the intervals $\mu(A_i)(x_j)=[\underline{\mu_{A_i}}(x_j),\overline{\mu_{A_i}}(x_j)]$ provides the membership value of criteria x_j to the interval valued fuzzy set A_i.

We now introduce in the set-valued case corresponding notions, which generalize the ones from Definition 2.3. In order to do this, we consider on $\mathcal{P}_0(\mathbb{R})$ the Hausdorff-Pompeiu pseudo-metric h [29] defined for every $A,B\in\mathcal{P}_0(\mathbb{R})$ by

$$h(A,B)=\max\{\sup_{x\in A}\inf_{y\in B}|x-y|,\sup_{y\in B}\inf_{x\in A}|x-y|\}.$$

On $\mathcal{P}_{kc}(\mathbb{R}), h$ has the particular form:

$$h([a,b],[c,d])=\max\{|a-c|,|b-d|\},\forall a,b,c,d\in\mathbb{R},a\le b,c\le d.$$

If, particularly, $0<a<c$, then $h([0,a],[0,c])=c-a$.

According to [29], $(\mathcal{P}_{kc}(\mathbb{R}),h)$ is a complete metric space.

For every $M\in\mathcal{P}_{kc}(\mathbb{R}), M=[a,b]$, we denote by $|M|=h(M,\{0\})(=\max\{|a|,|b|\})$. If $M\in\mathcal{P}_{kc}(\mathbb{R}_+), M=[a,b]$, then $|M|=b$. An interval-valued profile means uncertain profile. Such intervals can be interpreted as interval-valued functions from $\{1,2,...,n\}$ to $\mathcal{P}_{kc}(\mathbb{R}_+)$ describing in this way uncertain consumption or uncertain income at each time point. So, **if "$\widetilde{\le}$" is an arbitrary order relation on $\mathcal{P}_0(\mathbb{R}_+)$**, we introduce the following corresponding notions that generalize the ones already recalled in Definition 2.3.

Definition 3.5. Let $\mu:\mathcal{A}\to\mathcal{P}_0(\mathbb{R}_+)$ be a set multifunction, with $\mu(\emptyset)=\{0\}$. I. μ is said to be:

(i) an additive multimeasure if $\mu(A\cup B)=\mu(A)+\mu(B)$, for every disjoint $A,B\in\mathcal{A}$;

(ii) $\widetilde{\le}$ $-$monotone if $\mu(A)\,\widetilde{\le}\,\mu(B)$, for every $A,B\in\mathcal{A}$, with $A\subseteq B$;

(iii) $\widetilde{\leq}$ −subadditive if $\mu(A \cup B) \widetilde{\leq} \mu(A) + \mu(B)$, for every disjoint $A, B \in \mathcal{A}$;

(iv) a $\widetilde{\leq}$ − multisubmeasure if it is $\widetilde{\leq}$ −monotone and $\widetilde{\leq}$ −subadditive;

(v) null-additive if $\mu(A \cup B) = \mu(A)$, for every $A, B \in \mathcal{A}$, with $\mu(B) = \{0\}$;

II. $A \in \mathcal{A}$ is an $\widetilde{\leq}$ −atom of μ if $\{0\} \widetilde{\leq} \mu(A), \{0\} \neq \mu(A)$ and for every $B \in \mathcal{A}$, with $B \subset A$, we have either $\mu(B) = \{0\}$ or $\mu(A \setminus B) = \{0\}$.

III. μ is said to be $\widetilde{\leq}$ −finitely purely atomic if $T = \bigcup_{i=1}^{p} A_i$, where $A_i \in \mathcal{A}$, $i = \overline{1,p}$ are pairwise disjoint $\widetilde{\leq}$ −atoms of μ.

Remark 3.6. i) If "$\widetilde{\leq}$" is the usual inclusion of sets $\subseteq$, then one obtains the corresponding notions from [21].

ii) If "$\widetilde{\leq}$" is the relation "$\preceq$" from Example 3.1, then by [24], $\mu : \mathcal{A} \to \mathcal{P}_{kc}(\mathbb{R}_+)$ if and only if there exist two set functions $m_1, m_2 : \mathcal{A} \to \mathbb{R}_+$ so that for every $A \in \mathcal{A}$, $m_1(A) \leq m_2(A)$ ("$\leq$" is the usual order on $\mathbb{R}$) and $\mu(A) = [m_1(A), m_2(A)]$. Moreover, in this case, μ is monotone, subadditive, a multisubmeasure, null-additive, respectively with respect to "$\preceq$" (called in what follows in this paper *q-monotone, q-subadditive, q-multisubmeasure, q-null-additive*), if and only if both m_1, m_2 are monotone, subadditive, submeasures (in the sense of Drewnowski [18]), null-additive, respectively, in the sense of Definition 2.3. The same for the notion of an atom.

For any arbitrary fixed set $A \in \mathcal{A}, m_1(A)$ and $m_2(A)$ can be interpreted as *the lower* and *the upper* respectively limits of the confidence interval $\mu(A) = [m_1(A), m_2(A)]$.

Example 3.7. If $m_1, m_2 : \mathcal{A} \to [0, 1]$ are two finite measures, with $m_1(A) \leq m_2(A), \forall A \in \mathcal{A}$ and if m_2 is a probability measure, then one can consider the multimeasure $\mu : \mathcal{A} \to \mathcal{P}_0([0, 1])$, $\mu(A) = [m_1(A), m_2(A)]$, $\forall A \in \mathcal{A}$. Since $1 \in \mu(T)$, this is the simplest example of a probability multimeasure, used in control, robotics and decision theory (in Bayesian estimation).

