A NEW WAY TO EXTEND FUZZY IMPLICATIONS

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Abstract. The main purpose of this paper is to use a new way to extend fuzzy implications I from a generalized sublattice M to a bounded lattice L, such that the extended implications preserve many of the considered properties of fuzzy implications on M. Furthermore, as a special case, we investigate the extension of $(S,N)-$implications. Results indicate that the extended implications preserve many of the considered properties of (S,N)-implications.

1. **Introduction**

How to extend a given function from a subset to a larger domain, such that the new function possesses the main properties of the original one? This is a well-known problem which has been studied by many researchers [14, 15, 23, 24].

In computer science and modern fuzzy logic, the problem of extending functions can be considered for lattice-valued fuzzy connectives, in particular, for $t-$norms, $t-$conorms, fuzzy negations and fuzzy implications. Saminger-Platz et al. [24] proposed a way to extend a given $t-$norm T from a complete sublattice (in the sense of the usual concept of sublattice) M to a bounded lattice L. In [22, 23] Palmeira et al. developed an alternative method to extend $t-$norms, $t-$conorms, fuzzy negations and fuzzy implications which, by considering a modified notion of sublattice, generalizes the method proposed in [24]. Recently, Palmeira et al. [21] presented a new way to extend fuzzy connectives, where M is not necessarily a subset of L but a generalized version of the concept of a sublattice.

As a subsequent work of [21], we would like to extend a given fuzzy implication from a complete sublattice M to a bounded lattice L, where M is a generalized version of sublattice, and study whether it could preserve most of properties of fuzzy implications.

We begin with Section 2 a specific formalization of the main concepts used along the paper such as lattice homomorphisms, retractions, sublattices and fuzzy connectives. Section 3 is devoted to the method of how to extend fuzzy implications. Section 4 focuses on a special case of the extension, i.e., the extension of $(S,N)-$implications. Section 5 is the conclusion.

Remarkably, we have originally considered a way of extending lattice valued fuzzy implications based on e-operators as in [13], that is $I^E_{\odot}(x,y) = I\Big(r_1(x), r_1(y)\Big)\odot I\Big(r_2(x), r_2(y)\Big)$. However, it turned out that the way defined in Section 3 of this paper,

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that is $I^E_{\odot}(x,y)=I\Big(r_2(x),r_1(y)\Big)\odot I\Big(r_1(x),r_2(y)\Big)$, is more suitable for extending fuzzy implications. After comparing these two methods we have found that the second method can preserve more properties of fuzzy implications and the results concerning this method are main subject of this study.

2. Preliminaries

In this section we list some definitions and results which will be necessary for our work. Some details will be omitted, but an appropriate description of them can be found in $[1-3, 6-8, 11, 18, 19, 21, 26, 27]$.

2.1. Bounded and Complete Lattices.

Let L be a lattice. If there are elements 0_L and 1_L in $\langle L,\wedge_L,\vee_L\rangle$ such that, for every $x\in L$, $x\wedge_L 1_L=x$ and $x\vee_L 0_L=x$, then $\langle L,\wedge_L,\vee_L,0_L,1_L\rangle$ is called a bounded lattice.

Moreover, it is well-known that, given a lattice L, the relation

$$x\leq_L y \text{ if and only if } x\wedge_L y=x$$

define a partial order on L. This order will be used to compare elements.

Recall also that a lattice L is called a complete lattice if every non-empty subset of it has a top and bottom element.

Throughout this paper we take L as a bounded lattice as defined above. If L represents another thing, then the appropriate distinction will be made.

Definition 2.1. Let $\langle L,\wedge_L,\vee_L,0_L,1_L\rangle$ and $\langle M,\wedge_M,\vee_M,0_M,1_M\rangle$ be bounded lattices. A mapping $f: L\to M$ is said to be a lattice homomorphism if, for all $x,y\in L$, we have,

(1) $f(x\wedge_L y)=f(x)\wedge_M f(y)$;
(2) $f(x\vee_L y)=f(x)\vee_M f(y)$;
(3) $f(0_L)=0_M$ and $f(1_L)=1_M$.

Remark 2.2. Recall that, an injective (a surjective) lattice homomorphism is called a monomorphism (an epimorphism) and a bijective lattice homomorphism is called an isomorphism. An automorphism is an isomorphism from a lattice to itself.

Proposition 2.3. [23] *Every lattice homomorphism preserves the order.*

Proposition 2.4. [21] *Let L be a bounded lattice. Then a function $\rho: L\to L$ is an automorphism if*

(1) *ρ is bijective and*
(2) *for all $x,y\in L$, $x\leq_L y$ if and only if $\rho(x)\leq_L\rho(y)$.*

Definition 2.5. [10] Given a function $f: L^n\to L$. The action of an $L-$automorphism ρ over f is the function $f^\rho: L^n\to L$ defined by

$$f^\rho(x_1,\cdots,x_n)=\rho^{-1}(f(\rho(x_1),\cdots,\rho(x_n))), x_1,x_2,\cdots,x_n\in L.$$

In this case, f^ρ is said to be a conjugate of f.

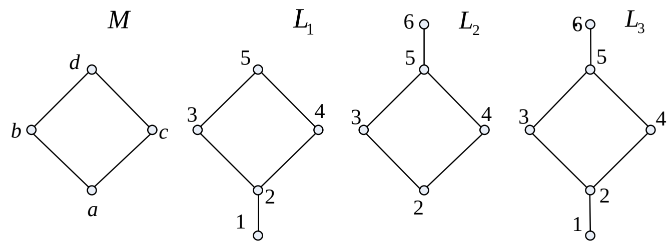

Figure 1. Hasse's Diagrams of Lattices M, L_1, L_2 and L_3

2.2. **Retracts and Sublattices.**

It is well known that the classical notion of a sublattice is given as follows, more details about it can be found in [7, 11].

Definition 2.6. [7] An (ordinary) sublattice of a lattice L is a subset M of L such that for all $x, y \in M$, $x \wedge_L y \in M$ and $x \vee_L y \in M$.

We would like to work with a relaxed notion of sublattice in which the condition $M \subseteq L$ is somewhat weakened.

Definition 2.7. [7] A homomorphism r of a lattice L onto a lattice M is said to be a retraction if there exists a homomorphism s of M into L which satisfies $r \circ s = id_M$. A lattice M is called a retract of a lattice L if there is a retraction r from L onto M, and s is then called a pseudo-inverse of r.

Notice that if a lattice M is a retract of a lattice L we have an identification of M with a subset $K = s(M)$ of L in which some properties of M can be carried onto K, including its lattice structure via retraction r. In this case, K works as an algebraic copy of M (i.e., they are isomorphic) embedded into L since r is an isomorphism when restricted to K.

Definition 2.8. [22] Let L and M be arbitrary bounded lattices. We say that M is an (r, s)-sublattice of L if M is a retract of L (i.e., M is a sublattice of L up to isomorphisms). In other words, M is an (r, s)-sublattice of L if there is a retraction r from L onto M with the pseudo-inverse $s : M \to L$.

Obviously, every sublattice in the classical sense is also an (r, s)-sublattice. Indeed, it is enough to consider as s the inclusion of M in L and as r the mapping which keeps M unchanged and sends x to $\sup M$ if the maximum of x and $\sup M$ is equal to x, sends x to $\inf M$ if the minimum of x and $\inf M$ is x, and send x to x in any other case.

