FURTHER RESULTS OF CONVERGENCE OF UNCERTAIN RANDOM SEQUENCES

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Abstract. Convergence is an issue being widely concerned about. Thus, in this paper, we mainly put forward two types of concepts of convergence in mean and convergence in distribution for the sequence of uncertain random variables. Then some of theorems are proved to show the relations among the three convergence concepts that are convergence in mean, convergence in measure and convergence in distribution. Furthermore, several examples are given to illustrate how we use the theorems to make sure the uncertain random sequence being convergent. Finally, several counterexamples are taken to explain the relations between these different types of convergence.

1. **Introduction**

The world is always filled with indeterminacy, usually one type is random phenomenon and the other is non-random phenomenon. For randomness associated the frequency, it can be analyzed by probability theory which is a branch of mathematics on the basis of normality axiom, duality axiom, countable additivity axiom and product theorem. A premise of applying probability theory is that the long run cumulative frequency probability distribution is closed enough to the true frequency. If we want to obtain cumulative probability distribution, we should possess lots of history data. While, history data are not always being there. For example, we cannot obtain the data of bearing capacity of a bridge being used. That is, there exist a lot of vague phenomena that do not behave with randomness. In this case, we should invite some domain experts to estimate the chance that possible events may appear. Nobelists Kahneman and Tversky [8] asserted that human beings tend to overweight unlikely events, so the estimated possibility generally has a much larger range than the real frequency. Hence we cannot use probability to model this phenomena.

As a tool that attempts to deal with non-random phenomena, Zadeh [24] introduced the concept of fuzzy set via membership function built on the normality, nonnegativity, and maximality axioms. Also, Zadeh [25] proposed the concept of possibility measure for measuring a fuzzy event. And he also provided a formula to calculate the union of fuzzy events, that is, the possibility measure of the union of fuzzy events is the maximum value of possibility measure of every fuzzy event no matter if they are independent. In order to describe the phenomena including both

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fuzziness and randomness, Kwakernaak [9, 10] proposed the concept of fuzzy random variable. As a different form of Kwakernaak's fuzzy random variable, the new concept is proposed by Liu [11, 12] via possibility measure. In order to study convergence theorems of fuzzy random variables, Cheng and Liu [3] introduced some convergence concepts for sequence of fuzzy random variables and study relations among these convergence concepts. Liu et al. [20] proved the convergence theorems for sequences of integrable fuzzy random variables such as dominated convergence theorem and bounded convergence theorem.

On the other hand, as a revision of fuzzy set theory, Liu [13] established uncertainty theory as a branch of axiomatic mathematics for modeling belief degrees to deal with imprecise phenomena as same as fuzzy set theory. On the other hand, Liu [13] introduced an uncertain measure as a set function satisfying normality, duality, subadditivity. Then Liu [14] proposed product axioms. And Liu [13] also given the value of uncertain measure of the union of uncertain events, that is, the uncertain measure of uncertain events is the maxim measure of every uncertain event, which only holds for independent ones. As a basic concept of uncertainty theory, Liu [13] presented the uncertain variable. For describing an uncertain variable, the concept of uncertainty distribution is introduced by Liu [13]. The concept of inverse uncertainty distribution is proposed by Liu[15]. After that, many scholars devoted their studies to the uncertainty theory and made significant progress. In addition, Liu [13] introduced the concept of independence of uncertain variables. After, the operational law of uncertain variables is presented by Liu [13]. In order to rank uncertain variables, Liu [13] proposed the concept of expected value of uncertain variables. The linearity of expected value operator was verified by Liu [15]. As an important contribution, Liu and Ha [19] derived a useful formula for calculating the expected values of strictly monotone functions of independent uncertain variables. Meanwhile, Liu [13] presented the concept of variance of uncertain variables. Furthermore, several authors proposed some formulas to calculate the variance through uncertainty distribution for more details see [16, 22].

As a mixture of uncertainty and randomness, Liu [17] first proposed chance theory, which is a mathematical methodology for modeling complex systems with both uncertainty and randomness, including chance measure, uncertain random variable, chance distribution, expected value, variance and so on. Following that, the operational law of uncertain random variable is presented by Liu [18]. Guo and Wang [7] proposed a formula for calculating the variance of uncertain random variables based on uncertainty distribution. Sheng and Yao [21] proved a formula to calculate the variance via chance distribution and inverse chance distribution. Ahmadzade et al. [2] established some results of moments of uncertain random variables. After that, many scholars devoted their studies to the uncertainty theory and made significant progress, such as [5, 6].