Now, if we suppose that $\mu : \mathcal{A} \to \mathcal{P}_0([0, 1])$ is an h-multisubmeasure (i.e., the set function $|\mu| : \mathcal{A} \to \mathbb{R}_+$, defined by $|\mu|(A) = |\mu(A)|$, for every $A \in \mathcal{A}$ is a submeasure in the sense of Drewnowski [18]), with $\overline{\mu}(T) > 0$, then one can generate a system of upper and lower probabilities with applications in statistical inference (Dempster [16]):

Let $\mathcal{M} = \{E \subset [0, 1]; \mu^{-1}(E), \mu^{+1}(E) \in \mathcal{A}\}$, where for every $E \subset [0, 1]$, $\mu^{-1}(E) = \{t \in T; \mu(\{t\}) \cap E \neq \emptyset\}$ and $\mu^{+1}(E) = \{t \in T; \mu(\{t\}) \subset E\}$.

For every $E \in \mathcal{M}$, we define *the upper probability* of $E, P^*(E) = \frac{\overline{\mu}(\mu^{-1}(E))}{\overline{\mu}(T)}$ and *the lower probability* of E, $P_*(E) = \frac{\overline{\mu}(\mu^{+1}(E))}{\overline{\mu}(T)}$.

We observe that P^*, $P_* : \mathcal{M} \to [0, 1]$ and $P_*(E) \leq P^*(E)$, for every $E \in \mathcal{M}$.

One may consider $\overline{\mu}(\mu^{-1}(E))$ as being the largest possible amount of probability from the measure $\overline{\mu}$ that can be transferred to outcomes $x \in E$ and $\overline{\mu}(\mu^{+1}(E))$ as the minimal amount of probability that can be transferred to outcomes $x \in E$.

Example 3.8. Suppose $m : \mathcal{A} \to \mathbb{R}_+$ is a probability.

In Dempster-Shafer's ([16], [43]) mathematical theory of evidence, belief functions *Belief* (*Bel*) and *Plausibility* (*Pl*) are defined by a probability distribution $m : \mathcal{P}(T) \to [0,1]$, with $m(\emptyset) = 0$ and $\sum_{A \subseteq T} m(A) = 1$.

For every $A \subseteq T, Bel(A) = \sum_{B \subseteq A} m(B)$ and $Pl(A) = \sum_{B, B\cap A \neq \emptyset} m(B)$.

i) $Bel(A) + Pl(cA) = 1$; $Bel(A) \leq Pl(A)$.

ii) $Bel(T) = 1, Bel(\emptyset) = 0, Pl(T) = 1, Pl(\emptyset) = 0$.

iii) $Bel(\bigcup_{i=1}^{n} A_i) \geq \sum_{\emptyset \neq S \subseteq \{A_1,\dots,A_n\}} (-1)^{|S|-1} Bel(\bigcap_{A_i \in S} A_i)$ (a general version of superadditivity, called *infty-monotone*), for every $n \in \mathbb{N}^*$ and every $\{A_1, ..., A_n\} \subset T$. for $n = 2, Bel(A_1 \cup A_2) \geq Bel(A_1) + Bel(A_2) - Bel(A_1 \cap A_2)$.

iv) $Pl(\bigcup_{i=1}^{n} A_i) \leq \sum_{\emptyset \neq S \subseteq \{A_1,\dots,A_n\}} (-1)^{|S|-1} Pl(\bigcap_{A_i \in S} A_i)$ (a general version of subadditivity, called *infty-alternating*).

for $n = 2, Pl(A_1 \cup A_2) \leq Pl(A_1) + Pl(A_2) - Pl(A_1 \cap A_2)$.

Belief and Plausibility non-additive measures identify a family of probability distribution for which they are lower and upper probability measures: for every $A \subseteq T, P(A) \in [Bel(A), Pl(A)]$.

The *Belief Interval of A* is the range defined by the minimum and maximum values which could be assigned to A: $[Bel(A), Pl(A)]$. This interval probability representation contains the precise probability of a set of interest (in the classical sense). The probability is uniquely determined if $Bel(A) = Pl(A)$. In this case, which corresponds to the classical probability, all the probabilities, $P(A)$, are uniquely determined for all subsets of T.

In what follows, our aim is to introduce an integral of an interval valued multifunction $F : T \to \mathcal{P}_{kc}(\mathbb{R}_+)$ with respect to an interval-valued set multifunction $\mu : \mathcal{A} \to \mathcal{P}_{kc}(\mathbb{R}_+)$, in both cases $\mathcal{P}_{kc}(\mathbb{R}_+)$ being endowed with the order relation "$\preceq$" from Example 3.1.

Taking into account Remark 3.6-ii), we are led to multifunctions $\mu : \mathcal{A} \to \mathcal{P}_{kc}(\mathbb{R}_+)$ defined for every $A \in \mathcal{A}$, by $\mu(A) = [m_1(A), m_2(A)]$, with $m_1(A) \leq m_2(A)$, for every $A \in \mathcal{A}$ and $F : T \to \mathcal{P}_{kc}(\mathbb{R}_+)$, defined by $F(t) = [f_1(t), f_2(t)]$, for every $t \in T$, with $f_1, f_2 : T \to \mathbb{R}_+$, $f_1(t) \leq f_2(t)$, for every $t \in T$. This will be the framework in what follows. Also, suppose **the relation "$\widetilde{\leq}$" is "$\preceq$" from Example 3.1**.

Remark 3.9. I. If $\nu : \mathcal{A} \to \mathbb{R}_+$ is finitely additive, then $m_1, m_2 : \mathcal{A} \to \mathbb{R}_+$, defined by $m_1(A) = \frac{\nu(A)}{1+\nu(A)}$ and $m_2(A) = \sqrt{\nu(A)}$, for every $A \in \mathcal{A}$ are submeasures (in the sense of Drewnowski [18]), so one can immediately generate a q-multisubmeasure (in the sense of [42] - see also [24]) $\mu : \mathcal{A} \to \mathcal{P}_{kc}(\mathbb{R}_+)$, defined by $\mu(A) = [m_1(A), m_2(A)], \forall A \in \mathcal{A}$.