Definition 2.9. [22] Every retraction $r : L \to M$ (with the pseudo-inverse s) which satisfies $s \circ r \leq id_L$ ($id_L \leq s \circ r$) is called a lower (an upper) retraction. In this case, M is a lower (an upper) retract of L.

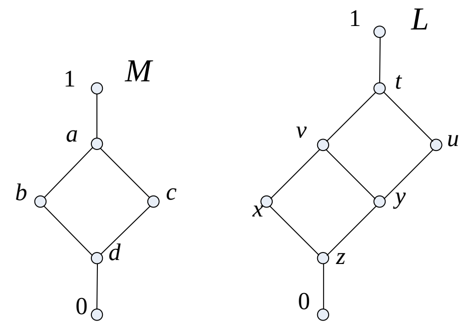

FIGURE 2. Hasse's Diagrams of Lattices M and L

Example 2.10. [22] Let M and L be bounded lattices as shown in Figure 2. A function $r : L{\to}M$ given by $r(x) = \sup\{z{\in}M|s(z){\leq}_L x\}$ is a lower retraction with its pseudo-inverse given by the mapping $s : M{\to}L$ defined by $s(1_M) = 1_L$, $s(a) = v$, $s(b) = x$, $s(c) = y$, $s(d) = z$ and $s(0_M) = 0_L$. Therefore, it follows that M is an (r, s)-sublattice of L in the sense of Definition 2.8.

Remark 2.11. From Figure 2, we can conclude that there may be more than one retraction from L onto M with the same pseudo-inverse. In other words, for a given lower retraction r with a pseudo-inverse s, we can define an upper retraction r' with the same pseudo-inverse s. For instance, if r is a lower retraction with pseudo-inverse s as defined in Example 2.10, then the function r' given by $r'(x) = \inf\{z{\in}M|s(z){\geq}_L x\}$ is an upper retraction since $id_L{\leq}s{\circ}r'$. It is easy to check that s is also a pseudo-inverse of r'.

Definition 2.12. [21] Let M be an (r_1, s)-sublattice of L. Then we say that

(1) M is a lower (r_1, s)-sublattice of L if r_1 is a lower retraction. Notation: $M{<}L$ with respect to (r_1, s);

(2) M is an upper (r_1, s)-sublattice of L whenever r_1 is an upper retraction. Notation: $M{>}L$ with respect to (r_1, s);

(3) If r_1 is a lower retraction with pseudo-inverse s and $r_2 : L{\to}M$ is an upper retraction with the same pseudo-inverse s, then M is called a full (r_1, r_2, s)-sublattice of L. Notation: $M{\trianglelefteq}L$ with respect to (r_1, r_2, s).

2.3. $t-$**Norms, Negations and Fuzzy Implications.**

In what follows we present a formalization of some fuzzy operators, namely, $t-$norms, $t-$conorms, fuzzy negations and implications. These concept can be founded in $[5, 9-12, 16, 17, 19]$.

Definition 2.13. [23] Let L be a bounded lattice. A binary operation $T : L{\times}L{\to}L$ is a $t-$norm if, it satisfies for all $x, y, z{\in}L$,

(1) $T(x, y) = T(y, x)$ (commutativity);

(2) $T(x, T(y, z)) = T(T(x, y), z)$ (associativity);

(3) If $x \leq_L y$, then $T(x,z) \leq_L T(y,z)$ (monotonicity);
(4) $T(x, 1_L) = x$ (boundary condition).

Definition 2.14. [23] Let L be a bounded lattice. A binary operation $S : L \times L \to L$ is a $t-$conorm if, it satisfies for all $x, y, z \in L$,
(1) $S(x,y) = S(y,x)$ (commutativity);
(2) $S(x, S(y,z)) = S(S(x,y), z)$ (associativity);
(3) If $x \leq_L y$, then $S(x,z) \leq_L S(y,z)$ (monotonicity);
(4) $S(x, 0_L) = x$ (boundary condition).
In addition, if $S(x,x) = x$ for all $x \in L$, then we say that S satisfies idempotency.

Definition 2.15. [21] A function $N : L \to L$ is called a fuzzy negation if it satisfies:
(1) $N(0_L) = 1_L$ and $N(1_L) = 0_L$;
(2) For all $x, y \in L$, if $x \leq_L y$, then $N(y) \leq_L N(x)$.
In addition, fuzzy negations satisfying the involution property

$$\forall x \in L,\ N(N(x)) = x$$

are called strong fuzzy negations.

Definition 2.16. Let S be a $t-$conorm and N be a fuzzy negation on L. We say that the pair (S, N) satisfies the law of excluding middle if for any $x \in L$, $S(N(x), x) = 1_L$.

Definition 2.17. [21] Let T be a $t-$norm, S be a $t-$conorm and N be a fuzzy negation, all defined on the same bounded lattice L. We say that (T, S, N) is a De Morgan triple if for all $x, y \in L$,
(1) $N(T(x,y)) = S(N(x), N(y))$;
(2) $N(S(x,y)) = T(N(x), N(y))$.

Definition 2.18. [23]. Let L be a bounded lattice. A binary operation $I : L \times L \to L$ is a fuzzy implication if it satisfies:
$(I1)$ I is decreasing in the first variable;
$(I2)$ I is increasing in the second variable;
$(I3)$ $I(0_L, 0_L) = 1_L,\ I(1_L, 1_L) = 1_L,\ I(1_L, 0_L) = 0_L$.

In addition to properties $(I1) - (I3)$, we always study other properties as following:
(NC) normality condition, $I(0_L, 1_L) = 1_L$;
(LB) left boundary condition, for all $y \in L$, $I(0_L, y) = 1_L$;
(RB) right boundary condition, for all $x \in L$, $I(x, 1_L) = 1_L$;
(NP) left neutrality property, for all $y \in L$, $I(1_L, y) = y$;
(EP) exchange principle, for all $x, y, z \in L$, $I(x, I(y,z)) = I(y, I(x,z))$;
(CP) law of contraposition, for all $x, y \in L$, $I(x,y) = I(N(y), N(x))$;
(LCP) law of left contraposition, for all $x, y \in L$, $I(N(x), y) = I(N(y), x)$;
(RCP) law of right contraposition, for all $x, y \in L$, $I(x, N(y)) = I(y, N(x))$;
(OP) ordering principle, for all $x, y \in L$, $I(x,y) = 1_L \Leftrightarrow x \leq_L y$;
(SN) strong fuzzy negation principle, the mapping N_I defined as $N_I(x) = I(x, 0_L)$ for all $x \in L$ is a strong fuzzy negation;

(CB) consequent boundary, for all $x, y \in L$, $y \leq_L I(x, y)$;
(IP) identity property, for all $x \in L$, $I(x, x) = 1_L$;
(P) positivity, $I(x, y) = 0_L$ if and only if $x = 1_L$ and $y = 0_L$;
(LI) law of importation, for all $x, y, z \in L$, $I(x, I(y, z)) = I(T(x, y), z)$, where T is a $t-$norm;
(BL) Boolean-like law, $\forall x, y \in L$, $I(x, I(y, x)) = 1_L$;
(IBL) iterative Boolean-like law, for all $x, y \in L$, $I(x, y) = I(x, I(x, y))$.
Some more details about these properties can be founded in [1−3, 10, 12, 15−17, 20].

Proposition 2.19. [22] *Let L be a bounded lattice. If a function $I : L \times L \to L$ satisfies $(I1)$ and $(I3)$, then the function $N_I : L \to L$ defined for each $x \in L$ by*

$$N_I(x) = I(x, 0_L)$$

is a fuzzy negation on L, called the natural negation of I.

