FINITE-TIME PASSIVITY OF DISCRETE-TIME T-S FUZZY NEURAL NETWORKS WITH TIME-VARYING DELAYS

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Abstract. This paper focuses on the problem of finite-time boundedness and finite-time passivity of discrete-time T-S fuzzy neural networks with time-varying delays. A suitable Lyapunov–Krasovskii functional (LKF) is established to derive sufficient condition for finite-time passivity of discrete-time T-S fuzzy neural networks. The dynamical system is transformed into a T-S fuzzy model with uncertain parameters. Furthermore, the obtained passivity criteria is established in terms of Linear matrix inequality (LMI), which can be easily checked by using the efficient MATLAB LMI toolbox. Finally, some numerical cases are given to illustrate the effectiveness of the proposed approach.

1. Introduction

In the midst of the earlier decades, a speedy development has been made in the examination of artificial neural networks (NNs) which have wide applications in various locales, for instance, reproducing moving pictures, signal processing pattern recognition and optimization issues. For the late progress, since time delays can’t be kept up a key separation from and they frequently provoke insecurity of neural networks, it essentially focuses on progressing different sorts of stability conditions of different sorts of delayed neural frameworks. Thus, the artificial delayed neural frameworks have been widely studied by many authors and a assortment of results have been derived (see [14, 10, 22, 42, 37, 38, 39, 40, 41, 1] and references therein).

In addition, in actualizing the continuous-time neural system for computer simulation, experimental or computational purposes, it is fundamental to detail a discrete-time framework which is a simple of the continuous-time neural system. In any case, as a rule, the discretization can’t preserve the dynamics of the continuous-time counterpart. In this manner, it is critical to study the dynamics of the discrete-time frameworks and numerous outcomes have been accomplished in the literature [24, 2, 29, 12, 19, 6, 15, 16]. Further, it is notable that the connection weights of the neurons are inherently dependent on certain resistance and capacitance values that unavoidably acquire uncertainties amid the parameter identification process. The deviations and perturbations in parameters affect the execution of neural systems. In this way, it is vital to study the dynamical behaviors of neural frameworks

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by considering the uncertainties. Many researchers have talked about the flow of delayed systems with uncertainties, (see [24, 2, 12] and references therein.)

It is worth pointing out that the theory of “Fuzzy Sets” plays a vital role in the modeling and controlling of nonlinear systems [27]. The T-S fuzzy dynamic model is described by a family of fuzzy IF-THEN rules that represent local linear input-output relations of a nonlinear system. The T-S fuzzy model introduced in [27] is essentially a multi-model approach in which some linear models are blended into an overall single model through nonlinear membership functions. Specifically, the T-S fuzzy neural network has the advantages of both the fuzzy logic and neural networks which has many important applications in moving image processing and pattern classification [27, 11, 31, 43, 30, 25, 3, 32, 33, 21].

In practice, the passivity theory, which was proposed first in circuit investigation, has likewise pulled in impressive consideration since it is a valuable instrument to the stability analysis of linear and nonlinear systems, especially for high-order systems. Passive properties of frameworks can keep the frameworks inside stable. Because of its significance, the issue of passivity analysis for delayed dynamic systems has been investigated [7, 34, 4, 26, 13, 18, 20]. For example, the passivity condition for discrete-time switched neural systems with different functions and mixed time delays was derived [34]. The problem of robust passivity for discrete-time delayed standard neural system model with norm-bounded parameter uncertainties was investigated [4, 26]. In [13], the delay-dependent robust passivity criterion for uncertain discrete-time neural networks with interval time-varying delay was proposed.

However, it should be mentioned that all these existing studies about the passivity analysis are performed using the conventional Lyapunov asymptotic stability theory, which is defined over the infinite-time interval. But in many practical applications, the transient behavior of system is concerned over a fixed finite-time interval, in which the system states need to grip below a prescribed upper bound and larger values are not permitted during this time-interval [8, 23, 35]. Recently, many biologists are focusing on the transient values of the actual network states. Many interesting results for finite-time stability of various types of systems can be found (see [23, 35, 36, 17, 28] and references therein). However, there is no work is studied in finite-time passivity of discrete-time T-S fuzzy neural networks with time-varying delays.