As an important role of convergence theorems in chance theory, several authors devoted their studies to this field. For instance, Yao and Gao [23] proved a law of large numbers for uncertain random variables with a common chance distribution. Gao and Sheng [4] proved another law of large numbers for uncertain random variables with different chance distributions. Ahmadzade et al. [1] established several convergence theorems for uncertain random sequences. In this paper, we introduce two concepts of convergence of uncertain random variables such as convergence in mean and convergence in distribution. Furthermore, relations among convergence concepts are explained by several examples.

The rest of this paper is organized as follows. Section 2 presents some basic concepts and theorems about chance theory including some concepts of uncertain random variables, expected value, variance and so on. Some convergence concepts of uncertain random variable is introduced in Section 3. Furthermore, relation among convergence concepts are explained by several theorems and examples in Section 4. Finally, some conclusions are given in Section 5.

2. **Preliminaries**

In this section, we review some concepts of uncertainty theory and chance theory, including uncertain measure, uncertain variable, uncertainty distribution, operational law and expected value, uncertain random variable, chance distribution and so on.

2.1. **Uncertainty Theory.** This subsection recalls some concepts of uncertainty theory, uncertain measure, uncertainty distribution and operational law.

Definition 2.1. (Liu [13]) Let $\mathcal{L}$ be a σ-algebra on a nonempty set Γ. A set function $\mathcal{M} : \mathcal{L} \to [0, 1]$ is called an uncertain measure if it satisfies the following axioms:
Axiom 1: (Normality Axiom) $\mathcal{M}\{\Gamma\} = 1$ for the universal set Γ.
Axiom 2: (Duality Axiom) $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ for any event Λ.
Axiom 3: (Subadditivity Axiom) For every countable sequence of events $\Lambda_1, \Lambda_2, \cdots$, we have

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} \le \sum_{i=1}^{\infty} \mathcal{M}\left\{\Lambda_i\right\}.$$

Then we call $\{\Gamma, \mathcal{L}, \mathcal{M}\}$ an uncertain space.

Besides, the product uncertain measure on the product σ-algebra $\mathcal{L}$ is defined by the following product axiom.
Axiom 4: (Product Axiom) (Liu [15]*) Let* $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ *be uncertainty spaces for* $k = 1, 2, \cdots$ *. The product uncertain measure* $\wedge$ *is an uncertain measure satisfying*

$$\mathcal{M}\left\{\prod_{k=1}^{\infty} \Lambda_k\right\} = \bigwedge_{k=1}^{\infty} \mathcal{M}_k\{\Lambda_k\}$$

where Λ_k *are arbitrarily chosen events from* $\mathcal{L}_k$ *for* $k = 1, 2, \cdots$*, respectively.*

Roughly speaking, an uncertain variable is a measurable function from an uncertainty space to the set of real numbers.

Definition 2.2. (Liu [13]) An uncertain variable ξ is a measurable function from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers, *i.e.*, for any Borel set B of real numbers, the set

$$\{\xi \in B\} = \{\gamma \in \Gamma | \xi(\gamma) \in B\}$$

is an event.

In order to describe an uncertain variable, a concept of uncertainty distribution is defined as follows.

Definition 2.3. (Liu [13]) The uncertainty distribution of an uncertain variable ξ is defined by

$$\Phi(x) = \mathcal{M}\{\xi \leq x\}$$

for any $x \in \Re$.

Definition 2.4. (Liu [14]) The uncertain variables $\xi_1, \xi_2, \cdots, \xi_n$ are said to be independent if

$$\mathcal{M}\left\{\bigcap_{i=1}^{n}(\xi_i \in B_i)\right\} = \bigwedge_{i=1}^{n} \mathcal{M}\{\xi_i \in B_i\}$$

for any Borel sets $B_1, B_2, \cdots, B_n$ of real numbers.

Definition 2.5. (Liu [13]) Let ξ be an uncertain variable with regular uncertainty distribution $\Phi(x)$. Then the inverse function $\Phi^{-1}(\alpha)$ is called the inverse uncertainty distribution of ξ.

The distribution of a monotonous function of uncertain variables can be obtained by the following theorem.

