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# SOLUTION AND STABILITY OF QUATTUORVIGINTIC FUNCTIONAL EQUATION IN INTUITIONISTIC FUZZY NORMED SPACES

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Abstract. In this paper, we investigate the general solution and the generalized Hyers-Ulam stability of a new functional equation satisfied by $f(x) = x^{24}$, which is called quattuorvigintic functional equation in intuitionistic fuzzy normed spaces by using the fixed point method. These results can be regarded as an important extension of stability results corresponding to functional equations on normed spaces.

## 1. Introduction

In 1940, a question that was given a serious thought by S. M. Ulam [19] concerning the stability of group homomorphisms, gave rise to the stability problem of functional equations.
In 1941, D. H. Hyers [10] solved the stability problem of Ulam in Banach spaces. Since the last few decades several stability problems of functional equations have been investigated [4, 5, 8, 9, 14, 15, 16, 17, 18, 20].
The notion of fuzzy sets was first introduced by L. A. Zadeh [21] in 1965 which is a powerful hand set for modeling uncertainty and vagueness in various problems.
In 1984, A. K. Katsaras [11] defined the notion of fuzzy norm on a linear space to construct a fuzzy vector topological structure. Thereafter, some mathematicians introduced and discussed several notions of fuzzy norms from various points of view [7, 13]. In particular in 2003, T. Bag and S. K. Samanta [1], following Cheng and Mordeson [3], gave an idea of a fuzzy norm in such a way that the corresponding fuzzy metric is of Kramosil and Michalek type [12]. They described some nice properties of the fuzzy norm in [2]. In [6], M. Eshaghi Gordji et al. investigated the generalized Hyers-Ulam stability of Jordan homomorphisms and Jordan derivations on fuzzy Banach algebras.

**Definition 1.1.** A binary operation $* : [0,1] \times [0,1] \to [0,1]$ ($\diamond : [0,1] \times [0,1] \to [0,1]$) is said to be a continuous $t$-norm (continuous $t$-conorm) if it satisfies the following conditions:

(1) $*$ is associative and commutative ($\diamond$ is associative and commutative),
(2) $*$ is continuous ($\diamond$ is continuous),

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(3) $a * 1 = a$ for all $a \in [0,1]$ ($a \diamond 0 = a$ for all $a \in [0,1]$),
(4) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0,1]$ ($a \diamond b \leq c \diamond d$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0,1]$).

**Definition 1.2.** The five-tuple $(X, \mu, \nu, *, \diamond)$ is said to be an intuitionistic fuzzy normed spaces (for short, IFNS) if $X$ is a vector space, $*$ is a continuous $t$-norm, $\diamond$ is a continuous $t$-conorm and $\mu, \nu$ are fuzzy sets on $X \times (0, \infty)$ satisfying the following conditions for all $x, y \in X$ and $s, t > 0$:
(i) $\mu(x,t) + \nu(x,t) \leq 1$, (ii) $\mu(x,t) > 0$, (iii) $\mu(x,t) = 1$ if and only if $x = 0$, (iv) $\mu(\alpha x, t) = \mu\left(x, \frac{t}{|\alpha|}\right)$ for all $\alpha \neq 0$, (v) $\mu(x,t) * \mu(y,s) \leq \mu(x+y, t+s)$, (vi) $\mu(x,.) : (0,\infty) \to [0,1]$ is continuous, (vii) $\lim_{t\to\infty} \mu(x,t) = 1$ and $\lim_{t\to 0} \mu(x,t) = 0$, (viii) $\nu(x,t) < 1$, (ix) $\nu(x,t) = 0$ if and only if $x = 0$, (x) $\nu(\alpha x, t) = \nu\left(x, \frac{t}{|\alpha|}\right)$ for all $\alpha \neq 0$, (xi) $\nu(x,t) \diamond \nu(y,s) \geq \nu(x+y, t+s)$, (xii) $\nu(x,.) : (0,\infty) \to [0,1]$ is continuous, (xiii) $\lim_{t\to\infty} \nu(x,t) = 0$ and $\lim_{t\to 0} \nu(x,t) = 1$.

Let $(X, \mu, \nu, *, \diamond)$ be an IFNS. Then, a sequence $(x_n)$ is said to be intuitionistic fuzzy convergent to $x \in X$ if $\lim_{n\to\infty} \mu(x_n - x, t) = 1$ and $\lim_{n\to\infty} \nu(x_n - x, t) = 0$ for all $t > 0$.
Let $(X, \mu, \nu, *, \diamond)$ be an IFNS. Then, a sequence $(x_n)$ is said to be intuitionistic fuzzy Cauchy sequence if $\lim_{n\to\infty} \mu(x_{n+p} - x_n, t) = 1$ and $\lim_{n\to\infty} \nu(x_{n+p} - x_n, t) = 0$ for all $t > 0$ and $p \in \mathbb{N}$.
Let $(X, \mu, \nu, *, \diamond)$ be an IFNS. Then $(X, \mu, \nu, *, \diamond)$ is said to be complete if every intuitionistic fuzzy Cauchy sequence in $(X, \mu, \nu, *, \diamond)$ is intuitionistic fuzzy convergent in $(X, \mu, \nu, *, \diamond)$.
Let $X$ be a linear space and $(Y, \mu, \nu)$ be an intuitionistic fuzzy Banach space and $f : X \to Y$ be a mapping. Define a mapping $Df : X^2 \to Y$ by

$$\begin{aligned}
&Df(x,y) := f(x+12y) - 24f(x+11y) + 276f(x+10y) - 2024f(x+9y) \\
&+ 10626f(x+8y) - 42504f(x+7y) + 134596f(x+6y) - 346104f(x+5y) \\
&+ 735471f(x+4y) - 1307504f(x+3y) + 1961256f(x+2y) - 2496144f(x+y) \\
&+ 2704156f(x) - 2496144f(x-y) + 1961256f(x-2y) - 1307504f(x-3y) \\
&+ 735471f(x-4y) - 346104f(x-5y) + 134596f(x-6y) - 42504f(x-7y) \\
&+ 10626f(x-8y) - 2024f(x-9y) + 276f(x-10y) - 24f(x-11y) + f(x-12y) \\
&- 24!f(y),
\end{aligned}$$

where 24 !=620448401733239439360000.
In this paper, we introduce the following quattuorvigintic functional equation

$$Df(x,y) = 0, \tag{1}$$

for all $x, y \in X$. We study the general solution and the generalized Hyers-Ulam stability of the functional equation (1) in intuitionistic fuzzy normed spaces by using the standard fixed point method.

## 2. General Solution of the Functional Equation (1)

In the following result, a solution of the quattuorvigintic functional equation (1) is given.