Motivated by the above discussion, in this paper the problem of finite-time stability for discrete-time T-S fuzzy neural networks with time-varying delays based on passive theory is investigated. Delay-dependent results for finite-time boundedness and finite-time passivity are derived by using finite-time stability method and Lyapunov-Krasovskii functional approach. Finite-time bounded condition for the considered system with norm bounded uncertainties is given separately. Furthermore, the obtained passivity criteria is established in terms of Linear matrix inequality (LMI), which can be easily checked by using the efficient MATLAB LMI toolbox. Finally, numerical examples are provided to demonstrate the effectiveness of the proposed method.
Finite-time Passivity of Discrete-time T-S Fuzzy Neural Networks with Time-varying Delays

**Notations**: Throughout the paper, \( \mathbb{R}^n \) denotes the \( n \) dimensional Euclidean space, and \( \mathbb{R}^{m \times n} \) is the set of all \( m \times n \) real matrices. For symmetric matrices \( \mathcal{X} \) and \( \mathcal{Y} \), the notation \( \mathcal{X} \geq \mathcal{Y} \) means that \( \mathcal{X} - \mathcal{Y} \) is positive-semi definite; \( \mathcal{M}^T \) is transpose of the matrix \( \mathcal{M} \); \( I \) is the identity matrix with appropriate dimension; In symmetric block matrices, we use ‘*’ to represent a term that is induced by symmetry; \( \lambda_{max} \) and \( \lambda_{min} \) denote the maximum and minimum eigenvalue of a matrix.

### 2. Problem Statement and Preliminaries

Consider the following uncertain discrete-time delayed neural networks described as

\[
\begin{align*}
  y(k+1) &= -(\mathcal{E} + \Delta\mathcal{E}(k))y(k) + (\mathcal{A} + \Delta\mathcal{A}(k))y(k) + u(k), \\
  z(k) &= \mathcal{D}f(y(k)), \\
  y(k) &= \phi(k), \quad k \in [-\tau_M, 0],
\end{align*}
\]

where \( y(k) = [y_1(k), y_2(k), ..., y_n(k)]^T \in \mathbb{R}^n \) is the neuron state vector, \( u(k) \) is the exogenous disturbance input vector belongs to \( L_2[0, \infty) \), and \( z(k) \) is the output vector of the neural network. The nonlinear function \( f(y(k)) = [f_1(y_1(k)), f_2(y_2(k)), ..., f_n(y_n(k))]^T \in \mathbb{R}^n \) denotes the neuron activation function, \( \phi(k) \in \mathbb{R}^n \) is a vector-valued initial condition function. \( \mathcal{E} = \text{diag}\{e_1, ..., e_n\} \) represents the state feedback coefficient matrix with \( |e_i| < 1 \). \( \mathcal{A}, \mathcal{B}, \mathcal{D} \) are interconnection weight matrices with appropriate dimensions. \( \Delta\mathcal{E}(k), \Delta\mathcal{A}(k), \) and \( \Delta\mathcal{B}(k) \) are parametric uncertainties. \( \tau(k) \) are the time-varying delays assumed to satisfy

\[
0 \leq \tau(k) \leq \tau_M, \text{ for all } k \geq 0,
\]

where \( \tau_M \) is constant.

**Assumption (H1).** [9]. The uncertain parameters \( \Delta\mathcal{E}(k), \Delta\mathcal{A}(k), \) and \( \Delta\mathcal{B}(k) \) in DNNs (1) are time-varying and norm bounded, which admit

\[
[\Delta\mathcal{E}(k) \ \Delta\mathcal{A}(k) \ \Delta\mathcal{B}(k)] = \mathcal{H}(k)[\mathcal{G}_1 \ \mathcal{G}_2 \ \mathcal{G}_3],
\]

where \( \mathcal{H} \) and \( \mathcal{G}_j, (j = 1, 2, 3) \) are known constant matrices of appropriate dimensions and \( \mathcal{H} \) is an unknown time-varying matrix with Lebesgue measurable elements bounded by

\[
\mathcal{H}^T(k)\mathcal{H}(k) \leq I.
\]