II. A set $A \in \mathcal{A}$ is a q-atom of μ if and only if it is an atom of both m_1 and m_2 (in the sense of Definition 2.3-II).

III. Suppose μ is null-additive and has q-atoms. If A is a q-atom of μ, then every $B \in \mathcal{A}$, with $B \subseteq A$ and $\{0\} \prec \mu(A)$, is also a q-atom of μ and $\mu(A \setminus B) = \{0\}$, so $\mu(A) = \mu(B)$ also holds.

IV. $\overline{\mu} = \overline{m_2}$ (on $\mathcal{A}$).

V. If μ is a q-multisubmeasure and if $A \in \mathcal{A}$ is a q-atom of μ, then $\overline{\mu}(A) = |\mu(A)| = m_2(A) = \overline{m_2}(A)$.

4. Gould Integrability

In this section, we study Gould integrability of interval-valued multifunctions with respect to interval-valued set multifunctions, pointing out various properties of this integral.

For the beginning, let be $m : \mathcal{A} \to \mathbb{R}_+$, with $m(\emptyset) = 0$.

Definition 4.1. [27] A real function $f : T \to \mathbb{R}$ is said to be:

I. $\overline{m}$-totally measurable (on $(T, \mathcal{A}, m)$) if for every $\varepsilon > 0$, there exists $P_\varepsilon = \{A_i\}_{i=\overline{0,n}} \in \mathcal{P}$ such that:

(i) $\overline{m}(A_0) < \varepsilon$ and

(ii) $\operatorname{osc}(f, A_i) = \sup\limits_{t,s \in A_i} |f(t) - f(s)| < \varepsilon$, for every $i = \overline{1,n}$.

II. $\overline{m}$-totally measurable on $B \in \mathcal{A}$ if the restriction $f|_B$ of f to B is $\overline{m}$-totally measurable on $(B, \mathcal{A}_\mathcal{B}, \mu_B)$, where $m_B = m|_{\mathcal{A}_B}$ and $\mathcal{A}_B = \{A \cap B; A \in \mathcal{A}\}$.

We consider $\sigma_{f,m}(P)$ (for short, $\sigma(P)) = \sum\limits_{i=1}^{n} f(t_i)m(A_i)$, for every $P = \{A_i\}_{i=\overline{1,n}} \in \mathcal{P}$ and every $t_i \in A_i, i = \overline{1,n}$.

Definition 4.2. [27] f is said to be:

I. Gould m-integrable on T if the net $(\sigma(P))_{P \in (\mathcal{P}, \leq)}$ is convergent in $\mathbb{R}$.
In this case, its limit is called the Gould integral of f on T with respect to m, denoted by $\int_T f dm$.

II. Gould m-integrable on $B \in \mathcal{A}$ if $f|_B$ is Gould m-integrable on $(B, \mathcal{A}_\mathcal{B}, m_B)$.

Remark 4.3. I. If it exists, the integral of f is unique.
II f is Gould m-integrable on T if and only if there exists $\alpha \in \mathbb{R}$ such that for every $\varepsilon > 0$, there exists $P_\varepsilon \in \mathcal{P}$ so that for every $P = \{A_i\}_{i=\overline{1,n}} \in \mathcal{P}$, with $P \geq P_\varepsilon$ and every $t_i \in A_i$, $i = \overline{1,n}$, we have $|\sigma(P) - \alpha| < \varepsilon$.
III. If $f \geq 0$ on T and f is m-integrable on T, then $\int_T f dm \geq 0$.

We now introduce total-measurability and Gould integrability for an interval-valued multifunction with respect to an interval-valued set multifunction. For this, unless stated otherwise, in what follows suppose $\mu : \mathcal{A} \to \mathcal{P}_{kc}(\mathbb{R}_+)$ is a set multifunction, defined by $\mu(A) = [m_1(A), m_2(A)]$, where $m_1, m_2 : \mathcal{A} \to \mathbb{R}_+$, with $m_1(\emptyset) = m_2(\emptyset) = 0$, $m_1(A) \leq m_2(A)$, for every $A \in \mathcal{A}$. Also, let $F : T \to \mathcal{P}_{kc}(\mathbb{R}_+)$ be an interval-valued multifunction, defined by $F(t) = [f_1(t), f_2(t)]$, for every $t \in T$, with $f_1, f_2 : T \to \mathbb{R}_+$, $f_1(t) \leq f_2(t)$, for every $t \in T$.

Definition 4.4. To the set multifunction μ we associate the set function $\overline{\mu} : \mathcal{P}(T) \to [0, \infty]$ (called the variation of μ) defined for every $E \subseteq T$, by $\overline{\mu}(E) = \sup\{\sum\limits_{i=1}^{n} |\mu(E_i)|\}$, where the supremum is extended over all finite families of pairwise disjoint sets $\{E_i\}_{i=1}^{n} \subset \mathcal{A}$, with $E_i \subseteq E$, for every $i = \overline{1,n}$.

μ is said to be of finite variation on $\mathcal{A}$ if $\overline{\mu}(T) < \infty$.

Definition 4.5. F is said to be:

I. $\overline{\mu}$-totally measurable (on $(T, \mathcal{A}, \mu)$) if for every $\varepsilon > 0$, there exists $P_\varepsilon = \{A_i\}_{i=\overline{0,n}} \in \mathcal{P}$ such that:

(i) $\overline{\mu}(A_0) < \varepsilon$ and

(ii) $\operatorname{osc}(F, A_i) = \sup\limits_{t,s \in A_i} h(F(t), F(s)) < \varepsilon$, for every $i = \overline{1,n}$.