Theorem 2.6. *(Liu* [15]*) Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent uncertain variables with uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively. If $f(\xi_1, \xi_2, \cdots, \xi_n)$ is strictly increasing with respect to $\xi_1, \xi_2, \cdots, \xi_m$ and strictly decreasing with respect to $\xi_{m+1}, \xi_{m+2}, \cdots, \xi_n$, then $\xi = f(\xi_1, \xi_2, \cdots, \xi_n)$ is an uncertain variable with an inverse uncertainty distribution*

$$\Phi^{-1}(\alpha) = f(\Phi_1^{-1}(\alpha), \cdots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \cdots, \Phi_n^{-1}(1-\alpha)).$$

Definition 2.7. (Liu [13]) The expected value of an uncertain variable ξ is defined by

$$E[\xi] = \int_0^{+\infty} \mathcal{M}\{\xi \geq x\}\mathrm{d}x - \int_{-\infty}^{0} \mathcal{M}\{\xi \leq x\}\mathrm{d}x$$

provided that at least one of the two integrals is finite.

Theorem 2.8. *(Liu* [13]*) Let ξ be an uncertain variable with uncertainty distribution $\Phi(x)$. If the expected value exists, then*

$$E[\xi] = \int_0^{+\infty} (1 - \Phi(x))\mathrm{d}x - \int_{-\infty}^{0} \Phi(x)\mathrm{d}x.$$

Liu and Ha [19] *proposed a generalized formula for expected value by inverse uncertainty distribution.*

Theorem 2.9. *(Liu and Ha* [19]*) Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively. If $f(\xi_1, \xi_2, \cdots, \xi_n)$ is strictly increasing with respect to $\xi_1, \xi_2, \cdots, \xi_m$ and strictly decreasing* *with respect to* $\xi_{m+1}, \xi_{m+2}, \cdots, \xi_n$*, then the uncertain variable* $\xi = f(\xi_1, \xi_2, \cdots, \xi_n)$ *has an expected value*

$$E[\xi] = \int_0^1 f(\Phi_1^{-1}(\alpha), \cdots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \cdots, \Phi_n^{-1}(1-\alpha))\mathrm{d}\alpha.$$

2.2. **Chance Theory.** *In this subsection, we review some fundamental concepts of chance theory, including chance measure, uncertain random variable, chance distribution, operational law, and expected value, and variance and so on.*

The chance space is refer to the product $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, \Pr)$*, in which* $(\Gamma, \mathcal{L}, \mathcal{M})$ *is an uncertainty space and* $(\Omega, \mathcal{A}, \Pr)$ *is a probability space.*

Definition 2.10. (Liu [17]) Let $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, \Pr)$ be a chance space, and let $\Theta \in \mathcal{L} \times \mathcal{A}$ be an uncertain random event. Then the chance measure of Θ is defined as

$$\mathrm{Ch}\{\Theta\} = \int_0^1 \Pr\{\omega \in \Omega \mid \mathcal{M}\{\gamma \in \Gamma | (\gamma, \omega) \in \Theta\} \geq r\}\mathrm{d}r.$$

Liu [17] *proved that a chance measure satisfies normality, duality, and monotonicity properties, that is*
(i) $\mathrm{Ch}\{\Gamma \times \Omega\} = 1$ *for the universal set* $\Gamma \times \Omega$*;*
(ii) $\mathrm{Ch}\{\Theta\} + \mathrm{Ch}\{\Theta^c\} = 1$ *for any event* Θ*;*
(iii) $\mathrm{Ch}\{\Theta_1\} \leq \mathrm{Ch}\{\Theta_2\}$ *for any real number set* $\Theta_1 \subset \Theta_2$.

Definition 2.11. (Liu [17]) An uncertain random variable is a measurable function ξ from a chance space $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, \Pr)$ to the set of real numbers, i.e., the set

$$\{(\gamma, \omega) \mid \xi(\gamma, \omega) \in B\}$$

is an event for any Borel set B.

Theorem 2.12. *(Liu* [17]*) Let* $f : R^n \to R$ *be a measurable function, and let* $\xi_1, \cdots, \xi_n$ *be uncertain random variables on the chance space* $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, \Pr)$. *Then* $\xi = f(\xi_1, \xi_2, \cdots, \xi_n)$ *is an uncertain random variable determined by*

$$\xi(\gamma, \omega) = f(\xi_1(\gamma, \omega), \xi_2(\gamma, \omega), \cdots, \xi_n(\gamma, \omega))$$

for all $(\gamma, \omega) \in \Gamma \times \Omega$.