As mentioned before, the uncertainties and vagueness are always unavoidable in practical implementation. To reflect such a reality, we use the fuzzy system approach to model the vagueness, in which the system parameters are with uncertainties. The \( i^{th} \) rule of the T-S fuzzy model is of the following form:

**Model i**: IF \( \{w_1 \text{ is } \zeta_{i1}\} \) and, ..., and \( \{w_p \text{ is } \zeta_{ip}\} \), THEN

\[
\begin{align*}
  y(k+1) &= -(\mathcal{E}_i + \Delta\mathcal{E}_i(k))y(k) + (\mathcal{A}_i + \Delta\mathcal{A}_i(k))y(k) + u(k), \\
  z(k) &= \mathcal{D}_i f(y(k)), \\
  y(k) &= \phi(k), \quad k \in [-\tau_M, 0],
\end{align*}
\]

where \( w_1(k), ..., w_p(k) \) are premise variables. \( \zeta_{ij} \ (i = 1, 2, ..., r, \ j = 1, 2, ..., p) \) are fuzzy sets and \( r \) is the number of IF-THEN rules.
Using a standard fuzzy inference method, the system (4) is inferred as follows

\[
\begin{align*}
y(k+1) &= \sum_{i=1}^{r} h_i(w(k))\{(e_i + \Delta e_i(k))y(k) + (a_i + \Delta a_i(k))f(y(k)) \\
& \quad + (b_i + \Delta b_i(k))f(y(k - \tau(k))) + u(k)\}, \\
z(k) &= \sum_{i=1}^{r} h_i(w(k))2\eta_f(y(k)), \quad i = 1, ..., r,
\end{align*}
\]

where \( h_i(w(k)) \) is the normalised membership function of the inferred fuzzy set \( \beta_i(w(k)) \), that is,

\[
h_i(w(k)) = \frac{\beta_i(w(k))}{\sum_{i=1}^{r} \beta_i(w(k))}, \quad \beta_i(w(k)) = \prod_{j=1}^{p} \zeta_{ij}(w_j(k)),
\]

and \( \zeta_{ij}(\cdot) \) is the grade membership function of \( w_p(k) \) in \( \zeta_{ij} \). We assume \( \beta_i(w(k)) \geq 0, \quad i = 1, ..., r, \quad \sum_{i=1}^{r} \beta_i(w(k)) > 0 \) for any \( w(k) \).

Hence \( h_i(w(k)) \) satisfies

\[
h_i(w(k)) \geq 0, \quad i = 1, ..., r, \quad \sum_{i=1}^{r} h_i(w(k)) = 1,
\]

for any \( w(k) \).

**Assumption (H2).** [14] In this brief, we assume that activation functions \( f_i(\cdot) \), \( i \in [1, m] \) are bounded and satisfy

\[
F_i^- \leq \frac{f_i(a) - f_i(b)}{a - b} \leq F_i^+, \quad a \neq b, \quad a, b \in \mathbb{R}, \quad i = 1, 2, ..., m,
\]

where \( F_i^- \) and \( F_i^+ \) are known real scalars. For notation convenience, we define the following matrices

\[
\begin{align*}
F_1 &= \text{diag}\left\{F_1^- F_1^+, F_2^- F_2^+, ..., F_m^- F_m^+\right\}, \\
F_2 &= \text{diag}\left\{\frac{F_1^- + F_1^+ + F_2^- + F_2^+ + ... + F_m^- + F_m^+}{2}\right\}
\end{align*}
\]

**Assumption (H3).** [18] The disturbance input vector \( u(k) \) is time-varying and for a given \( \vartheta > 0 \), satisfies \( u^T(k)u(k) \leq \vartheta \).

The following definitions and Lemmas will be used in the proof of main results.