Lemma 2.20. [5] *Let L be a bounded lattice. If a function $I : L \times L \to L$ satisfies (OP), then I satisfies (IP).*

Lemma 2.21. [5] *Let L be a bounded lattice. If a function $I : L \times L \to L$ satisfies (EP) and (OP), then I satisfies (NP), $(I1)$ and $(I3)$.*

Lemma 2.22. [22] *Let L be a bounded lattice. If a function $I : L \times L \to L$ satisfies (EP) and (OP), then*
(1) *N_I is a fuzzy negation;*
(2) *$x \leq_L N_I(N_I(x))$ for each $x \in L$;*
(3) *$N_I \circ N_I \circ N_I = N_I$.*

Proposition 2.23. [5] *Let ρ be an automorphism on L, $I : L \times L \to L$ be a function and $Q \in \{(I1), (I2), (NC), (LB), (RB)\}$. Then I satisfies Q if and only if so does I^ρ.*

Proposition 2.24. [5] *Let $I : L \times L \to L$ be a function and ρ an automorphism on L. Then, I is an $(S, N)-$implication on L generated by S and N if and only if I^ρ is an $(S, N)-$implication on L generated by S^ρ and N^ρ. In other words, $(I_{S,N})^\rho = I_{S^\rho, N^\rho}$.*

2.4. **Extention of $t-$norms, $t-$conorms and Negations.**

In what follows we present the extention of $t-$norms, $t-$conorms and negations. These concepts can be founded in [21].

Definition 2.25. [21] Let $M \trianglelefteq L$ with respect to (r_1, r_2, s). A mapping $\odot : M \times M \to L$ is called an e-operator on M, if it is isotonic and satisfies the following conditions for each $a, b \in M$ and $x \in L$,

$$r_1(a \odot b) = a \wedge_M b \text{ and } r_2(a \odot b) = a \vee_M b,$$
$$r_1(x) \odot r_2(x) = x.$$

Remark 2.26. In other words, if $M \trianglelefteq L$ with respect to (r_1, r_2, s) (by Definition 2.12, there are two retractions $r_1, r_2 : L \to M$), then e-operator $\odot$ describes an isotonic way to relate retractions r_1 and r_2 with the meet and join operators on M.

Proposition 2.27. [21] *Consider $M \trianglelefteq L$ with respect to (r_1, r_2, s) and let $\odot$ be an e-operator on M. Then the following properties hold for all $a, b \in M$ and $x, y \in L$,*
(1) *$a \leq_M b$ if and only if $r_1(a \odot b) = a$ and $r_2(a \odot b) = b$;*
(2) *For every $a \in M$ we have $s(a) = a \odot a$;*
(3) *$r_1(x) \leq_M r_1(y)$ and $r_2(x) \leq_M r_2(y)$ iff $x \leq_L y$;*
(4) *$r_1(x) = r_1(y)$ and $r_2(x) = r_2(y)$ if and only if $x = y$;*
(5) *$\odot$ is commutative.*

Definition 2.28. [21] Take $M \trianglelefteq L$ with respect to (r_1, r_2, s) and let $\odot$ be an $e-$operator on M. If ρ is an automorphism on M, then its extension is given by $\rho^E(x) = \rho \circ r_1(x) \odot \rho \circ r_2(x)$, for any $x \in L$.

Proposition 2.29. [21] *Let $M \trianglelefteq L$ with respect to (r_1, r_2, s), $\odot$ be an $e-$operator on M and $\rho : M \to M$ be an automorphism. Thus,*
(1) *ρ^E is an automorphism on L;*
(2) *The inverse of ρ^E is given by*

$$(\rho^E)^{-1}(x) = \rho^{-1} \circ r_1(x) \odot \rho^{-1} \circ r_2(x)$$

for all $x \in L$, i.e., $(\rho^E)^{-1} = (\rho^{-1})^E$.

Lemma 2.30. [21] *Let $M \trianglelefteq L$ with respect to (r_1, r_2, s) and $\odot$ be an $e-$operator on M. Thus, given an automorphism $\rho : M \to M$, ρ^E satisfies the following properties,*
(1) *For all $x \in M$ we have that $\rho^E(x \odot x) = \rho(x) \odot \rho(x)$;*
(2) *$\rho \circ r_1 = r_1 \circ \rho^E$ and $\rho^{-1} \circ r_1 = r_1 \circ (\rho^E)^{-1}$;*
(3) *$\rho \circ r_2 = r_2 \circ \rho^E$ and $\rho^{-1} \circ r_2 = r_2 \circ (\rho^E)^{-1}$.*

Proposition 2.31. [21] *Let $M \trianglelefteq L$ with respect to (r_1, r_2, s) and let $\odot$ be an e-operator on M. Then, given a $t-$norm T on M, the function $T^E_\odot : L^2 \to L$ defined by*

$$T^E_\odot(x, y) = T\Big(r_1(x), r_1(y)\Big) \odot T\Big(r_2(x), r_2(y)\Big)$$

is a $t-$norm on L. In this case, $T^E_\odot$ is said to be generated by T.

Proposition 2.32. [21] *Let $M \trianglelefteq L$ with respect to (r_1, r_2, s) and let $\odot$ be an e-operator on M. Then*
(1) *if S is a $t-$conorm on M, then*

$$S^E_\odot(x, y) = S\Big(r_1(x), r_1(y)\Big) \odot S\Big(r_2(x), r_2(y)\Big)$$

is a $t-$conorm on L. In this case, $S^E_\odot$ is said to be generated by S.
(2) *if N is a fuzzy negation on M, then*

$$N^E_\odot(x) = N\Big(r_1(x)\Big) \odot N\Big(r_2(x)\Big)$$

is a fuzzy negation on L. In this case, $N^E_\odot$ is said to be generated by N. Moreover, if N is involutive, then $N^E_\odot$ is also involutive.

Proposition 2.33. [21] *Let $M \trianglelefteq L$ with respect to (r_1, r_2, s) and let $\odot$ be an e-operator on M. If (T, S, N) is a De Morgan triple on M, then $(T^E_\odot, S^E_\odot, N^E_\odot)$ is a De Morgan triple.*

Remark 2.34. By this way, we can obtain three new fuzzy operators that preserve many properties of fuzzy connectives. More details can be found in [21].

3. Extension of Fuzzy Implications

Let M be an (r, s)-sublattice of L (in this case, there are a retraction $r : L \to M$ and a pseudo-inverse $s : M \to L$ such that $r \circ s = id_M$). In this section, we would like to extend fuzzy implications from M to L in a similar way, and see if they can preserve many of the considered properties.

3.1. The Extension Method for Fuzzy Implications.

Firstly, we present a new way to extend fuzzy implications. Secondly, we discuss the properties of the extended fuzzy implications.

Definition 3.1. Let $M \trianglelefteq L$ with respect to (r_1, r_2, s) and let $\odot$ be an e-operator on M, $I : M \times M \to M$ be a binary operator on M. Then a binary operator $I^E_\odot : L \times L \to L$ can be defined as following:

$$I^E_\odot(x, y) = I\Big(r_2(x), r_1(y)\Big) \odot I\Big(r_1(x), r_2(y)\Big), \forall x, y \in L.$$

In this case, $I^E_\odot$ is said to be generated by I.