**Theorem 2.1.** *If $X$ and $Y$ are real vector spaces and $f : X \to Y$ is a mapping satisfying quattuorvigintic functional equation (1) for all $x, y \in X$, then $f$ is a quattuorvigintic mapping, i.e. $f(x) = x^{24}$.*

*Proof.* Let $f_i : X \to Y (i = 1, \dots, 14)$, be mappings defined by the following relations:

$$\begin{aligned} f_1(x) := {} & f(13x) - 24f(12x) + 276f(11x) - 2024f(10x) + 10626f(9x) - 42504f(8x) \\ & +134596f(7x) - 346104f(6x) + 735471f(5x) - 1307504f(4x) + 1961256f(3x) \\ & -2496144f(2x) + 2704156f(x) - 2496144f(0) + 1961256f(-x) - 1307504f(-2x) \\ & +735471f(-3x) - 346104f(-4x) + 134596f(-5x) - 42504f(-6x) + 10626(-7x) \\ & -2024f(-8x) + 276f(-9x) - 24f(-10x) + f(-11x), \end{aligned} \tag{2}$$

$$\begin{aligned} f_2(x) := {} & f(24x) - 24f(22x) + 276f(20x) - 2024f(18x) + 10626f(16x) \\ & -42504f(14x) + 134596f(12x) - 346104f(10x) + 735471f(8x) - 1307504f(6x) \\ & +1961256f(4x) - 310224200866619722176144f(2x), \end{aligned} \tag{3}$$

$$\begin{aligned} f_3(x) := {} & 24f(23x) - 300f(22x) + 2024f(21x) - 10350f(20x) + 42504f(19x) \\ & -136620f(18x) + 346104f(17x) - 724845f(16x) + 1307504f(15x) \\ & -2003760f(14x) + 2496144f(13x) - 2569560f(12x) + 2496144f(11x) \\ & -2307360f(10x) + 1307504f(9x) + 346104f(7x) - 1442100f(6x) \\ & +42504f(5x) + 1950630f(4x) + 2024f(3x) \\ & -310224200866619722176420f(2x) + 620448401733239439360024f(x), \end{aligned} \tag{4}$$

$$\begin{aligned} f_4(x) := {} & 276f(22x) - 4600f(21x) + 38226f(20x) - 212520f(19x) + 883476f(18x) \\ & -2884200f(17x) + 7581651f(16x) - 16343800f(15x) + 29376336f(14x) \\ & -44574000f(13x) + 57337896f(12x) - 62403600f(11x) + 57600096f(10x) \\ & -45762640f(9x) + 31380096f(8x) - 17305200f(7x) + 6864396f(6x) \\ & -3187800f(5x) + 2970726f(4x) - 253000f(3x) \\ & -310224200866619722127844f(2x) \\ & +155112100433309859839993376f(x), \end{aligned} \tag{5}$$

$$\begin{aligned} f_5(x) := {} & 2024f(21x) - 37950f(20x) + 346104f(19x) - 2049300f(18x) \\ & +8846904f(17x) - 29566845f(16x) + 79180904f(15x) - 173613660f(14x) \\ & +316297104f(13x) - 483968760f(12x) + 626532144f(11x) - 688746960f(10x) \\ & +643173104f(9x) - 509926560f(8x) + 343565904f(7x) - 196125600f(6x) \\ & +92336904f(5x) - 34177770f(4x) + 11478104f(3x) \\ & -310224200866619725060896f(2x) \\ & +186754968921705071247918624f(x), \end{aligned} \tag{6}$$

$$\begin{aligned} f_6(x) := {} & 10626f(20x) - 212520f(19x) + 2047276f(18x) - 12660120f(17x) \\ & +56461251f(16x) - 193241400f(15x) + 526900836f(14x) - 1172296200f(13x) \\ & +2162419336f(12x) - 3343050000f(11x) + 4363448496f(10x) \end{aligned}$$

$$
\begin{gathered}
-4830038640f(9x)+4542268896f(8x)-3626016240f(7x)+2450262496f(6x)\\
-1396256400f(5x)+666336726f(4x)-260946224f(3x)\\
-3102242008666196389842224f(2x)+1442542534029781696490492976f(x), \quad (7)
\end{gathered}
$$

$$
\begin{gathered}
f_7(x):=42504f(19x)-885500f(18x)+8846904f(17x)-56450625f(16x)\\
+258406104f(15x)-903316260f(14x)+2505404904f(13x)-5652695510f(12x)\\
+10550487504f(11x)-16476857760f(10x)+21693987504f(9x)\\
-24192092760f(8x)+22898009904f(7x)-18390043760f(6x)\\
+12497281104f(5x)-7148788746f(4x)+3417009904f(3x)\\
-3102242008666210721340 96f(2x)+8035427250847183979603007504f(x), \quad (8)
\end{gathered}
$$

$$
\begin{gathered}
f_8(x):=134596f(18x)-2884200f(17x)+29577471f(16x)-193241400f(15x)\\
+903273756f(14x)-3215463480f(13x)+9058108906f(12x)\\
-20709971880f(11x)+39097292256f(10x)-61667237520f(9x)\\
+81904011816f(8x)-92039436720f(7x)+87706060816f(6x)\\
-70863986424f(5x)+48426381366f(4x)-27855180584f(3x)\\
-3102242008666062753015 84f(2x)+34406966118116793103987931616f(x), \quad (9)
\end{gathered}
$$

$$
\begin{gathered}
f_9(x):=346104f(17x)-7571025f(16x)+79180904f(15x)-526943340f(14x)\\
+2505404904f(13x)-9057974310f(12x)+25874242104f(11x)\\
-59894162460f(10x)+114317570864f(9x)-182073200760f(8x)\\
+243931561104f(7x)-276262654756f(6x)+265110241704f(5x)\\
-215587979706f(4x)+148402050104f(3x)\\
-3102242008667066969733 96f(2x)\\
+117916839197803888736391573984f(x), \quad (10)
\end{gathered}
$$

$$
\begin{gathered}
f_{10}(x):=735471f(16x)-16343800f(15x)+173571156f(14x)\\
-1172296200f(13x)+5652830106f(12x)-20709971880f(11x)\\
+59893816356f(10x)-140231884120f(9x)+270459163656f(8x)\\
-434867331624f(7x)+587671074716f(6x)-670904491224f(5x)\\
+649037957766f(4x)-534074197624f(3x)-3102242008662394538045 64f(2x)\\
+332656512831284991355511345016f(x), \quad (11)
\end{gathered}
$$

$$
\begin{gathered}
f_{11}(x):=1307504f(15x)-29418840f(14x)+316297104f(13x)\\
-2162284740f(12x)+10550487504f(11x)-39097638360f(10x)\\
+114317570864f(9x)-270459163656f(8x)+526781594064f(7x)\\
-854978826856f(6x)+1166425625904f(5x)-1347605474556f(4x)\\
+1333027785584f(3x)-3102242008677808921708 56f(2x)\\
+788978319302432336277230634384f(x), \quad (12)
\end{gathered}
$$

$$
\begin{gathered}
f_{12}(x):=1961256f(14x)-44574000f(13x)+484103356f(12x)\\
-3343050000f(11x)+16476511656f(10x)-61668545024f(9x)\\
+182104580856f(8x)-435210551424f(7x)+857234271256f(6x)
\end{gathered}
$$

$$\begin{gathered}
-1411817976624f(5x)+1971686940036f(4x)-2378651809424f(3x)\\
-3102242008640646415418 64f(2x)\\
+16002170863622498326722067 34976f(x),
\end{gathered} \tag{13}$$

$$\begin{gathered}
f_{13}(x):=2496144f(13x)-57203300f(12x)+626532144f(11x)\\
-4365755856f(10x)+21739750144f(9x)-82413938376f(8x)\\
+247557577344f(7x)-606052946576f(6x)+1235893313424f(5x)\\
-2138815370076f(4x)+3195724134064f(3x)-3102242008708106306333 76f(2x)\\
+28170752369519760900135703568 64f(x),
\end{gathered} \tag{14}$$

$$\begin{gathered}
f_{14}(x):=2704156f(12x)-64899744f(11x)+746347056f(10x)\\
-5473211744f(9x)+28734361656f(8x)-114937446624f(7x)\\
+363968580976f(6x)-935919208224f(5x)+1988828317476f(4x)\\
-3535694786624f(3x)-310224200861316177500064f(2x)\\
+4365803792247991305489858025536f(x).
\end{gathered} \tag{15}$$