Theorem 2.13. *(Liu* [17]*) Let* $\eta_1, \eta_2, \cdots, \eta_m$ *be independent random variables with probability distributions* $\Psi_1, \Psi_2, \cdots, \Psi_m$*, respectively, and let* $\tau_1, \tau_2, \cdots, \tau_n$ *be uncertain variables. Then the uncertain random variable*

$$\xi = f(\eta_1, \eta_2, \cdots, \eta_m, \tau_1, \tau_2, \cdots, \tau_n)$$

has a chance distribution

$$\Phi(x) = \int_{\Re^m} F(x, y_1, \cdots, y_m)\mathrm{d}\Psi_1(y_1) \cdots \mathrm{d}\Psi_m(y_m)$$

where $F(x, y_1, \cdots, y_m)$ *is the uncertainty distribution of uncertain variable*

$$f(\eta_1, \eta_2, \cdots, \eta_m, \tau_1, \tau_2, \cdots, \tau_n)$$

for any real numbers $y_1, y_2, \cdots, y_m$.

Definition 2.14. (Liu [17]) Let ξ be an uncertain random variable. Then its expected value is defined by

$$E[\xi] = \int_0^{+\infty} \mathrm{Ch}\{\xi \geq r\}\mathrm{d}r - \int_{-\infty}^0 \mathrm{Ch}\{\xi \leq r\}\mathrm{d}r$$

provided that at least one of the two integrals is finite.

Theorem 2.15. *(Liu [17]) Let $\eta_1, \eta_2, \cdots, \eta_m$ be independent random variables with probability distributions $\Psi_1, \Psi_2, \cdots, \Psi_m$, respectively, and let $\tau_1, \tau_2, \cdots, \tau_n$ be uncertain variables (not necessarily independent), then the uncertain random variable*

$$\xi = f(\eta_1, \eta_2, \cdots, \eta_m, \tau_1, \tau_2, \cdots, \tau_n)$$

has an expected value

$$E[\xi] = \int_{\mathbb{R}^m} E[f(y_1, \cdots, y_m, \tau_1, \cdots, \tau_n)] d\Psi_1(y_1) \cdots d\Psi_m(y_m)$$

where $E[f(y_1, \cdots, y_m, \tau_1, \cdots, \tau_n)]$ is the expected value of the uncertain variable

$$f(y_1, \cdots, y_m, \tau_1, \cdots, \tau_n)$$

for any real numbers $y_1, \cdots, y_m$.

For a random variable η and an uncertain variable τ, Liu [17] proved that $E[\eta + \tau] = E[\eta] + E[\tau]$ and $E[\eta \times \tau] = E[\eta] \times E[\tau]$.

Theorem 2.16. *(Liu [17]) Let ξ be an uncertain random variable, and f a non-negative function. If f is even and increasing on $[0, \infty)$; then for any given number $\epsilon > 0$, we have*

$$\mathrm{Ch}\{|\xi| > \epsilon\} \leq \frac{E[f(\xi)]}{f(\epsilon)}.$$

In order to establish some of convergence theorems, Ahmadzade et al. [1] introduced two types of convergence concepts.

Definition 2.17. (Ahamdzade et al. [1]) An uncertain random sequence $\{\xi_i, i \geq 1\}$ is said to be almost surely convergent ($a.s.$) to an uncertain random variable ξ if there exists an event Θ with $\mathrm{Ch}\{\Theta\} = 1$ such that

$$\lim_{i \to \infty} |\xi_i(\gamma, \omega) - \xi(\gamma, \omega)| = 0$$

for every $(\gamma, \omega) \in \Theta$.

Remark 2.18. When the uncertain random sequence degenerates to the uncertain sequence, the concept of almost sure convergence also holds. That is, almost sure convergence for an uncertain sequence is a special case of almost sure convergence for an uncertain random sequence.

Definition 2.19. (Ahmadzade et al. [1]) An uncertain random sequence $\{\xi_i, i \geq 1\}$ is said to be convergent in measure to an uncertain random variable ξ if

$$\lim_{i \to \infty} \mathrm{Ch}\{(\gamma, \omega) \in \Theta ||\xi_i(\gamma, \omega) - \xi(\gamma, \omega)| \geq \epsilon\} = 0$$

for any $\epsilon > 0$.

Remark 2.20. When the uncertain random sequence degenerates to the uncertain sequence, the concept of convergence in measure also holds. That is, convergence in measure for an uncertain sequence is a special case of convergence in measure for an uncertain random sequence.