Theorem 3.2. *Under the same conditions as in Definition 3.1, I is a mapping $M \times M \to M$. If I satisfies $Q \in \{I1, I2, I3\}$, then also $I^E_\odot$ satisfies properties Q.*

Proof. It is straightforward by the definition of fuzzy implications. □

Corollary 3.3. *Under the same conditions as in Definition 3.1, I is a mapping $M \times M \to M$. If I is a fuzzy implication, then $I^E_\odot$ is also a fuzzy implication.*

Example 3.4. [21] Given a bounded lattice $\langle L, \wedge_L, \vee_L, 0_L, 1_L \rangle$, it is possible to define a bounded lattice $\langle \mathbb{L}, \wedge, \vee, [0_L, 0_L], [1_L, 1_L] \rangle$, named the interval lattice ([4]) where

$$\mathbb{L} = \{[x, y] | x, y \in L \text{ and } x \leq_L y\}$$

and its operations are given by

$$[x, y] \wedge [a, b] = [x \wedge_L a, y \wedge_L b] \text{ and } [x, y] \vee [a, b] = [x \vee_L a, y \vee_L b].$$

The associated order for $\mathbb{L}$ agrees with the product order and can be expressed as

$$[x_1, y_1] \leq [x_2, y_2] \text{ if and only if } x_1 \leq_L x_2 \text{ and } y_1 \leq_L y_2.$$

Naturally, L can be seen as an (r_1, r_2, s)-sublattice of $\mathbb{L}$ by considering the lower retraction $r_1 : \mathbb{L} \to L$ given by $r_1([x, y]) = x$ with pseudo-inverse $s(x) = [x, x]$ for all $x \in L$. The function $r_2([x, y]) = y$ is an upper retraction from $\mathbb{L}$ to L, whose pseudo-inverse is also s. It is easy to check that $s \circ r_1 \leq id \leq s \circ r_2$. Therefore, $L \trianglelefteq \mathbb{L}$ with respect to (r_1, r_2, s). By Definition 2.25, the mapping $\odot : L \times L \to \mathbb{L}$ defined by $a \odot b = [a \wedge_L b, a \vee_L b]$ for each $a, b \in L$, is trivially an e-operator on L.

Thus, if I is a fuzzy implication on L, then $I^E_\odot$ defined in Definition 3.1 is a fuzzy implication on $\mathbb{L}$. In fact, for each $[x,y],[a,b]\in\mathbb{L}$,

$$\begin{aligned} I^E_\odot([x,y],[a,b]) &= I(r_2([x,y]),r_1([a,b]))\odot I(r_1([x,y]),r_2([a,b])) \\ &= I(y,a)\odot I(x,b) \\ &= [I(y,a),I(x,b)]. \end{aligned}$$

For all $[x_1,y_1],[x_2,y_2]\in\mathbb{L}$ and $[x_1,y_1]\leq[x_2,y_2]$, we have $I^E_\odot([x_1,y_1],[a,b])=[I(y_1,a),I(x_1,b)]$ and $I^E_\odot([x_2,y_2],[a,b])=[I(y_2,a),I(x_2,b)]$. Since I is a fuzzy implication on L, then $I(y_2,a)\leq_L I(y_1,a)$ and $I(x_2,b)\leq_L I(x_1,b)$. So $I^E_\odot([x_2,y_2],[a,b])\leq I^E_\odot([x_1,y_1],[a,b])$, i.e., $I^E_\odot$ satisfies $(I1)$. Similarly, we can prove that $I^E_\odot$ satisfies $(I2)$. Furthermore, we have,

$$I^E_\odot([0_L,0_L],[0_L,0_L])=[I(0_L,0_L),I(0_L,0_L)]=[1_L,1_L],$$

$$I^E_\odot([1_L,1_L],[1_L,1_L])=[I(1_L,1_L),I(1_L,1_L)]=[1_L,1_L],$$

$$I^E_\odot([1_L,1_L],[0_L,0_L])=[I(1_L,0_L),I(1_L,0_L)]=[0_L,0_L].$$

i.e., $I^E_\odot$ satisfies $(I3)$. Hence, $I^E_\odot$ is a fuzzy implication on $\mathbb{L}$.

Theorem 3.5. *Let $M\trianglelefteq L$ with respect to (r_1,r_2,s) and $\odot$ be an e-operator on M. Let I be a fuzzy implication on M and $Q\in\{(NC),(LB),(RB),(NP),(EP),(CP),(SN),(CB),(P)\}$. If I satisfies Q then so does $I^E_\odot$.*

Proof. Suppose that I satisfies one of the properties (NC), (LB), (RB), (NP), (EP), (CP), (SN), (CB) and (P), respectively. It is obvious that properties (NC), (LB) and (RB) hold for any fuzzy implication. Next, we need only prove the properties (EP), (CP), (SN) and (P), for (NP) and (CB) are easy to check.
(EP). For every $x,y,z\in L$, we know

$$I(r_2(x),r_1(z))\leq_M I(r_1(x),r_2(z)) \text{ and } I(r_2(y),r_1(z))\leq_M I(r_1(y),r_2(z)),$$

thus

$$\begin{aligned} I^E_\odot\left(x,I^E_\odot(y,z)\right) &= I^E_\odot\Bigg(x,I\Big(r_2(y),r_1(z)\Big)\odot I\Big(r_1(y),r_2(z)\Big)\Bigg) \quad \text{by Def. 3.1} \\ &= I\Bigg(r_2(x),I\Big(r_2(y),r_1(z)\Big)\Bigg)\odot I\Bigg(r_1(x),I\Big(r_1(y),r_2(z)\Big)\Bigg) \quad \text{by 3.1 and 2.27(1)} \\ &= I\Bigg(r_2(y),I\Big(r_2(x),r_1(z)\Big)\Bigg)\odot I\Bigg(r_1(y),I\Big(r_1(x),r_2(z)\Big)\Bigg) \quad \text{by (EP) of I} \\ &= I^E_\odot\Bigg(y,I\Big(r_2(x),r_1(z)\Big)\odot I\Big(r_1(x),r_2(z)\Big)\Bigg) \\ &= I^E_\odot\Big(y,I^E_\odot(x,z)\Big). \end{aligned}$$

(CP). For every $x, y \in L$, $r_1(x) \leq_M r_2(x)$ and $r_1(y) \leq_M r_2(y)$,

$$\begin{aligned}
&I^E_{\odot}\Big(N^E_{\odot}(y), N^E_{\odot}(x)\Big)\\
&= I\bigg(r_2\Big(N^E_{\odot}(y)\Big), r_1\Big(N^E_{\odot}(x)\Big)\bigg) \odot I\bigg(r_1\Big(N^E_{\odot}(y)\Big), r_2\Big(N^E_{\odot}(x)\Big)\bigg) \quad \text{by Def. 3.1}\\
&= I\bigg(N\Big(r_1(y)\Big), N\Big(r_2(x)\Big)\bigg) \odot I\bigg(N\Big(r_2(y)\Big), N\Big(r_1(x)\Big)\bigg) \quad \text{by 2.27(1) and 2.32(2)}\\
&= I\Big(r_2(x), r_1(y)\Big) \odot I\Big(r_1(x), r_2(y)\Big) \quad \text{by (CP) of I}\\
&= I^E_{\odot}(x, y). \quad \text{by Def. 3.1}
\end{aligned}$$