Replacing $(x, y)$ by $(x, x)$ in (1), we get

$$f_1(x) = 24!f(x), \tag{16}$$

for all $x \in X$. Replacing $(x, y)$ by $(0, 2x)$ in (1), we obtain

$$f_2(x) = 0, \tag{17}$$

for all $x \in X$. Replacing $(x, y)$ by $(ix, x)$(i=1,...,12) in (1), we get

$$f_i(x) = 0 \quad (i = 3, \ldots, 14), \tag{18}$$

for all $x \in X$. Replacing $(x, y)$ by $(0, x)$ in (1), we have

$$f(2x) = 2^{24}f(x),$$

for all $x \in X$. Hence $f$ is a quattuorvigintic mapping. □

## 3. Hyers-Ulam Intuitionistic Fuzzy Stability of the Functional Equation (1)

In this section, we investigate the generalized Hyers-Ulam stability of the quattuorvigintic functional equation (1) in intuitionistic fuzzy normed spaces. Throughout this section let $X$ be a linear space and $(Y, \mu, \nu)$ be an intuitionistic fuzzy Banach space.

**Lemma 3.1.** *Fix $a > 0$. Let $\psi, \psi' : X \times (0, \infty) \to [0, 1]$ be mappings, such that $\psi(2x, t) \geq \psi(x, Lt)$ and $\psi'(2x, t) \leq \psi'(x, Lt)$ for some $L > \frac{1}{2^{24}}$ and all $x \in X$ and $t > 0$. Moreover let $\lim_{t\to\infty} \psi(x, t) = 1$ and $\lim_{t\to\infty} \psi'(x, t) = 0$ for all $x \in X$. Suppose that $f : X \to Y$ is a mapping satisfying*

$$\mu(f(2x) - 2^{24}f(x), at) \geq \psi(x, t),$$

$$\nu(f(2x) - 2^{24}f(x), at) \leq \psi'(x, t),$$

*for all $x \in X$ and $t > 0$. Then there exists a mapping $Q : X \to Y$, such that*

$$\mu(f(x) - Q(x), t) \geq \psi\left(x, (2^{24}L - 1)\frac{t}{aL}\right), \tag{19}$$

$$\nu(f(x) - Q(x), t) \leq \psi'\left(x, (2^{24}L - 1)\frac{t}{aL}\right), \tag{20}$$

*for all $x \in X$ and $t > 0$.*

*Proof.* Let $E = \{f : X \to Y\}$, and introduce the complete generalized metric $d$ defined on $E$ by

$d(g, h) = \inf\{c \in (0, \infty) : \mu(g(x) - h(x), ct) \geq \psi(x, t), \nu(g(x) - h(x), ct) \leq \psi'(x, t)$ $(x \in X, t > 0)\}$.

We now define a mapping $J : E \to E$ by

$$Jg(x) := \frac{1}{2^{24}} g(2x).$$

It is easy to show that $J$ is a strictly contractive mapping with Lipschitz constant $\frac{1}{2^{24}L}$ and $d(Jf, f) \leq \frac{a}{2^{24}}$. So there exists a mapping $Q : X \to Y$, such that

$$Q(x) := \lim_{n\to\infty} \frac{f(2^n x)}{2^{24n}},$$

and (19) and (20) are satisfied. □

**Theorem 3.2.** *Let $(Z, \mu', \nu')$ be an intuitionistic fuzzy normed space. Let $\phi : X^2 \to Z$ be a mapping, such that*

$$\mu'(\phi(2x, 2x), t) \geq \mu'(\phi(x, x), Lt), \tag{21}$$

$$\nu'(\phi(2x, 2x), t) \leq \nu'(\phi(x, x), Lt), \tag{22}$$

*and*

$$\lim_{n\to\infty} \mu'\left(\frac{1}{2^{24n}}\phi(2^n x, 2^n y), t\right) = 1, \tag{23}$$

$$\lim_{n\to\infty} \nu'\left(\frac{1}{2^{24n}}\phi(2^n x, 2^n y), t\right) = 0, \tag{24}$$

*for some $L > \frac{1}{2^{24}}$ and all $x, y \in X$ and $t > 0$. Let $f : X \to Y$ be a mapping satisfying*

$$\mu(Df(x, y), t) \geq \mu'(\phi(x, y), t), \tag{25}$$

$$\nu(Df(x, y), t) \leq \nu'(\phi(x, y), t), \tag{26}$$

*for all $x, y \in X$ and $t > 0$. Then there exists a unique quattuorvigintic mapping $Q : X \to Y$, such that*

$$\mu(f(x) - Q(x), t) \geq \psi\left(x, (2^{24}L - 1)155112100433309859840000\frac{t}{L}\right), \tag{27}$$

$$\nu(f(x) - Q(x), t) \leq \psi'\left(x, (2^{24}L - 1)155112100433309859840000\frac{t}{L}\right), \tag{28}$$

*for all* $x \in X$ *and* $t > 0$, *where*

$$\psi(x,t) := \psi_1\left(x, \frac{t}{4096}\right) * \psi_2\left(x, \frac{t}{4096}\right) * \psi_3\left(x, \frac{t}{2048}\right) * \psi_4\left(x, \frac{t}{1024}\right)$$
$$*\psi_5\left(x, \frac{t}{512}\right) * \psi_6\left(x, \frac{t}{256}\right) * \psi_7\left(x, \frac{t}{128}\right) * \psi_8\left(x, \frac{t}{64}\right) * \psi_9\left(x, \frac{t}{32}\right)$$
$$*\psi_{10}\left(x, \frac{t}{16}\right) * \psi_{11}\left(x, \frac{t}{8}\right) * \psi_{12}\left(x, \frac{t}{4}\right) * \psi_{13}\left(x, \frac{t}{2}\right) * \psi_{14}(x,t), \tag{29}$$