3. Relations Among Convergence Concepts

In order to study the properties of uncertain random sequences, we introduce the following concepts of convergence:

Definition 3.1. An uncertain random sequence $\{\xi_i, i \geq 1\}$ is said to be convergent in mean to an uncertain random variable ξ if

$$\lim_{i\to\infty} E[|\xi_i - \xi|] = 0.$$

Remark 3.2. When the uncertain random sequence degenerates to the uncertain sequence, the concept of convergence in mean also holds. That is, convergence in mean for an uncertain sequence is a special case of convergence in mean for an uncertain random sequence.

Definition 3.3. Let $\Phi, \Phi_1, \Phi_2, \cdots$ be the chance distributions of uncertain random variables $\xi, \xi_1, \xi_2, \cdots$, respectively. The uncertain random sequence $\{\xi_i, i \geq 1\}$ is said to be convergent in distribution to the uncertain random variable ξ, if

$$\lim_{i\to\infty} \Phi_i(x) = \Phi(x),$$

for all x at which $\Phi(x)$ is continuous.

Remark 3.4. When the uncertain random sequence degenerates to the uncertain sequence, the concept of convergence in distribution also holds. That is, convergence in distribution for an uncertain sequence is a special case of convergence in distribution for an uncertain random sequence.

Theorem 3.5. *If the uncertain sequence $\{\xi_i, i \geq 1\}$ converges in mean to ξ, then $\{\xi_i, i \geq 1\}$ converges in measure to ξ.*

Proof. By invoking Theorem 2.16, we obtain

$$\mathrm{Ch}\{|\xi_i - \xi| > \epsilon\} \leq \frac{E[|\xi_i - \xi|]}{\epsilon} \to 0, \ i \to \infty,$$

which implies that $\{\xi_i, i \geq 1\}$ converges in measure to ξ. $\square$

The following example satisfies the above theorem.

Example 3.6. Let $\{\eta_i, i \geq 1\}$ be a sequence of random variables defined by

$$\Pr(\eta_i = 0) = 1 - \frac{1}{i^\alpha}, \ \Pr(\eta_i = i) = \frac{1}{i^\alpha}, \ \alpha > 1$$

Take an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be with $\{\gamma_1, \gamma_2, \cdots\}$, where

$$\mathcal{M}\{\Lambda\} = \begin{cases} \sup\limits_{\gamma_i \in \Lambda} 1/i, & \text{if } \sup_{\gamma_i \in \Lambda} 1/i < 0.5, \\ 1 - \sup\limits_{\gamma_i \notin \Lambda} 1/i, & \text{if } \sup_{\gamma_i \notin \Lambda} 1/i < 0.5, \\ 0.5, & \textit{otherwise}. \end{cases}$$

The uncertain variables are defined by

$$\tau_i(\gamma_j) = \begin{cases} i, & \text{if } i = j, \\ 0, & otherwise \end{cases}$$

for $i = 1, 2, \cdots$ and consider $\xi_i = \tau_i \eta_i$. For some small number $\epsilon > 0$ and $\xi \equiv 0$, we have

$$\begin{aligned} \mathrm{Ch}\{|\xi_i - \xi| > \epsilon\} &= E_{\Pr}[\mathcal{M}\{|\xi_i - \xi| > \epsilon\}] \\ &= E_{\Pr}[\mathcal{M}\{|\eta_i \tau_i| > \epsilon\}] \\ &= \frac{1}{i} \Pr(\eta_i = i) \\ &= \frac{1}{i} \frac{1}{i^\alpha} \\ &= \frac{1}{i^{\alpha+1}} \to 0, \ as \ i \to \infty. \end{aligned}$$

That is, $\{\xi_i, i \geq 1\}$ converges in measure to ξ. On the other hand,

$$E[|\xi_i - \xi|] = E[\tau_i]E[\eta_i] = 1 \times \frac{1}{i^{\alpha-1}} \to 0, \ as \ i \to \infty.$$

That is, the sequence $\{\xi_i, i \geq 1\}$ converges in mean to ξ.

The following example is a counterexample showing that the converse of the Theorem 3.5 is not true in general.