II. $\overline{\mu}$-totally measurable on $B \in \mathcal{A}$ if $F|_B$ is $\overline{\mu}$-totally measurable on $(B, \mathcal{A}_{\mathcal{B}}, \mu_B)$.

Remark 4.6. I. If F is $\overline{\mu}$-totally measurable on T, then it is $\overline{\mu}$-totally measurable on every $A \in \mathcal{A}$.

II. F is $\overline{\mu}$-totally measurable on T if and only if the functions f_1, f_2 are $\overline{m_2}$-totally measurable.

Definition 4.7. For a multifunction $F : T \to \mathcal{P}_{kc}(\mathbb{R}_+)$ we denote $\sigma_{F,\mu}(P)$ (for short, $\sigma(P)) = \sum\limits_{i=1}^{n} F(t_i)\mu(A_i)$, for every $P = \{A_i\}_{i=\overline{1,n}} \in \mathcal{P}$ and every $t_i \in A_i$, $i = \overline{1,n}$.

F is said to be Gould μ-integrable on T if the net $(\sigma(P))_{P \in (\mathcal{P}, \leq)}$ is convergent in $(\mathcal{P}_{kc}(\mathbb{R}), h)$. In this case, its limit is called the Gould integralof F on T with respect to μ, denoted by $\int_T F d\mu$.

The Gould integral on a subset E of T is defined in classical manner.

Remark 4.8. I. If it exists, the integral is unique.

II. If F μ-integrable on T, then $\int_T F d\mu \in \mathcal{P}_{kc}(\mathbb{R})$, so $\int_T F d\mu = [a, b]$, where $0 \leq a \leq b$. In consequence, F is μ-integrable on T if and only if there exists $A = [a, b] \in \mathcal{P}_{kc}(\mathbb{R}_+)$ such that for every $\varepsilon > 0$, there exists $P_\varepsilon \in \mathcal{P}$, so that for every $P = \{A_i\}_{i \in \overline{1,n}} \in \mathcal{P}$ with $P \geq P_\varepsilon$ and for every $t_i \in A_i$, $i = \overline{1,n}$, we have $h(\sum\limits_{i=1}^{n} F(t_i)\mu(A_i), A) < \varepsilon$.

III. If $\mu = \{0\}$, then F is μ-integrable on T and $\int_T F d\mu = \{0\}$.

In what follows, suppose, moreover, that $F : T \to \mathcal{P}_{kc}(\mathbb{R}_+)$ is **bounded** (i.e., there exists $M \geq 0$ so that $|F(t)|(= f_2(t)) \leq M$, for every $t \in T$) and $\mu : \mathcal{A} \to \mathcal{P}_{kc}(\mathbb{R}_+)$ is **of finite variation** (so $\overline{\mu}(T)(= \overline{m_2}(T)) < \infty$).

Proposition 4.9. *F is μ-integrable on T if and only if f_1 is m_1-integrable and f_2 is m_2-integrable on T in the sense of* [19] *and, in this case,*

$$\int_T F d\mu = [\int_T f_1 dm_1, \int_T f_2 dm_2].$$

Proof. . The statements easily follow since

$$\sum_{i=1}^{n} F(t_i)\mu(A_i) = \sum_{i=1}^{n} [f_1(t_i)m_1(A_i), f_2(t_i)m_2(A_i)] =$$

$$= [\sum_{i=1}^{n} f_1(t_i)m_1(A_i), \sum_{i=1}^{n} f_2(t_i)m_2(A_i)] = [\sigma_{f_1,m_1}(P), \sigma_{f_2,m_2}(P)],$$

for every $P = \{A_i\}_{i=\overline{1,n}} \in \mathcal{P}$ and every $t_i \in A_i, i = \overline{1,n}$. □

So, the following definitions are motivated:

Definition 4.10. I. Let $F : T \to \mathcal{P}_{kc}(-\infty, 0])$ be an interval-valued multifunction, defined by $F(t) = [f_1(t), f_2(t)]$, for every $t \in T$, with $f_1, f_2 : T \to (-\infty, 0]$, $f_1(t) \le f_2(t)$, for every $t \in T$. The Gould integral of F with respect to μ on T is defined by $\int_T F d\mu = [-\int_T f_2 dm_1, -\int_T f_1 dm_2]$.

II. For an interval-valued multifunction $F : T \to \mathcal{P}_{kc}(\mathbb{R})$, defined by $F(t) = [f_1(t), f_2(t)]$, for every $t \in T$, with $f_1, f_2 : T \to \mathbb{R}$, $f_1(t) \le f_2(t)$, for every $t \in T$ such that f_1 or f_2 does not have constant sign, we define the Gould integral of F with respect to μ on T this way:

$$\int_T F d\mu = [\int_T f_1^+ dm_1, \int_T f_2^+ dm_2] - [\int_T f_2^- dm_1, \int_T f_1^- dm_2],$$

where $f_i^+ = \max\{f_i, 0\}$ and $f_i^- = \min\{-f_i, 0\}$, $i = \overline{1,2}$.

Thus, without any loss of generality, in what follows we limit ourselves to the case when $F : T \to \mathcal{P}_{kc}(\mathbb{R}_+)$ and $\mu : \mathcal{A} \to \mathcal{P}_{kc}(\mathbb{R}_+)$.

Corollary 4.11. *If μ is a q-multisubmeasure and F is μ-integrable on T, then*

$$\int_T F d\mu = [(G)\int_T f_1 d\overline{m}_1, (G)\int_T f_2 d\overline{m}_2],$$

where $(G)\int_T f_1 d\overline{m}_1, (G)\int_T f_2 d\overline{m}_2$ are the Gould integrals [27].