*and the mappings* $\psi_i : X \times (0,\infty) \to [0,1] (i = 1, \ldots, 14)$, *are defined as follows:*

$$\psi_1(x,t) := \mu'\left(\phi(0,0), \frac{24!}{2704156}\frac{t}{7}\right) * \mu'\left(\phi(0,2x), \frac{t}{7}\right)$$
$$*\mu'\left(\phi(24x,24x), \frac{24!}{14}t\right) * \mu'\left(\phi(24x,-24x), \frac{24!}{14}t\right) * \mu'\left(\phi(22x,22x), \frac{24!}{14}\frac{t}{24}\right)$$
$$*\mu'\left(\phi(22x,-22x), \frac{24!}{14}\frac{t}{24}\right) * \mu'\left(\phi(20x,20x), \frac{24!}{14}\frac{t}{276}\right)$$
$$*\mu'\left(\phi(20x,-20x), \frac{24!}{14}\frac{t}{276}\right) * \mu'\left(\phi(18x,18x), \frac{24!}{14}\frac{t}{2024}\right)$$
$$*\mu'\left(\phi(18x,-18x), \frac{24!}{14}\frac{t}{2024}\right) * \mu'\left(\phi(16x,16x), \frac{24!}{14}\frac{t}{10626}\right)$$
$$*\mu'\left(\phi(16x,-16x), \frac{24!}{14}\frac{t}{10626}\right) * \mu'\left(\phi(14x,14x), \frac{24!}{14}\frac{t}{42504}\right)$$
$$*\mu'\left(\phi(14x,-14x), \frac{24!}{14}\frac{t}{42504}\right) * \mu'\left(\phi(12x,12x), \frac{24!}{14}\frac{t}{134596}\right)$$
$$*\mu'\left(\phi(12x,-12x), \frac{24!}{14}\frac{t}{134596}\right) * \mu'\left(\phi(10x,10x), \frac{24!}{14}\frac{t}{346104}\right)$$
$$*\mu'\left(\phi(10x,-10x), \frac{24!}{14}\frac{t}{346104}\right) * \mu'\left(\phi(8x,8x), \frac{24!}{14}\frac{t}{735471}\right)$$
$$*\mu'\left(\phi(8x,-8x), \frac{24!}{14}\frac{t}{735471}\right) * \mu'\left(\phi(6x,6x), \frac{24!}{14}\frac{t}{1307504}\right)$$
$$*\mu'\left(\phi(6x,-6x), \frac{24!}{14}\frac{t}{1307504}\right) * \mu'\left(\phi(4x,4x), \frac{24!}{14}\frac{t}{1961256}\right)$$
$$*\mu'\left(\phi(4x,-4x), \frac{24!}{14}\frac{t}{1961256}\right) * \mu'\left(\phi(2x,2x), \frac{24!}{14}\frac{t}{2496144}\right)$$
$$*\mu'\left(\phi(2x,-2x), \frac{24!}{14}\frac{t}{2496144}\right), \tag{30}$$

$$\psi_2(x,t) := \mu'\left(\phi(12x,x), \frac{t}{2}\right) * \mu'\left(\phi(0,0), 24!\frac{t}{2}\right), \tag{31}$$

$$\psi_3(x,t) := \mu'\left(\phi(0,0), \frac{24!}{14976}t\right) * \mu'\left(\phi(x,x), \frac{24!}{2}\frac{t}{624}\right)$$
$$*\mu'\left(\phi(x,-x), \frac{24!}{2}\frac{t}{624}\right) * \mu'\left(\phi(11x,x), \frac{t}{26}\right), \tag{32}$$

$$\begin{aligned}\psi_4(x,t) := \mu'\left(\phi(0,0), 24!\frac{t}{21253104}\right) * \mu'\left(\phi(x,x), \frac{24!}{2}\frac{t}{1848096}\right)\\ *\mu'\left(\phi(x,-x), \frac{24!}{2}\frac{t}{1848096}\right) * \mu'\left(\phi(2x,2x), \frac{24!}{2}\frac{t}{77004}\right)\\ *\mu'\left(\phi(2x,-2x), \frac{24!}{2}\frac{t}{77004}\right) * \mu'\left(\phi(10x,x), \frac{t}{279}\right),\end{aligned} \tag{33}$$

$$\begin{aligned}\psi_5(x,t) := \mu'\left(\phi(0,0), 24!\frac{t}{8307856128}\right)\\ *\mu'\left(\phi(x,x), \frac{24!}{2}\frac{t}{1132889472}\right) * \mu'\left(\phi(x,-x), \frac{24!}{2}\frac{t}{1132889472}\right)\\ *\mu'\left(\phi(2x,2x), \frac{24!}{2}\frac{t}{98512128}\right) * \mu'\left(\phi(2x,-2x), \frac{24!}{2}\frac{t}{98512128}\right)\\ *\mu'\left(\phi(3x,3x), \frac{24!}{2}\frac{t}{4104672}\right) * \mu'\left(\phi(3x,-3x), \frac{24!}{2}\frac{t}{4104672}\right)\\ *\mu'\left(\phi(9x,x), \frac{t}{2028}\right),\end{aligned} \tag{34}$$

$$\begin{aligned}\psi_6(x,t) := \mu'\left(\phi(0,0), 24!\frac{t}{1200366153756}\right)\\ *\mu'\left(\phi(x,x), \frac{24!}{2}\frac{t}{228641172144}\right) * \mu'\left(\phi(x,-x), \frac{24!}{2}\frac{t}{228641172144}\right)\\ *\mu'\left(\phi(2x,2x), \frac{24!}{2}\frac{t}{31178341656}\right) * \mu'\left(\phi(2x,-2x), \frac{24!}{2}\frac{t}{31178341656}\right)\\ *\mu'\left(\phi(3x,3x), \frac{24!}{2}\frac{t}{2711160144}\right) * \mu'\left(\phi(3x,-3x), \frac{24!}{2}\frac{t}{2711160144}\right)\\ *\mu'\left(\phi(4x,4x), \frac{24!}{2}\frac{t}{112965006}\right) * \mu'\left(\phi(4x,-4x), \frac{24!}{2}\frac{t}{112965006}\right)\\ *\mu'\left(\phi(8x,x), \frac{t}{10631}\right),\end{aligned} \tag{35}$$

$$\begin{aligned}\psi_7(x,t) := \mu'\left(\phi(0,0), 24!\frac{t}{76798141580160}\right) * \mu'\left(\phi(x,x), \frac{24!}{2}\frac{t}{19199535395040}\right)\\ *\mu'\left(\phi(x,-x), \frac{24!}{2}\frac{t}{19199535395040}\right) * \mu'\left(\phi(2x,2x), \frac{24!}{2}\frac{t}{3657054360960}\right)\\ *\mu'\left(\phi(2x,-2x), \frac{24!}{2}\frac{t}{3657054360960}\right) * \mu'\left(\phi(3x,3x), \frac{24!}{2}\frac{t}{498689231040}\right)\\ *\mu'\left(\phi(3x,-3x), \frac{24!}{2}\frac{t}{498689231040}\right) * \mu'\left(\phi(4x,4x), \frac{24!}{2}\frac{t}{43364280960}\right)\\ *\mu'\left(\phi(4x,-4x), \frac{24!}{2}\frac{t}{43364280960}\right) * \mu'\left(\phi(5x,5x), \frac{24!}{2}\frac{t}{1806845040}\right)\\ *\mu'\left(\phi(5x,-5x), \frac{24!}{2}\frac{t}{1806845040}\right) * \mu'\left(\phi(7x,x), \frac{t}{42510}\right),\end{aligned} \tag{36}$$