Example 3.7. Suppose that $(\Omega, \mathcal{A}, \Pr)$ is probability space on the interval $[0, 1]$ with Borel algebra and Lebesgue measure. The random variables are considered as follows:

$$\eta_i(\omega) = \begin{cases} 0, & \text{if } \frac{1}{i} < \omega < 1, \\ e^i, & \text{if } 0 < \omega < \frac{1}{i}, \end{cases}$$

Take an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be with $\{\gamma_1, \gamma_2, \cdots\}$ and

$$\mathcal{M}\{\Lambda\} = \begin{cases} \sup\limits_{\gamma_i \in \Lambda} 1/i, & \text{if } \sup_{\gamma_i \in \Lambda} 1/i < 0.5, \\ 1 - \sup\limits_{\gamma_i \notin \Lambda} 1/i, & \text{if } \sup_{\gamma_i \notin \Lambda} 1/i < 0.5, \\ 0.5, & otherwise. \end{cases}$$

The uncertain variables are defined by

$$\tau_i(\gamma_j) = \begin{cases} i, & \text{if } i = j, \\ 0, & otherwise. \end{cases}$$

Then we have

$$\begin{aligned} \mathrm{Ch}\{|\xi_i - \xi| > \epsilon\} &= E_{\Pr}[\mathcal{M}\{|\xi_i - \xi| > \epsilon\}] \\ &= \Pr(\eta_i = e^i)\mathcal{M}\{\tau_i = i\} \\ &= \frac{1}{i} \times \frac{1}{i} = \frac{1}{i^2} \to 0. \end{aligned}$$

This means the sequence $\{\xi_i, i \geq 1\}$ converges in measure to ξ. However,

$$E[|\xi_i - \xi|] = E[\tau_i]E[\eta_i] = 1 \times \frac{e^i}{i} \to \infty,\ as\ i \to \infty.$$

That is, the sequence $\{\xi_i, i \geq 1\}$ does not converge in mean to ξ.

Theorem 3.8. *If the uncertain random sequence $\{\xi_i, i \geq 1\}$ converges in measure to ξ, then $\{\xi_i, i \geq 1\}$ converges in distribution to ξ.*

Proof. Let x be a given continuity point of the chance distribution of uncertain random variable ξ. It is easy to see that

$$\begin{aligned}\{\xi_i \leq x\} &= \{\xi_i \leq x, \xi \leq y\} \cup \{\xi_i \leq x, \xi > y\}\\ &\subset \{\xi \leq y\} \cup \{|\xi - \xi| \geq y - x\},\ \forall\ y > x.\end{aligned}$$

Monotonicity and subadditivity of chance measure imply that

$$\mathrm{Ch}\{\xi_i \leq x\} \leq \mathrm{Ch}\{\xi \leq y\} + \mathrm{Ch}\{|\xi_i - \xi| > y - x\},\ \forall\ y > x.$$

Convergence in measure implies that

$$\mathrm{Ch}\{|\xi_i - \xi| > y - x\} \to 0,\ as\ i \to \infty.$$

Thus, we have

$$\limsup_{i\to\infty} \mathrm{Ch}\{\xi_i \leq x\} \leq \mathrm{Ch}\{\xi \leq y\},\ \forall\ y > x.$$

Finally, we have

$$\limsup_{i\to\infty} \mathrm{Ch}\{\xi_i \leq x\} \leq \mathrm{Ch}\{\xi \leq x\},\ as\ y \to x. \tag{1}$$

On the other hand, it is clear that

$$\begin{aligned}\{\xi \leq z\} &= \{\xi \leq z, \xi_i \leq x\} \cup \{\xi \leq z, \xi_i > x\}\\ &\subset \{\xi_i \leq x\} \cup \{|\xi_i - \xi| \geq x - z\},\ \forall\ x > z,\end{aligned}$$

which implies that

$$\mathrm{Ch}\{\xi \leq z\} \leq \mathrm{Ch}\{\xi_i \leq x\} + \mathrm{Ch}\{|\xi_i - \xi| > x - z\},\ \forall\ x > z.$$

Convergence in measure implies that

$$\mathrm{Ch}\{|\xi_i - \xi| > x - z\} \to 0,\ as\ i \to \infty.$$

Thus, we obtain

$$\mathrm{Ch}\{\xi \leq z\} \leq \liminf_{i\to\infty} \mathrm{Ch}\{\xi_i \leq x\},\ \forall\ x > z.$$

Letting $z \to x$,

$$\mathrm{Ch}\{\xi \leq x\} \leq \liminf_{i\to\infty} \mathrm{Ch}\{\xi_i \leq x\}. \tag{2}$$

Equations (1) and (2) imply that

$$\mathrm{Ch}\{\xi \leq x\} \leq \liminf_{i\to\infty} \mathrm{Ch}\{\xi_i \leq x\} \leq \limsup_{i\to\infty} \mathrm{Ch}\{\xi_i \leq x\} \leq \mathrm{Ch}\{\xi \leq x\}.$$

Hence we have

$$\lim_{i\to\infty} \mathrm{Ch}\{\xi_i \leq x\} = \mathrm{Ch}\{\xi \leq x\}.$$

The proof is finished. □

The following example is an evidence of Theorem 3.8.