Proof. It follows by Theorem 2.15 [19]. □

By the above corollary and by Theorem 5.1 [13], we get:

Corollary 4.12. *If, moreover, both m_1 and m_2 are complete and σ-additive, then*

$$\int_T F d\mu = [(B)\int_T f_1 dm_1, (B)\int_T f_2 dm_2],$$

where $(B)\int_T f_1 dm_1, (B)\int_T f_2 dm_2$ are the Birkhoff integrals [5], *which, in this case, are equal to the Lebesgue integrals.*

Example 4.13. By [11], one gets:

i) If $T = \{t_1, ..., t_n\}$ is a finite set and $\mathcal{A} = \mathcal{P}(T)$, then any $F = [f_1, f_2]$ is μ-integrable and

$$\int_T F d\mu = [\sum_{i=1}^{n} f_1(t_i) m_1(\{t_i\}), \sum_{i=1}^{n} f_2(t_i) m_2(\{t_i\})].$$

ii) If m_1, m_2 are finitely additive and f_1, f_2 are simple, $f_i = \sum_{j=1}^{n_i} a_j^i \cdot 1_{A_j^i}, i = \overline{1,2}$, then $F = [f_1, f_2]$ is μ-integrable and

$$\int_T F d\mu = [\sum_{i=1}^{n_1} a_i^1 m_1(A_i^1), \sum_{i=1}^{n_2} a_i^2 m_2(A_i^2)].$$

Now, we establish some results concerning the relation between $\overline{\mu}$-totally measurability and μ-integrability.

Theorem 4.14. *Let $\mu : \mathcal{A} \to \mathcal{P}_{kc}(\mathbb{R}_+)$ be a q-multisubmeasure. The following statements are equivalent:*
I. F is μ-integrable on T;
II. F is $\overline{\mu}$-integrable on T;
III. F is $\overline{\mu}$-totally measurable on T.

Proof. By Proposition 4.9, F is μ-integrable if and only if f_1 is m_1-integrable and f_2 is m_2-integrable. Now, the equivalences I $\Leftrightarrow$ II $\Leftrightarrow$ III are immediate by Theorem 2.16 [19]. □

Remark 4.15. In fact, in this case, F is μ-integrable on T if and only if f_1 and f_2 are m_2-integrable, which, according to [19], is equivalent to the fact that f_1 and f_2 are $\overline{m_2}$-totally measurable.

Theorem 4.16. *Suppose that $\mu : \mathcal{A} \to \mathcal{P}_{kc}(\mathbb{R}_+)$ is q-monotone and null-additive. If $A \in \mathcal{A}$ is a q-atom of μ and if F is $\overline{\mu}$-totally-measurable on A, then F is μ-integrable on A.*

Proof. Let $A \in \mathcal{A}$ be a q-atom of μ. By Remarks 3.9-II and 4.6-II it is enough to prove that any $\overline{m}$-totally measurable real function on A is m-integrable on A, where $m : \mathcal{A} \to \mathbb{R}_+$ is a set function of finite variation. Indeed, if we consider the induced set multifunction [21] $\mu : \mathcal{A} \to \mathcal{P}_{kc}(\mathbb{R}_+)$, $\mu(A) = [0, m(A)]$, for every $A \in \mathcal{A}$, with $m : \mathcal{A} \to \mathbb{R}_+$, in Theorem 2.14 [23] the conclusion follows. □

Corollary 4.17. *Suppose $\mu : \mathcal{A} \to \mathcal{P}_{kc}(\mathbb{R}_+)$ is q-finitely purely atomic, q-monotone and q-null-additive. If F is $\overline{\mu}$-totally-measurable on T, then F is μ-integrable on T.*

By Proposition 4.9 and Theorem 3.7 [39] we get:

Theorem 4.18. *Suppose $F : T \to \mathcal{P}_{kc}(\mathbb{R}_+)$ is μ-integrable. Then F is μ-integrable on $A \in \mathcal{A}$ if and only if $F\chi_A$ is μ-integrable on T, where χ_A is the characteristic function of A. In this case, $\int_A F d\mu = \int_T F\chi_A d\mu$.*

The next results are straightforward by Theorem 2.7-i), ii) [19] and Proposition 4.9.

Theorem 4.19. *Let $F : T \to \mathcal{P}_{kc}(\mathbb{R}_+)$ be μ-integrable and $\alpha \in \mathbb{R}_+$. Then:*
I) αF is μ-integrable and $\int_T \alpha F d\mu = \alpha \int_T F d\mu$.
II) F is $\alpha\mu$-integrable and $\int_T F d(\alpha\mu) = \alpha \int_T F d\mu$.

Theorem 4.20. *Let $F, G : T \to \mathcal{P}_{kc}(\mathbb{R}_+)$ be μ-integrable. Then the following statements hold:*
I. $F + G$ is μ-integrable and $\int_T (F + G) d\mu = \int_T F d\mu + \int_T G d\mu$;
II. If μ is q-multisubmeasure, then $F \cdot G$ is μ-integrable;
III. If μ is q-multisubmeasure, then $F \vee G$ and $F \wedge G$ are μ-integrable.