(SN). Suppose that I satisfies (SN), i.e., for all $x \in M$, $N_I(x) = I(x, 0_L)$ is a strong negation. Then,

$$\begin{aligned}
N_{I^E_{\odot}}(x) &= I^E_{\odot}(x, 0_L) = I\Big(r_2(x), r_1(0_L)\Big) \odot I\Big(r_1(x), r_2(0_L)\Big)\\
&= I\Big(r_2(x), 0_M\Big) \odot I\Big(r_1(x), 0_M\Big)\\
&= N_I\Big(r_2(x)\Big) \odot N_I\Big(r_1(x)\Big) = (N_I)^E_{\odot}(x).
\end{aligned}$$

and

$$\begin{aligned}
N_{I^E_{\odot}}\Big(N_{I^E_{\odot}}(x)\Big) &= N_{I^E_{\odot}}\bigg(N_I\Big(r_1(x)\Big) \odot N_I\Big(r_2(x)\Big)\bigg)\\
&= N_I\bigg(N_I\Big(r_2(x)\Big)\bigg) \odot N_I\bigg(N_I\Big(r_1(x)\Big)\bigg) \quad \text{by Prop. 2.27(1)}\\
&= r_2(x) \odot r_1(x) = x. \quad \text{by (SN) of I}
\end{aligned}$$

Therefore, $I^E_{\odot}$ satisfies (SN).
(P). $(\Rightarrow)$ Suppose $I^E_{\odot}(x, y) = 0_L$ for some $x, y \in L$. Then

$$I\Big(r_2(x), r_1(y)\Big) \odot I\Big(r_1(x), r_2(y)\Big) = 0_L = s(0_M) = 0_M \odot 0_M. \tag{1}$$

When we apply mapping r_1 on the product of (1) we get $I(r_2(x), r_1(y)) = 0_M$, that implies $r_2(x) = 1_M$ and $r_1(y) = 0_M$. Similarly, we also can get $I(r_1(x), r_2(y)) = 0_M$, that implies $r_1(x) = 1_M$ and $r_2(y) = 0_M$. Thus, $x = r_1(x) \odot r_2(x) = 1_M \odot 1_M = 1_L$ and $y = r_1(y) \odot r_2(y) = 0_M \odot 0_M = 0_L$.
$(\Leftarrow)$ Suppose $x = 1_L$ and $y = 0_L$, then

$$\begin{aligned}
I^E_{\odot}(x, y) &= I^E_{\odot}(1_L, 0_L) = I\Big(r_2(1_L), r_1(0_L)\Big) \odot I\Big(r_1(1_L), r_2(0_L)\Big)\\
&= I\Big(1_M, 0_M\Big) \odot I\Big(1_M, 0_M\Big)\\
&= 0_M \odot 0_M = s(0_M) = 0_L.
\end{aligned}$$

□

Theorem 3.6. *Let $M \trianglelefteq L$ with respect to (r_1, r_2, s). Let T be a $t-$norm , $\odot$ be an e-operator and I be a fuzzy implication on M. Then, I satisfies (LI) with respect to T if and only if $I^E_{\odot}$ satisfies (LI) with respect to $T^E_{\odot}$.*

Proof. ($\Rightarrow$) For all $x, y, z \in L$, since I satisfies (LI) with respect to T, hence

$$I^E_\odot\Big(T^E_\odot(x,y),z\Big) = I^E_\odot\Bigg(T\Big(r_1(x),r_1(y)\Big)\odot T\Big(r_2(x),r_2(y)\Big),z\Bigg) \text{ by Prop. 2.31}$$
$$= I\Bigg(T\Big(r_2(x),r_2(y)\Big),r_1(z)\Bigg)\odot I\Bigg(T\Big(r_1(x),r_1(y)\Big),r_2(z)\Bigg) \text{ by 3.1 and 2.27(1)}$$
$$= I\Bigg(r_2(x),I\Big(r_2(y),r_1(z)\Big)\Bigg)\odot I\Bigg(r_1(x),I\Big(r_1(y),r_2(z)\Big)\Bigg) \text{ by (LI) of I}$$
$$= I^E_\odot\Bigg(x,I\Big(r_2(y),r_1(z)\Big)\odot I\Big(r_1(y),r_2(z)\Big)\Bigg) \text{ by Def. 3.1 and Prop. 2.27(1)}$$
$$= I^E_\odot(x,I^E_\odot(y,z)). \text{ by Def. 3.1}$$

So $I^E_\odot$ satisfies (LI) with respect to $T^E_\odot$.

($\Leftarrow$) For all $x, y, z \in L$, since $I^E_\odot$ satisfies (LI) with respect to $T^E_\odot$, i.e., $I^E_\odot\Big(T^E_\odot(x,y),z\Big) = I^E_\odot(x,I^E_\odot(y,z))$, hence

$$I\Bigg(T\Big(r_2(x),r_2(y)\Big),r_1(z)\Bigg)\odot I\Bigg(T\Big(r_1(x),r_1(y)\Big),r_2(z)\Bigg)$$
$$= I\Bigg(r_2(x),I\Big(r_2(y),r_1(z)\Big)\Bigg)\odot I\Bigg(r_1(x),I\Big(r_1(y),r_2(z)\Big)\Bigg).$$

We know from monotonicity of T and I that $I\Bigg(T\Big(r_2(x),r_2(y)\Big),r_1(z)\Bigg)\leq I\Bigg(T\Big(r_1(x), r_1(y)\Big),r_2(z)\Bigg)$ and $I\Bigg(r_2(x),I\Big(r_2(y),r_1(z)\Big)\Bigg)\leq I\Bigg(r_1(x),I\Big(r_1(y),r_2(z)\Big)\Bigg)$. Applying r_1 and r_2 to both sides and using Prop. 2.27(1) we get that

$$I\Bigg(T\Big(r_2(x),r_2(y)\Big),r_1(z)\Bigg) = I\Bigg(r_2(x),I\Big(r_2(y),r_1(z)\Big)\Bigg)$$

and

$$I\Bigg(T\Big(r_1(x),r_1(y)\Big),r_2(z)\Bigg) = I\Bigg(r_2(x),I\Big(r_2(y),r_1(z)\Big)\Bigg).$$

Thus for any $X, Y, Z \in M$ we get that

$$I\Bigg(T\Big(r_2(s(X)),r_2(s(Y))\Big),r_1(s(Z))\Bigg) = I\Bigg(r_2(s(X)),I\Big(r_2(s(Y)),r_1(s(Z))\Big)\Bigg),$$

i.e. $I(T(X,Y),Z) = I(X,I(Y,Z))$. So I satisfies (LI) with respect to T. $\square$

Theorem 3.7. *Let $M \trianglelefteq L$ with respect to (r_1, r_2, s). Let I be a fuzzy implication on M, $\odot$ be an e-operator on M. Then, I satisfies (IBL) if and only if $I^E_\odot$ satisfies (IBL).*

Proof. ($\Rightarrow$) For all $x, y \in L$, since I satisfies (IBL), hence

$$I_{\odot}^{E}\Big(x, I_{\odot}^{E}(x,y)\Big) = I_{\odot}^{E}\Bigg(x, I\Big(r_2(x), r_1(y)\Big)\odot I\Big(r_1(x), r_2(y)\Big)\Bigg) \text{ by Def. 3.1}$$
$$= I\Bigg(r_2(x), I\Big(r_2(x), r_1(y)\Big)\Bigg)\odot I\Bigg(r_1(x), I\Big(r_1(x), r_2(y)\Big)\Bigg) \text{ by 3.1 and 2.27(1)}$$
$$= I\Big(r_2(x), r_1(y)\Big)\odot I\Big(r_1(x), r_2(y)\Big) = I_{\odot}^{E}(x,y). \text{ by (IBL) of I}$$

So $I_{\odot}^{E}$ satisfies (IBL).