$$\psi_8(x,t) := \mu'\left(\phi(0,0), 24!\frac{t}{2438479149123248}\right) * \mu'\left(\phi(x,x), \frac{24!}{2}\frac{t}{770046047091552}\right)$$
$$*\mu'\left(\phi(x,-x), \frac{24!}{2}\frac{t}{770046047091552}\right) * \mu'\left(\phi(2x,2x), \frac{24!}{2}\frac{t}{192511511772888}\right)$$
$$*\mu'\left(\phi(2x,-2x), \frac{24!}{2}\frac{t}{192511511772888}\right) * \mu'\left(\phi(3x,3x), \frac{24!}{2}\frac{t}{36668859385312}\right)$$
$$*\mu'\left(\phi(3x,-3x), \frac{24!}{2}\frac{t}{36668859385312}\right) * \mu'\left(\phi(4x,4x), \frac{24!}{2}\frac{t}{5000299007088}\right)$$
$$*\mu'\left(\phi(4x,-4x), \frac{24!}{2}\frac{t}{5000299007088}\right) * \mu'\left(\phi(5x,5x), \frac{24!}{2}\frac{t}{434808609312}\right)$$
$$*\mu'\left(\phi(5x,-5x), \frac{24!}{2}\frac{t}{434808609312}\right) * \mu'\left(\phi(6x,6x), \frac{24!}{2}\frac{t}{18117025388}\right)$$
$$*\mu'\left(\phi(6x,-6x), \frac{24!}{2}\frac{t}{18117025388}\right) * \mu'\left(\phi(6x,x), \frac{t}{134603}\right), \tag{37}$$

$$\psi_9(x,t) := \mu'\left(\phi(0,0), 24!\frac{t}{41460056923963392}\right)$$
$$*\mu'\left(\phi(x,x), \frac{24!}{2}\frac{t}{16123355470430208}\right) * \mu'\left(\phi(x,-x), \frac{24!}{2}\frac{t}{16123355470430208}\right)$$
$$*\mu'\left(\phi(2x,2x), \frac{24!}{2}\frac{t}{5091585938030592}\right) * \mu'\left(\phi(2x,-2x), \frac{24!}{2}\frac{t}{5091585938030592}\right)$$
$$*\mu'\left(\phi(3x,3x), \frac{24!}{2}\frac{t}{1272896484507648}\right) * \mu'\left(\phi(3x,-3x), \frac{24!}{2}\frac{t}{1272896484507648}\right)$$
$$*\mu'\left(\phi(4x,4x), \frac{24!}{2}\frac{t}{242456473239552}\right) * \mu'\left(\phi(4x,-4x), \frac{24!}{2}\frac{t}{242456473239552}\right)$$
$$*\mu'\left(\phi(5x,5x), \frac{24!}{2}\frac{t}{33062246350848}\right) * \mu'\left(\phi(5x,-5x), \frac{24!}{2}\frac{t}{33062246350848}\right)$$
$$*\mu'\left(\phi(6x,6x), \frac{24!}{2}\frac{t}{2874977943552}\right) * \mu'\left(\phi(6x,-6x), \frac{24!}{2}\frac{t}{2874977943552}\right)$$
$$*\mu'\left(\phi(7x,7x), \frac{24!}{2}\frac{t}{119790747648}\right) * \mu'\left(\phi(7x,-7x), \frac{24!}{2}\frac{t}{119790747648}\right)$$
$$*\mu'\left(\phi(5x,x), \frac{t}{346112}\right), \tag{38}$$

$$\psi_{10}(x,t) := \mu'\left(\phi(0,0), 24!\frac{t}{397834070447218680}\right)$$
$$*\mu'\left(\phi(x,x), \frac{24!}{2}\frac{t}{187216033151632320}\right) * \mu'\left(\phi(x,-x), \frac{24!}{2}\frac{t}{187216033151632320}\right)$$
$$*\mu'\left(\phi(2x,2x), \frac{24!}{2}\frac{t}{72806235114523680}\right)$$
$$*\mu'\left(\phi(2x,-2x), \frac{24!}{2}\frac{t}{72806235114523680}\right)$$
$$*\mu'\left(\phi(3x,3x), \frac{24!}{2}\frac{t}{22991442667744320}\right)$$

$$
\begin{aligned}
&*\mu'\left(\phi(3x,-3x),\frac{24!}{2}\frac{t}{22991442667744320}\right)\\
&*\mu'\left(\phi(4x,4x),\frac{24!}{2}\frac{t}{5747860666936080}\right)*\mu'\left(\phi(4x,-4x),\frac{24!}{2}\frac{t}{5747860666936080}\right)\\
&*\mu'\left(\phi(5x,5x),\frac{24!}{2}\frac{t}{1094830603225920}\right)*\mu'\left(\phi(5x,-5x),\frac{24!}{2}\frac{t}{1094830603225920}\right)\\
&*\mu'\left(\phi(6x,6x),\frac{24!}{2}\frac{t}{149295082258080}\right)*\mu'\left(\phi(6x,-6x),\frac{24!}{2}\frac{t}{149295082258080}\right)\\
&*\mu'\left(\phi(7x,7x),\frac{24!}{2}\frac{t}{12982181065920}\right)*\mu'\left(\phi(7x,-7x),\frac{24!}{2}\frac{t}{12982181065920}\right)\\
&*\mu'\left(\phi(8x,8x),\frac{24!}{2}\frac{t}{540924211080}\right)*\mu'\left(\phi(8x,-8x),\frac{24!}{2}\frac{t}{540924211080}\right)\\
&*\mu'\left(\phi(4x,x),\frac{t}{735480}\right), \qquad (39)
\end{aligned}
$$

$$
\begin{aligned}
\psi_{11}(x,t) := {}&\mu'\left(\phi(0,0),24!\frac{t}{2235282407279860224}\right)\\
&*\mu'\left(\phi(x,x),\frac{24!}{2}\frac{t}{1257346354094921376}\right)\\
&*\mu'\left(\phi(x,-x),\frac{24!}{2}\frac{t}{1257346354094921376}\right)\\
&*\mu'\left(\phi(2x,2x),\frac{24!}{2}\frac{t}{591692401927021824}\right)\\
&*\mu'\left(\phi(2x,-2x),\frac{24!}{2}\frac{t}{591692401927021824}\right)\\
&*\mu'\left(\phi(3x,3x),\frac{24!}{2}\frac{t}{230102600749397376}\right)\\
&*\mu'\left(\phi(3x,-3x),\frac{24!}{2}\frac{t}{230102600749397376}\right)\\
&*\mu'\left(\phi(4x,4x),\frac{24!}{2}\frac{t}{72663979184020224}\right)\\
&*\mu'\left(\phi(4x,-4x),\frac{24!}{2}\frac{t}{72663979184020224}\right)\\
&*\mu'\left(\phi(5x,5x),\frac{24!}{2}\frac{t}{18165994796005056}\right)\\
&*\mu'\left(\phi(5x,-5x),\frac{24!}{2}\frac{t}{18165994796005056}\right)\\
&*\mu'\left(\phi(6x,6x),\frac{24!}{2}\frac{t}{3460189484953344}\right)\\
&*\mu'\left(\phi(6x,-6x),\frac{24!}{2}\frac{t}{3460189484953344}\right)*\mu'\left(\phi(7x,7x),\frac{24!}{2}\frac{t}{471844020675456}\right)\\
&*\mu'\left(\phi(7x,-7x),\frac{24!}{2}\frac{t}{471844020675456}\right)*\mu'\left(\phi(8x,8x),\frac{24!}{2}\frac{t}{41029914841344}\right)
\end{aligned}
$$

$$*\mu'\left(\phi(8x,-8x),\frac{24!}{2}\frac{t}{41029914841344}\right)*\mu'\left(\phi(9x,9x),\frac{24!}{2}\frac{t}{1709579785056}\right)$$
$$*\mu'\left(\phi(9x,-9x),\frac{24!}{2}\frac{t}{1709579785056}\right)*\mu'\left(\phi(3x,x),\frac{t}{1307514}\right), \quad (40)$$