Example 3.9. Let $\{\eta_i, i \geq 1\}$ be a sequence of random variables defined by

$$\Pr(\eta_i = 0) = 1 - \frac{1}{i},\ \Pr(\eta_i = 1) = \frac{1}{i}.$$

Take an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $\{\gamma_1, \gamma_2\}$ with

$$\mathcal{M}\{\gamma_1\} = \frac{1}{2},\ \mathcal{M}\{\gamma_2\} = \frac{1}{2}.$$

We define an uncertain variable as

$$\tau(\gamma) = \begin{cases} -1, & \text{if } \gamma = \gamma_1, \\ 1, & \text{if } \gamma = \gamma_2. \end{cases}$$

We also define $\tau_i = -\tau$, $\xi_i = \eta_i \tau_i$ and $\xi \equiv 0$. It is easy to see that

$$\mathrm{Ch}\{\xi_i \leq x\} = \begin{cases} 0, & \text{if } x < -1, \\ \dfrac{1}{2i}, & \text{if } -1 \leq x < 0, \\ 1 - \dfrac{1}{2i}, & \text{if } 0 \leq x < 1, \\ 1, & \text{if } x \geq 1. \end{cases}$$

Therefore,

$$\lim_{i \to \infty} \mathrm{Ch}\{\xi_i \leq x\} = \begin{cases} 0, & \text{if } x < 0, \\ 1, & \text{if } x \geq 0, \end{cases}$$

i.e., $\{\xi_i, i \geq 1\}$ converges in distribution to ξ. On the other hand,

$$\begin{aligned} \mathrm{Ch}\{|\xi_i - \xi| > \epsilon\} &= E_{\Pr}[\mathcal{M}\{|\xi_i - \xi| > \epsilon\}] \\ &= \Pr(\eta_i = 1)\mathcal{M}\{\gamma_1\} + P(\eta_i = 1)\mathcal{M}\{\gamma_2\} \\ &= \frac{1}{2} \times \frac{1}{i} + \frac{1}{2} \times \frac{1}{i} \\ &= \frac{1}{i} \to 0, as\ i \to \infty. \end{aligned}$$

That is, the sequence $\{\xi_i, i \geq 1\}$ converges in mean to ξ.

Following example is a counterexample showing that the converse of the Theorem 3.8 is not true in general.

Example 3.10. Consider a symmetric coin with $\eta = 1$ for heads and $\eta = 0$ for tails, and let $\eta_i = 1 - \eta,\ i \geq 1$. Take an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $\{\gamma_1, \gamma_2\}$ with $\mathcal{M}\{\gamma_1\} = \frac{1}{2}$ and $\mathcal{M}\{\gamma_2\} = \frac{1}{2}$. We define an uncertain variable as

$$\tau(\gamma) = \begin{cases} -1, & \text{if } \gamma = \gamma_1, \\ 1, & \text{if } \gamma = \gamma_2. \end{cases}$$

We also define $\tau_i = -\tau$, $\xi_i = \eta_i\tau_i$ and $\xi = \eta\tau$. Then ξ_i and ξ have the same chance distribution. Thus $\{\xi_i, i \geq 1\}$ converges in distribution to ξ. However, for some small number $\epsilon > 0$, we have

$$\begin{aligned}
\mathrm{Ch}\{|\xi_i - \xi| > \epsilon\} &= E_{\mathrm{Pr}}[\mathcal{M}\{|\xi_i - \xi| > \epsilon\}] \\
&= \Pr(\eta = 1)\mathcal{M}\{\gamma_1\} + \Pr(\eta = 1)\mathcal{M}\{\gamma_2\} \\
&\quad + \Pr(\eta = 0)\mathcal{M}\{\gamma_1\} + \Pr(\eta = 0)\mathcal{M}\{\gamma_2\} \\
&= \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} \\
&= 1.
\end{aligned}$$

That is, the sequence $\{\xi_i, i \geq 1\}$ does not converge in measure to ξ.