Proof. I. Since F, G are μ-integrable, then f_1, g_1 are m_1-integrable and f_2, g_2 are m_2-integrable. According to Theorem 2.7 - iv) [19] it follows that $f_1 + g_1$ is m_1-integrable, $f_2 + g_2$ is m_2-integrable and, moreover, $\int_T (f_i + g_i) dm_i = \int_T f_i dm_i +$

$\int_T g_i dm_i$, $i = \overline{1,2}$. Therefore, $F + G$ is μ-integrable and

$$\int_T (F+G)d\mu = [\int_T (f_1+g_1)dm_1, \int_T (f_2+g_2)dm_2] =$$
$$= [\int_T f_1 dm_1 + \int_T g_1 dm_1, \int_T f_2 dm_2 + \int_T g_2 dm_2] =$$
$$= [\int_T f_1 dm_1, \int_T f_2 dm_2] + [\int_T g_1 dm_1, \int_T g_2 dm_2] = \int_T F d\mu + \int_T G d\mu.$$

II. Based on Theorem 4.14, the Gould μ-integrability of F and G on T is equivalent to their $\overline{\mu}$-totally-measurability on T. Hence, f_1, g_1, f_2, g_2 are $\overline{m_2}$-totally measurable on T. Then, $f_1 \cdot g_1$ and $f_2 \cdot g_2$ are $\overline{m_2}$-totally measurable. Since $m_1 \leq m_2$ it follows that $f_1 \cdot g_1$ is $\overline{m_1}$-totally measurable and $f_2 \cdot g_2$ is $\overline{m_2}$-totally measurable. Applying again Theorem 4.14 and Remark 4.6-II we obtain that $F \cdot G$ is μ-integrable.

III. It follows by Theorem 4.14 and Theorem 2.18 [19]. □

Using Theorem 2.9-i) [19] we get:

Theorem 4.21. *Let $\mu_1, \mu_2 : \mathcal{A} \to \mathcal{P}_{kc}(\mathbb{R}_+)$ be of finite variation, with $\mu_1(\emptyset) = \mu_2(\emptyset) = \{0\}$. Suppose $F : T \to \mathcal{P}_{kc}(\mathbb{R}_+)$ is both μ_1-integrable and μ_2-integrable. If $\mu : \mathcal{A} \to \mathcal{P}_{kc}(\mathbb{R}_+)$ is defined by $\mu(A) = \mu_1(A) + \mu_2(A)$, for every $A \in \mathcal{A}$, then F is μ-integrable and*

$$\int_T F d(\mu_1 + \mu_2) = \int_T F d\mu_1 + \int_T F d\mu_2.$$

Theorem 4.22. *If $F, G : T \to \mathcal{P}_{kc}(\mathbb{R}_+)$ are μ-integrable, then*

$$h(\int_T F d\mu, \int_T G d\mu) \leq \sup_{t \in T} h(F(t), G(t)) \cdot \overline{\mu}(T).$$

Proof. Since F, G are μ-integrable then f_1, g_1 are m_1-integrable and f_2, g_2 are m_2-integrable. According to Theorem 2.7-iii) [19] we have

$$|\int_T f_1 dm_1 - \int_T g_1 dm_1| \leq \sup_{t \in T} |f_1(t) - g_1(t)| \overline{m_1}(T) \tag{1}$$

and

$$|\int_T f_2 dm_2 - \int_T g_2 dm_2| \leq \sup_{t \in T} |f_2(t) - g_2(t)| \overline{m_2}(T). \tag{2}$$

Therefore, by (1) and (2) it follows

$$\begin{aligned}
h(\int_T F d\mu, \int_T G d\mu) &= \max\{|\int_T f_1 dm_1 - \int_T g_1 dm_1|, |\int_T f_2 dm_2 - \int_T g_2 dm_2|\} \leq \\
&\leq \max\{\sup_{t \in T} |f_1(t) - g_1(t)| \overline{m_1}(T), \sup_{t \in T} |f_2(t) - g_2(t)| \overline{m_2}(T)\} \\
&\leq \max\{\sup_{t \in T} |f_1(t) - g_1(t)|, \sup_{t \in T} |f_2(t) - g_2(t)|\} \cdot \overline{m_2}(T) \leq \\
&\leq \sup_{t \in T} \max\{|f_1(t) - g_1(t)|, |f_2(t) - g_2(t)|\} \cdot \overline{m_2}(T) = \\
&= \sup_{t \in T} h(F(t), G(t)) \overline{\mu}(T).
\end{aligned}$$

By the previous theorem we obtain: □

Corollary 4.23. *If $F : T \to \mathcal{P}_{kc}(\mathbb{R}_+)$ is μ-integrable, then*

$$|\int_T F d\mu| \leq \sup_{t\in T} f_2(t) \cdot \overline{m_2}(T).$$

Theorem 4.24. *Let $F, G : T \to \mathcal{P}_{kc}(\mathbb{R}_+)$ be μ-integrable.*
I. If $F(t) \preceq G(t)$, for every $t \in T$, then $\int_T F d\mu \preceq \int_T G d\mu$.
II. If μ is a q-multisubmeasure and if $F(t) \subseteq G(t)$, for every $t \in T$, then $\int_T F d\mu \subseteq \int_T G d\mu$.