$$\psi_{12}(x,t):=\mu'\left(\phi(0,0),24!\frac{t}{7544062738469138112}\right)$$
$$*\mu'\left(\phi(x,x),\frac{24!}{2}\frac{t}{5029375158979425408}\right)$$
$$*\mu'\left(\phi(x,-x),\frac{24!}{2}\frac{t}{5029375158979425408}\right)$$
$$*\mu'\left(\phi(2x,2x),\frac{24!}{2}\frac{t}{2829023526925926792}\right)$$
$$*\mu'\left(\phi(2x,-2x),\frac{24!}{2}\frac{t}{2829023526925926792}\right)$$
$$*\mu'\left(\phi(3x,3x),\frac{24!}{2}\frac{t}{1331305189141612608}\right)$$
$$*\mu'\left(\phi(3x,-3x),\frac{24!}{2}\frac{t}{1331305189141612608}\right)$$
$$*\mu'\left(\phi(4x,4x),\frac{24!}{2}\frac{t}{517729795777293792}\right)$$
$$*\mu'\left(\phi(4x,-4x),\frac{24!}{2}\frac{t}{517729795777293792}\right)$$
$$*\mu'\left(\phi(5x,5x),\frac{24!}{2}\frac{t}{163493619719145408}\right)$$
$$*\mu'\left(\phi(5x,-5x),\frac{24!}{2}\frac{t}{163493619719145408}\right)$$
$$*\mu'\left(\phi(6x,6x),\frac{24!}{2}\frac{t}{40873404929786352}\right)$$
$$*\mu'\left(\phi(6x,-6x),\frac{24!}{2}\frac{t}{40873404929786352}\right)$$
$$*\mu'\left(\phi(7x,7x),\frac{24!}{2}\frac{t}{7785410462816448}\right)*\mu'\left(\phi(7x,-7x),\frac{24!}{2}\frac{t}{7785410462816448}\right)$$
$$*\mu'\left(\phi(8x,8x),\frac{24!}{2}\frac{t}{1061646881293152}\right)*\mu'\left(\phi(8x,-8x),\frac{24!}{2}\frac{t}{1061646881293152}\right)$$
$$*\mu'\left(\phi(9x,9x),\frac{24!}{2}\frac{t}{92317120112448}\right)*\mu'\left(\phi(9x,-9x),\frac{24!}{2}\frac{t}{92317120112448}\right)$$
$$*\mu'\left(\phi(10x,10x),\frac{24!}{2}\frac{t}{3846546671352}\right)*\mu'\left(\phi(10x,-10x),\frac{24!}{2}\frac{t}{3846546671352}\right)$$
$$*\mu'\left(\phi(2x,x),\frac{t}{1961267}\right), \quad (41)$$

$$\psi_{13}(x,t) := \mu'\left(\phi(0,0), 24!\frac{t}{15552886227004578816}\right)$$
$$*\mu'\left(\phi(x,x), \frac{24!}{2}\frac{t}{12220124892646454784}\right)$$
$$*\mu'\left(\phi(x,-x), \frac{24!}{2}\frac{t}{12220124892646454784}\right)$$
$$*\mu'\left(\phi(2x,2x), \frac{24!}{2}\frac{t}{8146749928430969856}\right)$$
$$*\mu'\left(\phi(2x,-2x), \frac{24!}{2}\frac{t}{8146749928430969856}\right)$$
$$*\mu'\left(\phi(3x,3x), \frac{24!}{2}\frac{t}{4582546834742420544}\right)$$
$$*\mu'\left(\phi(3x,-3x), \frac{24!}{2}\frac{t}{4582546834742420544}\right)$$
$$*\mu'\left(\phi(4x,4x), \frac{24!}{2}\frac{t}{2156492628114080256}\right)$$
$$*\mu'\left(\phi(4x,-4x), \frac{24!}{2}\frac{t}{2156492628114080256}\right)$$
$$*\mu'\left(\phi(5x,5x), \frac{24!}{2}\frac{t}{838636022044364544}\right)$$
$$*\mu'\left(\phi(5x,-5x), \frac{24!}{2}\frac{t}{838636022044364544}\right)$$
$$*\mu'\left(\phi(6x,6x), \frac{24!}{2}\frac{t}{264832428014009856}\right)$$
$$*\mu'\left(\phi(6x,-6x), \frac{24!}{2}\frac{t}{264832428014009856}\right)$$
$$*\mu'\left(\phi(7x,7x), \frac{24!}{2}\frac{t}{66208107003502464}\right)$$
$$*\mu'\left(\phi(7x,-7x), \frac{24!}{2}\frac{t}{66208107003502464}\right)$$
$$*\mu'\left(\phi(8x,8x), \frac{24!}{2}\frac{t}{12611068000667136}\right)$$
$$*\mu'\left(\phi(8x,-8x), \frac{24!}{2}\frac{t}{12611068000667136}\right)$$
$$*\mu'\left(\phi(9x,9x), \frac{24!}{2}\frac{t}{1719691091000064}\right) * \mu'\left(\phi(9x,-9x), \frac{24!}{2}\frac{t}{1719691091000064}\right)$$
$$*\mu'\left(\phi(10x,10x), \frac{24!}{2}\frac{t}{149538355739136}\right)$$
$$*\mu'\left(\phi(10x,-10x), \frac{24!}{2}\frac{t}{149538355739136}\right)$$
$$*\mu'\left(\phi(11x,11x), \frac{24!}{2}\frac{t}{6230764822464}\right) * \mu'\left(\phi(11x,-11x), \frac{24!}{2}\frac{t}{6230764822464}\right)$$
$$*\mu'\left(\phi(x,x), \frac{t}{2496156}\right), \tag{42}$$

$$
\begin{aligned}
\psi_{14}(x,t) := \mu'\left(\phi(0,0), 24!\frac{t}{19774126759681168784}\right)\\
*\mu'\left(\phi(x,x), \frac{24!}{2}\frac{t}{18253040085859540416}\right)\\
*\mu'\left(\phi(x,-x), \frac{24!}{2}\frac{t}{18253040085859540416}\right)\\
*\mu'\left(\phi(2x,2x), \frac{24!}{2}\frac{t}{14341674353175353184}\right)\\
*\mu'\left(\phi(2x,-2x), \frac{24!}{2}\frac{t}{14341674353175353184}\right)\\
*\mu'\left(\phi(3x,3x), \frac{24!}{2}\frac{t}{9561116235450235456}\right)\\
*\mu'\left(\phi(3x,-3x), \frac{24!}{2}\frac{t}{9561116235450235456}\right)\\
*\mu'\left(\phi(4x,4x), \frac{24!}{2}\frac{t}{5378127882440757444}\right)\\
*\mu'\left(\phi(4x,-4x), \frac{24!}{2}\frac{t}{5378127882440757444}\right)\\
*\mu'\left(\phi(5x,5x), \frac{24!}{2}\frac{t}{2530883709383885856}\right)\\
*\mu'\left(\phi(5x,-5x), \frac{24!}{2}\frac{t}{2530883709383885856}\right)\\
*\mu'\left(\phi(6x,6x), \frac{24!}{2}\frac{t}{984232553649288944}\right)\\
*\mu'\left(\phi(6x,-6x), \frac{24!}{2}\frac{t}{984232553649288944}\right)\\
*\mu'\left(\phi(7x,7x), \frac{24!}{2}\frac{t}{310810280099775456}\right)\\
*\mu'\left(\phi(7x,-7x), \frac{24!}{2}\frac{t}{310810280099775456}\right)\\
*\mu'\left(\phi(8x,8x), \frac{24!}{2}\frac{t}{77702570024943864}\right)\\
*\mu'\left(\phi(8x,-8x), \frac{24!}{2}\frac{t}{77702570024943864}\right)\\
*\mu'\left(\phi(9x,9x), \frac{24!}{2}\frac{t}{14800489528560736}\right)\\
*\mu'\left(\phi(9x,-9x), \frac{24!}{2}\frac{t}{14800489528560736}\right)\\
*\mu'\left(\phi(10x,10x), \frac{24!}{2}\frac{t}{2018248572076464}\right)\\
*\mu'\left(\phi(10x,-10x), \frac{24!}{2}\frac{t}{2018248572076464}\right)
\end{aligned}
$$