($\Leftarrow$) For all $x, y \in L$, since $I_{\odot}^{E}$ satisfies (IBL), i.e., $I_{\odot}^{E}\Big(x, I_{\odot}^{E}(x,y)\Big) = I_{\odot}^{E}(x,y)$, hence $I\Bigg(r_2(x), I\Big(r_2(x), r_1(y)\Big)\Bigg)\odot I\Bigg(r_1(x), I\Big(r_1(x), r_2(y)\Big)\Bigg) = I\Big(r_2(x), r_1(y)\Big)\odot I\Big(r_1(x), r_2(y)\Big)$. Applying r_1 and r_2 to both sides and using Prop. 2.27(1) we get

$$I\Bigg(r_2(x), I\Big(r_2(x), r_1(y)\Big)\Bigg) = I\Big(r_2(x), r_1(y)\Big)$$

and

$$I\Bigg(r_1(x), I\Big(r_1(x), r_2(y)\Big)\Bigg) = I\Big(r_1(x), r_2(y)\Big).$$

Thus for any $X, Y, Z \in M$ we get that $I\Bigg(r_2(s(X)), I\Big(r_2(s(X)), r_1(s(Y))\Big)\Bigg) = I\Big(r_2(s(X)), r_1(s(Y))\Big)$, i.e. $I(X, I(X,Y)) = I(X,Y)$. So I satisfies (IBL). $\square$

Remark 3.8. In Definition 3.1, we adopt a new way to extend fuzzy implications, which is similar to the way extending t-norms (or t-conorms) in [21]. Theorems 3.5, 3.6 and 3.7 indicate that $I_{\odot}^{E}$ preserves many of the properties of fuzzy implications. But the properties $(OP), (IP)$ and (BL) are not preserved, e.g. easy computation with the construction in Example 3.4, when the implication I on $[0,1]$ is standard Łukasiewicz implication $I(x,y) = \min\{1-x+y, 1\}$ and intervals in $\mathbb{L}$ are considered for example $[0, \frac{1}{2}]$ (as x) and $[\frac{1}{4}, \frac{3}{4}]$ (as y) in (OP) and (BL), and intervals $[\frac{1}{4}, \frac{1}{2}]$ (as x) and $[\frac{3}{4}, 1]$ (as y) are considered in (IP).

3.2. **Relationship between Negations and Implications.**

Now, we will discuss the relationship between the natural negation of $I_{\odot}^{E}$ and that of I.

Proposition 3.9. *Let $M \trianglelefteq L$ with respect to (r_1, r_2, s), $\odot$ be an e-operator on M and $I : M\times M \to M$ be a binary operator satisfying (EP) and (OP). Then*

(1) *For all $x \in L$, $N_{I_{\odot}^{E}}(x) := I_{\odot}^{E}(x, 0_L)$ is a fuzzy negation on L;*

(2) *For each $x \in L$, $N_{I_{\odot}^{E}}(x) = (N_I)_{\odot}^{E}(x)$;*

(3) *For each $x \in L$, $x \leq_L N_{I_{\odot}^{E}}(N_{I_{\odot}^{E}}(x))$;*

(4) $N_{I_{\odot}^{E}} \circ N_{I_{\odot}^{E}} \circ N_{I_{\odot}^{E}} = N_{I_{\odot}^{E}}$.

Proof. (1) Since I satisfies (EP) and (OP), by Lemma 2.21, I satisfies $(I1)$ and $(I3)$. By Theorem 3.2, $I_{\odot}^{E}$ satisfies $(I1)$ and $(I3)$. Based on the definition of $N_{I_{\odot}^{E}}$, we know that $N_{I_{\odot}^{E}}$ is decreasing with respect to x, and $N_{I_{\odot}^{E}}(0_L) = I_{\odot}^{E}(0,0) = 1_L$, $N_{I_{\odot}^{E}}(1_L) = I_{\odot}^{E}(1_L, 0) = 0_L$. Thus, $N_{I_{\odot}^{E}}(x) := I_{\odot}^{E}(x, 0_L)$ for all $x \in L$ is a fuzzy negation on L.
(2) and (3) are analogous to the Theorem 3.5 for (SN).
(4) By Lemma 2.22(3), we know that if I satisfies (EP) and (OP), then $N_I \circ N_I \circ N_I = N_I$. Thus for each $x \in L$,

$$\begin{aligned}
&N_{I_{\odot}^{E}}\left(N_{I_{\odot}^{E}}\left(N_{I_{\odot}^{E}}(x)\right)\right) = N_{I_{\odot}^{E}}\left(N_I\Big(N_I(r_1(x))\Big) \odot N_I\Big(N_I(r_2(x))\Big)\right) \text{ by Thm. 3.5}\\
&= I_{\odot}^{E}\left(N_I\Big(N_I(r_1(x))\Big) \odot N_I\Big(N_I(r_2(x))\Big), 0_L\right) \text{ by Prop. 3.9(1)}\\
&= I\left(N_I\Big(N_I(r_2(x))\Big), r_1(0_L)\right) \odot I\left(N_I\Big(N_I(r_1(x))\Big), r_2(0_L)\right) \text{ by Def. 3.1}\\
&= I\left(N_I\Big(N_I(r_2(x))\Big), 0_M\right) \odot I\left(N_I\Big(N_I(r_1(x))\Big), 0_M\right)\\
&= N_I\left(N_I\Big(N_I(r_2(x))\Big)\right) \odot N_I\left(N_I\Big(N_I(r_1(x))\Big)\right) \text{ by Prop. 2.19}\\
&= N_I(r_2(x)) \odot N_I(r_1(x)) \text{ by Lemma 2.22(3)}\\
&= (N_I)_{\odot}^{E}(x) = N_{I_{\odot}^{E}}(x). \text{ by Prop. 3.9(2)}
\end{aligned}$$

□

Remark 3.10. From Proposition 3.9(2), we can conclude that the natural negation of $I_{\odot}^{E}$ can also be obtained by the extension of N_I, i.e., for each $x \in L$, $N_{I_{\odot}^{E}}(x) = (N_I)_{\odot}^{E}(x)$.

3.3. **Implications and Automorphisms.**

Now, we pay attention to the following issue: Given $M \trianglelefteq L$ with respect to (r_1, r_2, s), $\odot$ is an $e-$operator on M, I is a fuzzy implication on M and ρ is an automorphism on M. Is the fuzzy implication $(I^{\rho})_{\odot}^{E}$ generated by the extension of the conjugate of I equal to the conjugate of the extension of $I_{\odot}^{E}$? In other words, does there exist a suitable automorphism ρ' on L such that the equation $(I^{\rho})_{\odot}^{E} = (I_{\odot}^{E})^{\rho'}$ holds ?

Proposition 3.11. *Let $M \trianglelefteq L$ with respect to (r_1, r_2, s), $\odot$ be an $e-$operator and ρ be an automorphism on M and let $I : M \times M \to M$ be a function. If I satisfies $Q \in \{(I1), (I2), (NC), (LB), (RB), (NP)\}$, then $(I^{\rho})_{\odot}^{E}$ also satisfies Q.*

Proof. Straightforwardly from Theorem 3.2 for $(I1)$ and $(I2)$. The rest follows from the easy computation of $(I^{\rho})_{\odot}^{E}$ and the facts that $r_1(0_L) = 0_M = r_2(0_L)$, $r_1(1_L) = 1_M = r_2(1_L)$ and $1_M \odot 1_M = 1_L$. □

Remark 3.12. Propositions 2.23 and 3.11 indicate that I^ρ and $(I^\rho)^E_\odot$ preserve many properties of I.