Proof. I. We apply Theorem 2.7-v) [19].
II. Since $F(t) = [f_1(t), f_2(t)] \subseteq [g_1(t), g_2(t)]$, we have $g_1(t) \leq f_1(t) \leq f_2(t) \leq g_2(t)$, for every $t \in T$. According to Theorem 4.14 and Remark 4.15 we have that f_1 and f_2 are $\overline{m_2}$-totally measurable. So, taking into account Theorem 2.7-v) [19] we obtain $\int_T g_1 dm_1 \leq \int_T f_1 dm_1$ and $\int_T f_2 dm_2 \leq \int_T g_2 dm_2$. On the other hand, since f_2 is $\overline{m_2}$-totally measurable and $m_1 \leq m_2$ it follows that f_2 is $\overline{m_1}$-totally measurable, hence f_2 is m_1-integrable too. Therefore, by Theorems 2.7-v) and 2.9-ii) [19], $\int_T f_1 dm_1 \leq \int_T f_2 dm_1 \leq \int_T f_2 dm_2$, which together with the previous inequalities finish the proof. □

Corollary 4.25. *If μ is a q-multisubmeasure and if $F, G : T \to \mathcal{P}_{kc}(\mathbb{R}_+)$ are μ-integrable, then:*

$$\int_T (F \wedge G) d\mu \preceq \int_T F d\mu \wedge \int_T G d\mu$$

and

$$\int_T F d\mu \vee \int_T G d\mu \preceq \int_T (F \vee G) d\mu.$$

Proof. It follows by Theorem 4.20-III and the previous theorem. □

By Theorems 2.7-v) and 2.9-ii) from [19], we analogously obtain:

Theorem 4.26. *Let $\mu_1, \mu_2 : \mathcal{A} \to \mathcal{P}_{kc}(\mathbb{R}_+)$ be interval-valued set multifunctions of finite variation, with $\mu_1(\emptyset) = \mu_2(\emptyset) = \{0\}$ and $F : T \to \mathcal{P}_{kc}(\mathbb{R}_+)$. The following statements hold:*
I. If $\mu_1 \preceq \mu_2$ and if F is simultaneously μ_1-integrable and μ_2-integrable, then $\int_T F d\mu_1 \preceq \int_T F d\mu_2$.
II. If μ_1, μ_2 are q-multisubmeasures such that $\mu_1 \subseteq \mu_2$ and if F is μ_2-integrable, then $\int_T F d\mu_1 \subseteq \int_T F d\mu_2$.

Remark 4.27. I. If μ is a q-multisubmeasure and if F is μ-integrable, then by Theorem 4.14 and Theorem 4.26, we have $\int_T F d\mu \preceq \int_T F d\overline{\mu}$.

II. If μ_1, μ_2 are q-multisubmeasures and F is μ_1-integrable and μ_2-integrable, then

$$\int_T F d(\mu_1 \wedge \mu_2) \preceq (\int_T F d\mu_1) \wedge (\int_T F d\mu_2).$$

5. **Important Properties of the Integral**

In what follows we establish results that highlight important properties of the set multifunction $\varphi : \mathcal{A} \to \mathcal{P}_{kc}(\mathbb{R}_+)$, defined by $\varphi(A) = \int_A F d\mu = [\int_A f_1 dm_1, \int_A f_2 dm_2]$, for every $A \in \mathcal{A}$, where $F : T \to \mathcal{P}_{kc}(\mathbb{R}_+)$, $F(t) = [f_1(t), f_2(t)], \forall t \in T$ is μ-integrable on T.

Theorem 5.1. *Let $\mu_1, \mu_2 : \mathcal{A} \to \mathcal{P}_{kc}(\mathbb{R}_+)$ be q-multisubmeasures of finite variation, defined by*

$$\mu_1(A) = [m_1(A), m_2(A)], \forall A \in \mathcal{A}$$

and

$$\mu_2(A) = [m_1^{'}(A), m_2^{'}(A)], \forall A \in \mathcal{A},$$

where $m_i, m_i^{'} : \mathcal{A} \to \mathbb{R}_+$, with $m_i(\emptyset) = m_i^{'}(\emptyset) = 0$, $i = \overline{1,2}$.
Suppose $F : T \to \mathcal{P}_{kc}(\mathbb{R}_+)$ is both μ_1-integrable and μ_2-integrable. Then:
I. F is $\mu_1 \wedge \mu_2$-integrable on T.
II. Let A be a q-atom of both μ_1 and μ_2. If $\mu : \mathcal{A} \to \mathcal{P}_{kc}(\mathbb{R}_+)$ is defined by

$$\mu(A) = \mu_1(A) \cdot \mu_2(A) = [(m_1 m_1^{'})(A), (m_2 m_2^{'})(A)], \ \forall A \in \mathcal{A},$$

then F is μ-integrable on A.

Proof. I. We observe that

$$\overline{m_i \wedge m_i^{'}} \leq \overline{m_1}, \overline{m_2}, \text{ for } i = \overline{1,2}.$$

Now, since F is μ_1-integrable and μ_2-integrable we have that f_1 is $\overline{m_1}$-totally measurable and $\overline{m_1^{'}}$-totally measurable, f_2 is $\overline{m_2}$-totally measurable and $\overline{m_2^{'}}$-totally measurable, respectively. Therefore, f_1 is $\overline{m_1 \wedge m_1^{'}}$-totally measurable and f_2 is $\overline{m_2 \wedge m_2^{'}}$-totally measurable. Applying Theorem 4.14, F is $\mu_1 \wedge \mu_2$-integrable.
II. First, we note that, by [13], if A is a q-atom of μ and if $\{A_i\}_{i=\overline{1,n}} \in \mathcal{P}_A$, then, there can exist only one set, for instance, without loss of generality, A_1, so that $\{0\} \prec \mu(A_1)$ and $\mu(A_i) = \{0\}, \forall i = \overline{2,n}$. Let $A \in \mathcal{A}$ be a q-atom of μ_1 and μ_2.
Since F is $\overline{\mu_1}$-totally-measurable on A, then for every $\varepsilon > 0$ there exists $P_\varepsilon = \{A_i\}_{i=\overline{0,n}} \in \mathcal{P}_A$ such that

$$\overline{\mu_1}(A_0) = m_2(A_0) < \sqrt{\varepsilon} \text{ and } \sup_{s,t \in A_i} h(F(s), F(t)) < \varepsilon, \forall i = \overline{1,n}. \tag{3}$$