$$*\mu'\left(\phi(11x,11x),\frac{24!}{2}\frac{t}{175499875832736}\right)$$
$$*\mu'\left(\phi(11x,-11x),\frac{24!}{2}\frac{t}{175499875832736}\right)$$
$$*\mu'\left(\phi(12x,12x),\frac{24!}{2}\frac{t}{7312494826364}\right)*\mu'\left(\phi(12x,-12x),\frac{24!}{2}\frac{t}{7312494826364}\right)$$
$$*\mu'\left(\phi(0,x),\frac{t}{2704169}\right). \tag{43}$$

*The mapping $\psi'$ is defined similar to the definition of $\psi$ with this difference that $\mu'$ and $*$ in the definition of $\psi,\psi_i(i=1,\dots,14)$, are replaced by $\nu'$ and $\diamond$ anywhere respectively.*

*Proof.* Let $f_i: X\to Y(i=1,\dots,14)$, be mappings defined in Theorem 2.1. Replacing $x=y=0$ in (25), we have

$$\mu(f(0),t)\ge\mu'(\phi(0,0),24!t), \tag{44}$$

for all $t>0$. Replacing $(x,y)$ by $(x,x)$ in (25), we have

$$\mu(f_1(x),t)\ge\mu'(\phi(x,x),t), \tag{45}$$

for all $x\in X$ and $t>0$. Replacing $(x,y)$ by $(x,-x)$ in (25) and using (45), we have

$$\mu(f(x)-f(-x),t)\ge\mu'\left(\phi(x,x),\frac{24!}{2}t\right)*\mu'\left(\phi(x,-x),\frac{24!}{2}t\right), \tag{46}$$

for all $x\in X$ and $t>0$. Replacing $(x,y)$ by $(0,2x)$ in (25) and using (44) and (46), we have

$$\mu(f_2(x),t)\ge\psi_1(x,t), \tag{47}$$

for all $x\in X$ and $t>0$. Replacing $(x,y)$ by $(12x,x)$ in (25) and using (47), we have

$$\mu(f_3(x),t)\ge\psi_1\left(x,\frac{t}{2}\right)*\psi_2\left(x,\frac{t}{2}\right), \tag{48}$$

for all $x\in X$ and $t>0$. Replacing $(x,y)$ by $(11x,x)$ in (25), then it follows from (48) that

$$\mu(f_4(x),t)\ge\psi_1\left(x,\frac{t}{4}\right)*\psi_2\left(x,\frac{t}{4}\right)*\psi_3\left(x,\frac{t}{2}\right), \tag{49}$$

for all $x\in X$ and $t>0$. Replacing $(x,y)$ by $(10x,x)$ in (25) and using (49), we have

$$\mu(f_5(x),t)\ge\psi_1\left(x,\frac{t}{8}\right)*\psi_2\left(x,\frac{t}{8}\right)*\psi_3\left(x,\frac{t}{4}\right)*\psi_4\left(x,\frac{t}{2}\right), \tag{50}$$

for all $x\in X$ and $t>0$. Replacing $(x,y)$ by $(9x,x)$ in (25) and using (50), we have

$$\mu(f_6(x),t)\ge\psi_1\left(x,\frac{t}{16}\right)*\psi_2\left(x,\frac{t}{16}\right)*\psi_3\left(x,\frac{t}{8}\right)*\psi_4\left(x,\frac{t}{4}\right)$$
$$*\psi_5\left(x,\frac{t}{2}\right), \tag{51}$$

for all $x \in X$ and $t > 0$. Replacing $(x, y)$ by $(8x, x)$ in (25) and using (51), we have

$$\mu(f_7(x), t) \geq \psi_1\left(x, \frac{t}{32}\right) * \psi_2\left(x, \frac{t}{32}\right) * \psi_3\left(x, \frac{t}{16}\right) * \psi_4\left(x, \frac{t}{8}\right) * \psi_5\left(x, \frac{t}{4}\right) * \psi_6\left(x, \frac{t}{2}\right), \tag{52}$$

for all $x \in X$ and $t > 0$. Replacing $(x, y)$ by $(7x, x)$ in (25), then it follows from (52) that

$$\mu(f_8(x), t) \geq \psi_1\left(x, \frac{t}{64}\right) * \psi_2\left(x, \frac{t}{64}\right) * \psi_3\left(x, \frac{t}{32}\right) * \psi_4\left(x, \frac{t}{16}\right) * \psi_5\left(x, \frac{t}{8}\right) * \psi_6\left(x, \frac{t}{4}\right) * \psi_7\left(x, \frac{t}{2}\right), \tag{53}$$

for all $x \in X$ and $t > 0$. Replacing $(x, y)$ by $(6x, x)$ in (25) and using (53), we have

$$\mu(f_9(x), t) \geq \psi_1\left(x, \frac{t}{128}\right) * \psi_2\left(x, \frac{t}{128}\right) * \psi_3\left(x, \frac{t}{64}\right) * \psi_4\left(x, \frac{t}{32}\right) * \psi_5\left(x, \frac{t}{16}\right) * \psi_6\left(x, \frac{t}{8}\right) * \psi_7\left(x, \frac{t}{4}\right) * \psi_8\left(x, \frac{t}{2}\right), \tag{54}$$

for all $x \in X$ and $t > 0$. Replacing $(x, y)$ by $(5x, x)$ in (25) and using (54), we have

$$\mu(f_{10}(x), t) \geq \psi_1\left(x, \frac{t}{256}\right) * \psi_2\left(x, \frac{t}{256}\right) * \psi_3\left(x, \frac{t}{128}\right) * \psi_4\left(x, \frac{t}{64}\right) * \psi_5\left(x, \frac{t}{32}\right) * \psi_6\left(x, \frac{t}{16}\right) * \psi_7\left(x, \frac{t}{8}\right) * \psi_8\left(x, \frac{t}{4}\right) * \psi_9\left(x, \frac{t}{2}\right), \tag{55}$$

for all $x \in X$ and $t > 0$. Replacing $(x, y)$ by $(4x, x)$ in (25), then it follows from (55) that

$$\mu(f_{11}(x), t) \geq \psi_1\left(x, \frac{t}{512}\right) * \psi_2\left(x, \frac{t}{512}\right) * \psi_3\left(x, \frac{t}{256}\right) * \psi_4\left(x, \frac{t}{128}\right) * \psi_5\left(x, \frac{t}{64}\right) * \psi_6\left(x, \frac{t}{32}\right) * \psi_7\left(x, \frac{t}{16}\right) * \psi_8\left(x, \frac{t}{8}\right) * \psi_9\left(x, \frac{t}{4}\right) * \psi_{10}\left(x, \frac{t}{2}\right), \tag{56}$$