4. Conclusions

This paper introduced two types of convergence concepts of uncertain random sequences. If an uncertain random sequence converges in mean, then it converges in measure. And if an uncertain random sequence converges in measure, then it converges in distribution. For illustrating relations among convergence concepts, several examples and theorems were stated and proved. It is mentioned that, if an uncertain random sequence degenerates in to the case of random variable, all results are hold. Furthermore, if an uncertain random sequence reduces to uncertain one, all theorems are satisfied.

References

[1] H. Ahmadzade, Y.H. Sheng, M. Esfahani, *On the convergence of uncertain random sequences*, Fuzzy Optimitization and Decision Making, **16(2)** (2017), 205-220.

[2] H. Ahmadzade, Y. H. Sheng and F. Hassantabar, *Some results of moments of uncertain random variables*, Iranian Journal of Fuzzy Systems, **14(2)** (2017), 1–21.

[3] X. W. Cheng and Y. K. Liu, *The convergence of fuzzy random variables*, Proceedings of the Eleventh International Fuzzy Systems Association World Congress, **1** (2005), 1–8.

[4] R. Gao and Y. H. Sheng, *Law of large numbers for uncertain random variables with different chance distributions*, Journal of Intelligent and Fuzzy Systems, **31** (2016), 1227–1234.

[5] R. Gao, Y. Sun and D. A. Ralescu, *Order statistics of uncertain random variables with application to k-out-of-n system*, Fuzzy Optimization and Decision Making, **16(2)** (2017), 159-181.

[6] R. Gao and K. Yao, *Importance index of components in uncertain random systems*, Knowledge-Based Systems, **109** (2016), 208–217.

[7] H. Y. Guo and X. S. Wang, *Variance of uncertain random variables*, Journal of Uncertainty Analysis and Applications, **2(6)** (2014), 1–7.

[8] D. Kahneman and A. Tversky, *Prospect theory: an analysis of decision under risk*, Econometrica, **47(2)** (1979), 263–292.

[9] H. Kwakernaak, *Fuzzy random variables-I: definitions and theorems*, Information Sciences, **15(1)** (1978), 1–29.

[10] H. Kwakernaak, *Fuzzy random variables-II: algorithms and examples for the discrete case*, Information Sciences, **17(3)** (1979), 253–278.

[11] B. Liu, *Random fuzzy dependent-chance programming and its hybrid intelligent algorithm*, Information Sciences, **141(3)** (2002), 259–271.

[12] B. Liu, Uncertainty Theory: an Introduction to Its Axiomatic Foundations, Springer, Berlin, 2004.
[13] B. Liu, Uncertainty Theory (2nd ed.), Springer, Berlin, 2007.
[14] B. Liu, *Some research problems in uncertainty theory*, Journal of Uncertain Systems, **3(1)** (2009), 3-10.
[15] B. Liu, Uncertainty Theory: A Branch of Mathematics for Modeling Human Uncertainty, Springer, Berlin, 2010.
[16] B. Liu, Uncertainty Theory (4th ed.), Springer, Berlin, 2015.
[17] Y. H. Liu, *Uncertain random variables: A mixture of uncertainty and randomness*, Soft Computing, **17(4)** (2013), 625–634.
[18] Y. H. Liu, *Uncertain random programming with applications*, Fuzzy Optimization and Decision Making, **12(2)** (2013), 153-169.
[19] Y. H. Liu and M. H. Ha, *Expected value of function of uncertain variables*, Journal of Uncertain Systems, **4(3)** (2010), 181-186.
[20] Y. K. Liu, Z. Q. Liu and J. W. Gao, *The modes of convergence in the approximation of fuzzy random optimization problems*, Soft Computing, **13(2)** (2009), 117-125.
[21] Y. H. Sheng and K. Yao, *Some formulas of variance of uncertain random variable*, Journal of Uncertainty Analysis and Applications, Article 12, **2** (2014).
[22] K. Yao, *A formula to calculate the variance of uncertain variable*, Soft Computing, **19(10)** (2014), 2947–2953.
[23] K. Yao and J. W. Gao, *Law of large numbers for uncertain random variables*, IEEE Transactions on Fuzzy Systems, **24(3)** (2016), 615–621.
[24] L. A. Zadeh, *Fuzzy sets*, Information and Control, **8** (1965), 338–353.
[25] L. A. Zadeh, *Fuzzy sets as a basis for a theory of possibility*, Fuzzy Sets and Systems, **1** (1978), 3–28.

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