Analogously, since F is $\overline{\mu_2}$-totally-measurable on A, there is $P_\varepsilon^{'} = \{B_j\}_{j=\overline{0,p}} \in \mathcal{P}_A$ such that

$$\overline{\mu_2}(B_0) = m_2^{'}(B_0) < \sqrt{\varepsilon} \text{ and } \sup_{s,t \in B_j} h(F(s), F(t)) < \varepsilon, \forall j = \overline{1,p}. \tag{4}$$

Consider $P = \{A_i \cap B_j\} \in \mathcal{P}_A$. Since $A_0 \cap B_0 \subseteq A$ and A is an atom of μ_1 and μ_2, we have two cases:
i) $\mu_1(A_0 \cap B_0) = \{0\}$ or $\mu_2(A_0 \cap B_0) = \{0\}$. Then $\overline{\mu_1 \cdot \mu_2}(A_0 \cap B_0) = 0 < \varepsilon$ and $\sup_{s,t \in A_i \cap B_j} h(F(s), F(t)) < \varepsilon$, for every $i = \overline{1,n}, j = \overline{1,p}$
ii) $\{0\} \prec \mu_1(A_0 \cap B_0)$ and $\{0\} \prec \mu_2(A_0 \cap B_0)$.
By Remark 3.9-III, $A_0 \cap B_0$ is a q-atom of μ_1 and μ_2 too, and $\mu_1(A_0 \cap B_0) = \mu_1(A)$, $\mu_2(A_0 \cap B_0) = \mu_2(A)$. Applying Remark 3.9-IV we have $\overline{\mu_1}(A_0 \cap B_0) = |\mu_1(A_0 \cap B_0)| = m_2(A_0 \cap B_0)$ and $\overline{\mu_2}(A_0 \cap B_0) = |\mu_2(A_0 \cap B_0)| = m_2^{'}(A_0 \cap B_0)$.
Hence, by (3) and (4) we get

$$\overline{\mu_1 \cdot \mu_2}(A_0 \cap B_0) = (m_2 \cdot m_2^{'})(A_0 \cap B_0) = m_2(A_0 \cap B_0) \cdot m_2^{'}(A_0 \cap B_0) < \varepsilon$$

and $\sup_{s,t \in A_i \cap B_j} h(F(s), F(t)) < \varepsilon$, for every $i = \overline{1,n}, j = \overline{1,p}$. Therefore, F is $\overline{\mu_1 \cdot \mu_2}$-totally measurable on A, so by Theorem 4.14, F is $\mu_1 \cdot \mu_2$-integrable on A. □

Theorem 5.2. *If $F : T \to \mathcal{P}_{kc}(\mathbb{R}_+)$ is μ-integrable on T, then:*

I. $\varphi \ll \overline{m_2}$ (i.e., for every $\varepsilon > 0$, there is $\delta > 0$ such that for every $A \in \mathcal{A}$ with $\overline{m_2}(A) < \delta$, it results $|\varphi(A)| < \varepsilon$).

II. If μ is q-monotone, then the same is φ.

Proof. . I is a consequence of Corollary 4.23.

II. It follows by Proposition 4.9 and Theorem 2.14 [19]. □

Theorem 5.3. *Let μ be a q-multisubmeasure and $F : T \to \mathcal{P}_{kc}(\mathbb{R}_+)$ μ-integrable. Then φ is a finitely additive multimeasure.*

Proof. Evidently, $\varphi(\emptyset) = 0$. We denote $\varphi_1(A) = \int_A f_1 dm_1$ and $\varphi_2(A) = \int_A f_2 dm_2$, for every $A \in \mathcal{A}$. Since m_1, m_2 are submeasures in Drewnowski's sense, then, by Theorem 2.12 [19], φ_1 and φ_2 are finitely additive, so $\varphi = [\varphi_1, \varphi_2]$ is also finitely additive. □

Theorem 5.4. *I. If $\mu : \mathcal{A} \to \mathcal{P}_{kc}(\mathbb{R}_+)$ and $F : T \to \mathcal{P}_{kc}(\mathbb{R}_+)$, $F(t) = [f_1(t), f_2(t)], \forall t \in T$ is μ-integrable on T, then:*
(i) $| \int_T F d\mu| = \int_T f_2 dm_2 = \int_T |F| d|\mu|$.
(ii)

$$\overline{\varphi}(T) = \sup\{\sum_{i=1}^{n} |\varphi(A_i)|\} = \sup\{\sum_{i=1}^{n} \int_{A_i} f_2 dm_2\} = \int_T f_2 dm_2,$$

where $\{A_i\}_{i=\overline{1,n}} \in \mathcal{P}$.
II. If $F : T \to \mathbb{R}_+$ is a single valued function and $\mu : \mathcal{A} \to \mathcal{P}_{kc}(\mathbb{R}_+)$ is monotone, then our integral is a particular case of [39].
III. If $F : T \to \mathbb{R}_+$ is a single valued function and $\mu : \mathcal{A} \to \mathbb{R}_+$ is a set function, then our integral is a particular case of [25].

Remark 5.5. We note that if $m : \mathcal{A} \to \mathbb{R}_+$ is countable additive, then Choquet integral coincides with Gould one and they are equal to Lebesgue integral. But if m is an arbitrary set function, the integral we proposed is different from the one introduced by Jang [33].
Indeed, according to [15], if $f : T \to [0, +\infty)$ is a simple function, then:

i) If m is subadditive, then $(C) \int_A f dm \leq (G) \int_A f dm, \forall A \in \mathcal{A}$;

ii) If m is finitely additive, then $(C) \int_A f dm = (G) \int_A f dm, \forall A \in \mathcal{A}$.

In conclusion, the lower and the upper, Gould and Choquet integrals (limits) are not always necessarily equal, even if f is a simple function.

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