for all $x \in X$ and $t > 0$. Replacing $(x, y)$ by $(3x, x)$ in (25) and using (56), we have

$$\mu(f_{12}(x), t) \geq \psi_1\left(x, \frac{t}{1024}\right) * \psi_2\left(x, \frac{t}{1024}\right) * \psi_3\left(x, \frac{t}{512}\right) * \psi_4\left(x, \frac{t}{256}\right) \psi_5\left(x, \frac{t}{128}\right) * \psi_6\left(x, \frac{t}{64}\right) * \psi_7\left(x, \frac{t}{32}\right) * \psi_8\left(x, \frac{t}{16}\right) * \psi_9\left(x, \frac{t}{8}\right) * \psi_{10}\left(x, \frac{t}{4}\right) * \psi_{11}\left(x, \frac{t}{2}\right), \tag{57}$$

for all $x \in X$ and $t > 0$. Replacing $(x, y)$ by $(2x, x)$ in (25) and using (57), we have

$$\begin{aligned}\mu(f_{13}(x), t) \geq \psi_1\left(x, \frac{t}{2048}\right) * \psi_2\left(x, \frac{t}{2048}\right) * \psi_3\left(x, \frac{t}{1024}\right) * \psi_4\left(x, \frac{t}{512}\right)\\ *\psi_5\left(x, \frac{t}{256}\right) * \psi_6\left(x, \frac{t}{128}\right) * \psi_7\left(x, \frac{t}{64}\right) * \psi_8\left(x, \frac{t}{32}\right) * \psi_9\left(x, \frac{t}{16}\right)\\ *\psi_{10}\left(x, \frac{t}{8}\right) * \psi_{11}\left(x, \frac{t}{4}\right) * \psi_{12}\left(x, \frac{t}{2}\right),\end{aligned} \tag{58}$$

for all $x \in X$ and $t > 0$. Replacing $(x, y)$ by $(x, x)$ in (25) and using (58), we have

$$\begin{aligned}\mu(f_{14}(x), t) \geq \psi_1\left(x, \frac{t}{4096}\right) * \psi_2\left(x, \frac{t}{4096}\right) * \psi_3\left(x, \frac{t}{2048}\right) * \psi_4\left(x, \frac{t}{1024}\right)\\ *\psi_5\left(x, \frac{t}{512}\right) * \psi_6\left(x, \frac{t}{256}\right) * \psi_7\left(x, \frac{t}{128}\right) * \psi_8\left(x, \frac{t}{64}\right) * \psi_9\left(x, \frac{t}{32}\right)\\ *\psi_{10}\left(x, \frac{t}{16}\right) * \psi_{11}\left(x, \frac{t}{8}\right) * \psi_{12}\left(x, \frac{t}{4}\right) * \psi_{13}\left(x, \frac{t}{2}\right),\end{aligned} \tag{59}$$

for all $x \in X$ and $t > 0$. Replacing $(x, y)$ by $(0, x)$ in (25) and using (59), we have

$$\mu\left(f(2x) - 2^{24} f(x), \frac{t}{155112100433309859840000}\right) \geq \psi(x, t), \tag{60}$$

for all $x \in X$ and $t > 0$. Similarly, we have

$$\nu\left(f(2x) - 2^{24} f(x), \frac{t}{155112100433309859840000}\right) \leq \psi'(x, t), \tag{61}$$

for all $x \in X$ and $t > 0$. By Lemma 3.1, there exists a mapping $Q : X \to Y$, such that

$$Q(x) := \lim_{n\to\infty} \frac{f(2^n x)}{2^{24n}},$$

and (27) and (28) are satisfied.
Replacing $x$ and $y$ by $2^n x$ and $2^n y$ in (25), we get

$$\mu\left(\frac{1}{2^{24n}} Df(2^n x, 2^n y), \frac{t}{2^{24n}}\right) \geq \mu'(\phi(2^n x, 2^n y), t), \tag{62}$$

for all $x, y \in X$ and $t > 0$. Similarly

$$\nu\left(\frac{1}{2^{24n}} Df(2^n x, 2^n y), \frac{t}{2^{24n}}\right) \leq \nu'(\phi(2^n x, 2^n y), t), \tag{63}$$

for all $x, y \in X$ and $t > 0$. Therefore, the mapping $Q : X \to Y$ is a quattuorvigintic mapping. The uniqueness assertion is easy and therefore we omit its pertinent proof. □

**Corollary 3.3.** *Fix $\alpha > 0$. Let $\mu', \nu'$ be fuzzy sets on $\mathbb{R} \times (0, \infty)$, such that*

$$\mu'(x,t) := \frac{t}{t+|x|},$$

$$\upsilon'(x,t) := \frac{|x|}{t+|x|}.$$

*Let $f : X \to Y$ be a mapping satisfying*

$$\mu(Df(x,y),t) \geq \mu'(\alpha,t), \tag{64}$$

$$\nu(Df(x,y),t) \leq \nu'(\alpha,t), \tag{65}$$

*for all $x, y \in X$ and $t > 0$.*

*Then there exists a unique quattuorvigintic mapping $Q : X \to Y$, such that*

$$\mu(f(x) - Q(x),t) \geq \psi(x,(2^{24}-1)155112100433309859840000t), \tag{66}$$

$$\nu(f(x) - Q(x),t) \leq \psi'(x,(2^{24}-1)155112100433309859840000t), \tag{67}$$

*for all $x \in X$ and $t > 0$, where*

$$\begin{aligned}\psi(x,t) := \psi_1\left(x, \frac{t}{4096}\right) * \psi_2\left(x, \frac{t}{4096}\right) * \psi_3\left(x, \frac{t}{2048}\right) * \psi_4\left(x, \frac{t}{1024}\right) \\ *\psi_5\left(x, \frac{t}{512}\right) * \psi_6\left(x, \frac{t}{256}\right) * \psi_7\left(x, \frac{t}{128}\right) * \psi_8\left(x, \frac{t}{64}\right) * \psi_9\left(x, \frac{t}{32}\right) \\ *\psi_{10}\left(x, \frac{t}{16}\right) * \psi_{11}\left(x, \frac{t}{8}\right) * \psi_{12}\left(x, \frac{t}{4}\right) * \psi_{13}\left(x, \frac{t}{2}\right) * \psi_{14}(x,t),\end{aligned} \tag{68}$$

*and the mappings $\psi_i : X \times (0, \infty) \to [0,1] (i = 1, \ldots, 14)$, are defined by the relations (30),...,(43) with $\phi(x,y) = \alpha$ for all $x, y \in X$. The mapping $\psi'$ is defined similar to the definition of $\psi$ with this difference that $\mu'$ and $*$ in the definition of $\psi, \psi_i (i = 1, \ldots, 14)$, are replaced by $\nu'$ and $\diamond$ anywhere respectively.*

*Proof.* Let $Z = \mathbb{R}$, $a * b = ab$ and $a \diamond b = min\{a+b, 1\}$ and $\phi(x,y) = \alpha$ for all $x, y \in X$ and $L = 1$ in Theorem 3.2, we get the desired result. □

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