Theorem 3.13. *Let $M{\trianglelefteq}L$ with respect to (r_1, r_2, s) , $\odot$ be an $e-$operator on M and I be a fuzzy implication on M. Given an automorphism $\rho : M{\to}M$. Then we have $(I^\rho)^E_\odot = (I^E_\odot)^{\rho^E}$.*

Proof. For all $x, y{\in}L$, we have

$$\begin{aligned}
(I^E_\odot)^{\rho^E}(x, y) &= (\rho^E)^{-1}\left(I^E_\odot\Big(\rho^E(x), \rho^E(y)\Big)\right) \quad \text{by Def. 2.5}\\
&= (\rho^E)^{-1}\left(I\Big(r_2(\rho^E(x)), r_1(\rho^E(y))\Big){\odot}I\Big(r_1(\rho^E(x)), r_2(\rho^E(y))\Big)\right) \quad \text{by Def. 3.1}\\
&= \rho^{-1}\left(I\Big(r_2(\rho^E(x)), r_1(\rho^E(y))\Big)\right){\odot}\rho^{-1}\left(I\Big(r_1(\rho^E(x)), r_2(\rho^E(y))\Big)\right) \quad \text{by 2.29(2)}\\
&= \rho^{-1}\left(I\Big(\rho(r_2(x)), \rho(r_1(y))\Big)\right){\odot}\rho^{-1}\left(I\Big(\rho(r_1(x)), \rho(r_2(y))\Big)\right) \quad \text{by 2.30(2)(3)}\\
&= I^\rho(r_2(x), r_1(y)){\odot}I^\rho(r_1(x), r_2(y)) \quad \text{by Def. 2.5}\\
&= (I^\rho)^E_\odot(x, y).
\end{aligned}$$

Therefore, we conclude that $(I^\rho)^E_\odot = (I^E_\odot)^{\rho^E}$. □

4. A Special Case of The Extended Fuzzy Implications

In this section, we will study a special type of fuzzy implication named (S, N)-implication, that is, an implication defined from a $t-$conorm S and a fuzzy negation N. More detailed results about the (S, N)-implication can be found in [5, 19, 22].

Definition 4.1. Let S be a $t-$conorm on M and N be a fuzzy negation on M. Then a binary operator $I_{S,N} : M{\times}M{\to}M$ can be defined as follows:

$$\forall x, y{\in}M, I_{S,N}(x, y) = S(N(x), y)$$

In this case, S and N are said to be the generators of $I_{S,N}$.

Example 4.2. Let S be any $t-$conorm on M and N be the negation

$$N(x) = \begin{cases} 1_M & \text{if } x = 0_M, \\ 0_M & \text{if } x{\neq}0_M, \end{cases}$$

Then, we can obtain a binary operator $I_{S,N}$ as following:

$$I_{S,N}(x, y) = \begin{cases} 1_M & \text{if } x = 0_M, \\ y & \text{if } x{\neq}0_M, \end{cases} \quad x, y{\in}M.$$

Proposition 4.3. [5] *Under the same conditions as in Definition 4.1, the function $I_{S,N}$ is a fuzzy implication called $(S, N)-$implication. If N is strong, then $I_{S,N}$ is called a strong implication or $S-$implication.*

Proposition 4.4. *Let $M \trianglelefteq L$ with respect to (r_1, r_2, s), $\odot$ be an $e-$operator on M and I be an $(S,N)-$implication on M. Then the function $I_{S^E_\odot, N^E_\odot} : L \times L \to L$ defined by $I_{S^E_\odot, N^E_\odot}(x,y) = S^E_\odot(N^E_\odot(x), y)$ is a fuzzy implication, named $(S^E_\odot, N^E_\odot)$-implication. Furthermore, for all $x, y \in L$, $I_{S^E_\odot, N^E_\odot}(x,y) = (I_{S,N})^E_\odot(x,y)$.*

Proof. For all $x, y \in L$, by Proposition 2.32 and Definition 3.1, we have

$$\begin{aligned}
&I_{S^E_\odot, N^E_\odot}(x,y) = S^E_\odot\Big(N^E_\odot(x), y\Big) = S^E_\odot\Big(N(r_1(x)) \odot N(r_2(x)), y\Big)\\
&= S\Bigg(r_1\Big(N(r_1(x)) \odot N(r_2(x))\Big), r_1(y)\Bigg) \odot S\Bigg(r_2\Big(N(r_1(x)) \odot N(r_2(x))\Big), r_2(y)\Bigg)\\
&= S\Big(N(r_2(x)), r_1(y)\Big) \odot S\Big(N(r_1(x)), r_2(y)\Big) \text{ by Prop. 2.27(1)}\\
&= I_{S,N}\Big(r_2(x), r_1(y)\Big) \odot I_{S,N}\Big(r_1(x), r_2(y)\Big)\\
&= (I_{S,N})^E_\odot(x,y).
\end{aligned}$$

So we proved that $I_{S^E_\odot, N^E_\odot}(x,y) = (I_{S,N})^E_\odot(x,y)$ and thus $I_{S^E_\odot, N^E_\odot}$ is a fuzzy implication. □

Remark 4.5. Proposition 4.4 shows that we can extend $I_{S,N}$ in Definition 4.1 to $(I_{S,N})^E_\odot$ by the way of Definition 3.1, which happens to be the same as $I_{S^E_\odot, N^E_\odot}$ defined by a $t-$conorm $S^E_\odot$ and a fuzzy negation $N^E_\odot$ in Definition 4.1. This fact also indicates the effectiveness of the way to extend fuzzy implications in this paper.

Proposition 4.6. *Let $M \trianglelefteq L$ with respect to (r_1, r_2, s), S be a $t-$conorm on M, N be a fuzzy negation on M, $\odot$ be an $e-$operator on M and $(I_{S,N})^E_\odot$ be an $(S^E_\odot, N^E_\odot)$-implication on L. Then,*

(1) *$(I_{S,N})^E_\odot$ satisfies (NP) and (EP);*

(2) *$N^E_\odot(x) = (I_{S,N})^E_\odot(x, 0_L) = (N_{I_{S,N}})^E_\odot(x)$;*

(3) *$(I_{S,N})^E_\odot$ satisfies (RCP);*

(4) *If N is a strong fuzzy negation, then $(I_{S,N})^E_\odot$ satisfies (CP) with respect to $N^E_\odot$.*

Proof. (1) It is straightforward from Theorem 3.2 and Theorem 3.5, since $I_{S,N}$ is a fuzzy implication satisfying (NP) and (EP).
(2) It is straightforward from Proposition 2.32(2).
(3) For all $x, y \in L$,

$$\begin{aligned}
&(I_{S,N})^E_\odot\Big(x, N^E_\odot(y)\Big) = S\Big(N(r_2(x)), r_1(N^E_\odot(y))\Big) \odot S\Big(N(r_1(x)), r_2(N^E_\odot(y))\Big)\\
&= S\Big(N(r_2(x)), N(r_2(y))\Big) \odot S\Big(N(r_1(x)), N(r_1(y))\Big) \text{ by 2.27(1) and 2.32(2)}\\
&= S\Big(N(r_2(y)), N(r_2(x))\Big) \odot S\Big(N(r_1(y)), N(r_1(x))\Big) \text{ by Def. 2.14(1)}\\
&= S\Big(N(r_2(y)), r_1(N^E_\odot(x))\Big) \odot S\Big(N(r_1(y)), r_2(N^E_\odot(x))\Big) \text{ by 2.27(1) and 2.32(2)}\\
&= (I_{S,N})^E_\odot\Big(y, N^E_\odot(x)\Big), \text{ by Prop. 4.4}
\end{aligned}$$

i.e., $(I_{S,N})^E_\odot$ satisfies (RCP).
(4) Suppose that N is a strong fuzzy negation. Then for all $x, y\in L$,

$$\begin{aligned}
&(I_{S,N})^E_\odot\Big(N^E_\odot(y), N^E_\odot(x)\Big)\\
&= S\Bigg(N\Big(r_2(N^E_\odot(y))\Big), r_1\Big(N^E_\odot(x)\Big)\Bigg)\odot S\Bigg(N\Big(r_1(N^E_\odot(y))\Big), r_2\Big(N^E_\odot(x)\Big)\Bigg) \text{ by 4.4}\\
&= S\Bigg(N\Big(N(r_1(y))\Big), N\Big(r_2(x)\Big)\Bigg)\odot S\Bigg(N\Big(N(r_2(y))\Big), N\Big(r_1(x)\Big)\Bigg)\\
&= S\Big(r_1(y), N(r_2(x))\Big)\odot S\Big(r_2(y), N(r_1(x))\Big)\\
&= (I_{S,N})^E_\odot(x,y),
\end{aligned}$$

i.e., $(I_{S,N})^E_\odot$ satisfies (CP). □

Proposition 4.7. *Let $M\trianglelefteq L$ with respect to (r_1, r_2, s), S be a $t-$conorm on M, N be a strong fuzzy negation on M, $\odot$ be an $e-$operator on M and $I^E_\odot : L\times L\to L$ be an $(S^E_\odot, N^E_\odot)-$implication. We define a binary operator $S_{I^E_\odot} : L\times L\to L$ as following:*

$$S_{I^E_\odot}(x,y) = I^E_\odot\Big(N^E_\odot(x), y\Big).$$

Then, for all $x, y\in L$, $S_{I^E_\odot}(x,y) = S^E_\odot(x,y)$.

Proof. For all $x, y\in L$,

$$\begin{aligned}
S_{I^E_\odot}(x,y) &= I^E_\odot\Big(N^E_\odot(x), y\Big) = I^E_\odot\Big(N(r_1(x))\odot N(r_2(x)), y\Big)\\
&= S\Bigg(N\Big(N(r_1(x))\Big), r_1(y)\Bigg)\odot S\Bigg(N\Big(N(r_2(x))\Big), r_2(y)\Bigg)\\
&= S\Big(r_1(x), r_1(y)\Big)\odot S\Big(r_2(x), r_2(y)\Big)\\
&= S^E_\odot(x,y).
\end{aligned}$$

□

Proposition 4.8. *Let $M\trianglelefteq L$ with respect to (r_1, r_2, s) and $\odot$ be an $e-$operator on M. ρ, S, N be respectively an automorphism, a $t-$conorm and a fuzzy negation on M, I be an $(S,N)-$implication on M generated from S and N. Then for all $x, y\in L$, $(I_{S^\rho,N^\rho})^E_\odot(x,y) = ((I_{S,N})^\rho)^E_\odot(x,y)$.*

Proof. For all $x, y\in L$, we have

$$\begin{aligned}
(I_{S^\rho,N^\rho})^E_\odot(x,y) &= I_{S^\rho,N^\rho}\Big(r_2(x), r_1(y)\Big)\odot I_{S^\rho,N^\rho}\Big(r_1(x), r_2(y)\Big)\\
&= (I_{S,N})^\rho(r_2(x), r_1(y))\odot(I_{S,N})^\rho(r_1(x), r_2(y)) \text{ by Prop. 2.24}\\
&= ((I_{S,N})^\rho)^E_\odot(x,y).
\end{aligned}$$

□

Remark 4.9. From Propositions 4.4, 4.6, 4.7 and 4.8, we can consider $(S^E_\odot, N^E_\odot)$-implication on L as the extension of $(S,N)-$implication on M, which indicates that$(I_{S,N})^E_\odot$ preserves many of the considered properties of (S,N)-implications.

Next, we will discuss some results of the properties (LI) and (IBL) with respect to $(I_{S,N})^E_\odot$ on L. More details about these proporties we can see [1, 7, 25].

Proposition 4.10. *Let $M \trianglelefteq L$ with respect to (r_1, r_2, s). Let T be a $t-$norm, S be a $t-$conorm and let N be a strong fuzzy negation on M, such that (T, S, N) is a De Morgan triple. Let $(I_{S,N})^E_{\odot}$ be an $(S^E_{\odot}, N^E_{\odot})$-implication. Then $(I_{S,N})^E_{\odot}$ satisfies (LI) with respect to $T^E_{\odot}$.*

Proof. It is immediate from Theorem 3.6, since $I_{S,N}$ satisfies (LI) with respect to T. □

Proposition 4.11. *Let $M \trianglelefteq L$ with respect to (r_1, r_2, s). Let S be a $t-$conorm, N be a fuzzy negation on M, $S^E_{\odot}$ be a $t-$conorm, $N^E_{\odot}$ be a fuzzy negation, $(I_{S,N})^E_{\odot}$ be an $(S^E_{\odot}, N^E_{\odot})-$implication on L. If for all $x \in M$, $S(x, x) = x$, then $(I_{S,N})^E_{\odot}$ satisfies (IBL).*

Proof. It is immediate from Theorem 3.7, since $I_{S,N}$ satisfies (IBL). □

5. Conclusion and Remarks

In this paper, we paid attention to the extension of fuzzy implications on a bounded lattice and discussed whether they could preserve many of the considered properties. In Section 3, we gave the extension of fuzzy implications from the general sublattice M to a bounded lattice L, and proved that the new fuzzy implications obtained on L preserves many of the considered properties of fuzzy implications on M except $(OP), (IP)$ and (BL). In Subsection 3.2, we discussed the relationship between the extension of negations and implications. We proved that $N_{I^E_{\odot}}$ is a fuzzy negation but not a strong one, and obtained $N_{I^E_{\odot}} = (N_I)^E_{\odot}$ and $N_{I^E_{\odot}} \circ N_{I^E_{\odot}} \circ N_{I^E_{\odot}} = N_{I^E_{\odot}}$ in Proposition 3.9. In Subsection 3.3, we investigated the automorphisms of the new fuzzy implications. We proved that $(I^{\rho})^E_{\odot}$ preserve many of the considered properties of I in Proposition 3.11 and $(I^{\rho})^E_{\odot} = (I^E_{\odot})^{\rho^E}$ in Theorem 3.13. In Section 4, we discussed an example of the extended fuzzy implications, $(I_{S,N})^E_{\odot}$-implication, which can be considered as the extension of (S, N)-implication on M, and we proved that $I_{S^E_{\odot}, N^E_{\odot}}(x, y) = (I_{S,N})^E_{\odot}(x, y)$. Also we paid attention to whether $(S^E_{\odot}, N^E_{\odot})$-implications preserving many of the considered properties of (S, N)-implications.

However, there are still many open problems to be solved. For example, we can use the similar way to extend other fuzzy implications such as R-implications, QL-implications, D-implications and implications defined from uninorms. In future, it may we can find some better ways to extend fuzzy implications such that they preserve most of their properties.

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