**Definition 2.1.** [18] The DNN (1) is said to be robustly finite-time bounded with respect to \((c_1, c_2, L, N, \vartheta)\), where \( 0 < c_1 < c_2 \) and \( L > 0 \), if

\[
y^T(k_1)Ly(k_1) \leq c_1 \quad \Rightarrow \quad y^T(k_2)Ly(k_2) \leq c_2, \\
\forall k_1 \in [-\tau_M, \tau_M + 1, ..., 0], \quad k_2 \in \{1, 2, ..., N\},
\]

holds for any nonzero \( u(k) \) satisfies Assumption (H3).

**Definition 2.2.** [18] The DNN (1) is said to be robustly finite-time passive with respect to \((c_1, c_2, L, N, \gamma, \vartheta)\), where \( 0 < c_1 < c_2, \gamma \) is a prescribed positive scalar
and $L > 0$, if DNN (1) with output (2) is robustly finite-time bounded with respect to $(c_1, c_2, L, N, \vartheta)$, and under the zero initial condition the output $z(k)$ satisfies

$$2 \sum_{k=0}^{N} z^T(k)u(k) \geq -\gamma \sum_{k=0}^{N} u^T(k)u(k)$$

holds for any nonzero $u(k)$ satisfies Assumption (H3).

**Lemma 2.3.** [5] Let $\mathcal{M}$, $\mathcal{P}$, $\mathcal{Q}$ be given matrices such that $\mathcal{Q} > 0$, then

$$\begin{bmatrix} \mathcal{P} & \mathcal{M}^T \\ \mathcal{M} & -\mathcal{Q} \end{bmatrix} < 0 \iff \mathcal{M} + \mathcal{M}^T \mathcal{Q}^{-1} \mathcal{M} < 0.$$

**Lemma 2.4.** [9] Given matrices $\mathcal{Q} = \mathcal{Q}^T, \mathcal{H}, \mathcal{E}$ and $\mathcal{R} = \mathcal{R}^T > 0$ with appropriate dimensions

$$\mathcal{Q} + \mathcal{H} \mathcal{F}(k) \mathcal{E} + \mathcal{E}^T \mathcal{F}^T(k) \mathcal{H}^T < 0$$

for all $\mathcal{F}(k)$ satisfying $\mathcal{F}^T(k) \mathcal{F}(k) \leq \mathcal{R}$ if and only if there exists a scalar $\varepsilon > 0$ such that

$$\mathcal{Q} + \varepsilon^{-1} \mathcal{H} \mathcal{H}^T + \varepsilon \mathcal{E}^T \mathcal{R} \mathcal{E} < 0.$$

### 3. Finite-time Boundedness

In this section, we provide the condition for the following nominal system of (5), (without uncertainties)

$$y(k+1) = \sum_{i=1}^{r} h_i(w(k))\{-\mathcal{E}_i y(k) + \mathcal{A}_i f(y(k)) + \mathcal{B}_i f(y(k-\tau(k))) + u(k)\},$$

$$y(k) = \phi(k), \ k \in [-\tau_M, 0], \ i = 1, ..., r,$$

(7)

to be finite-time bounded with respect to $(c_1, c_2, R, N, \vartheta)$.

**Theorem 3.1.** Under assumptions (H2) & (H3), for given scalars $\mu \geq 1$, $\tau_M$, the system (7) is finite-time bounded with respect to $(c_1, c_2, R, N, \vartheta)$, if there exist matrices $\mathcal{P} > 0$, $\mathcal{Q} > 0$, diagonal matrices $\mathcal{W}_i > 0$, $(i = 1, 2, 3)$, and matrix $\mathcal{W}$ of appropriate dimension, such that the following inequalities hold:

$$\Omega_i = \begin{bmatrix} (1.1) & \mathcal{F}_2 \mathcal{W}_3 & 0 & \mathcal{E}_i^T \mathcal{P} \mathcal{A}_i + \mathcal{F}_2 \mathcal{W}_1 + \mathcal{F}_1 \mathcal{W}_3 & \mathcal{E}_i^T \mathcal{P} \mathcal{A}_i - \mathcal{F}_2 \mathcal{W}_3 & \mathcal{E}_i^T \mathcal{P} \mathcal{W}_3 & \mathcal{E}_i^T \mathcal{P} \\ + & -\mathcal{F}_1 \mathcal{W}_1 - 2 \mathcal{W}_3 & 0 & -\mathcal{F}_2 \mathcal{W}_1 & 0 & 0 & 0 \\ + & + & + & -\mu^T \mathcal{Q} & 0 & 0 & 0 \\ + & + & + & \mathcal{A}_i^T \mathcal{P} \mathcal{A}_i - \mathcal{F}_1 \mathcal{W}_3 - \mathcal{W}_1 & \mathcal{A}_i^T \mathcal{P} \mathcal{A}_i + \mathcal{F}_2 \mathcal{W}_3 & \mathcal{A}_i^T \mathcal{P} \\ + & + & + & + & + & + & -\mathcal{P} - \mathcal{W} \end{bmatrix} < 0,$$

(8)

$$0 \leq \overline{\mathcal{Q}} \leq \lambda_{\text{max}}(\overline{\mathcal{Q}}) I, \ i = 1, 2, ..., r,$$

$$\mu^N \left( (\lambda_{\text{max}}(\overline{\mathcal{P}}) + \tau_M \mu^T \mathcal{M}^{-1} \lambda_{\text{max}}(\overline{\mathcal{Q}})) c_1 + \lambda_{\text{max}}(\mathcal{W}) \vartheta \right) \lambda_{\text{min}}(\overline{\mathcal{P}}) < c_2,$$

where

$$\mu^N \left( \lambda_{\text{max}}(\overline{\mathcal{P}}) + \tau_M \mu^T \mathcal{M}^{-1} \lambda_{\text{max}}(\overline{\mathcal{Q}}) \right) c_1 + \lambda_{\text{max}}(\mathcal{W}) \vartheta \lambda_{\text{min}}(\overline{\mathcal{P}}) < c_2,$$

$$0 \leq \overline{\mathcal{Q}} \leq \lambda_{\text{max}}(\overline{\mathcal{Q}}) I, \ i = 1, 2, ..., r,$$

$$\mu^N \left( \lambda_{\text{max}}(\overline{\mathcal{P}}) + \tau_M \mu^T \mathcal{M}^{-1} \lambda_{\text{max}}(\overline{\mathcal{Q}}) \right) c_1 + \lambda_{\text{max}}(\mathcal{W}) \vartheta \lambda_{\text{min}}(\overline{\mathcal{P}}) < c_2.$$
Proof. We consider the LKF candidate as

$$V(y(k), k) = V_1(y(k), k) + V_2(y(k), k)$$

where

$$V_1(y(k), k) = y(k)^T P y(k),$$

$$V_2(y(k), k) = \sum_{i=k-\tau_M}^{k-1} \mu^{k-i-1} y(i)^T Q y(i)$$

Calculating the forward difference of $V(y(k), k)$ by defining $\Delta V(y(k), k) = V(y(k+1), k) - V(y(k), k)$ along the solutions of (7), we obtain

$$\Delta V_1(y(k), k) - (\mu - 1) V_1(y(k), k) = \Delta V_1(y(k), k) - (\mu - 1) V_1(y(k), k)$$

then

$$\Delta V_1(y(k), k) - (\mu - 1) V_1(y(k), k) = y^T (k+1) P y(k+1) - \mu y^T (k) P y(k)$$

$$= \sum_{i=1}^{r} f_i(w(k)) \otimes \left[ \begin{array}{c} y(k) \\ f(y(k)) \\ f(y(k-\tau(k))) \\ u(k) \end{array} \right]^T \left[ \begin{array}{cccc} E^T P E_i & E^T P A_i & E^T P B_i & E^T P C_i \\ A_i^T P E_i & A_i^T P A_i & A_i^T P B_i & A_i^T P C_i \\ B_i^T P E_i & B_i^T P A_i & B_i^T P B_i & B_i^T P C_i \\ P E_i & P A_i & P B_i & P C_i \end{array} \right] \left[ \begin{array}{c} y(k) \\ f(y(k)) \\ f(y(k-\tau(k))) \\ u(k) \end{array} \right] - \mu y^T (k) P y(k).$$

$$\Delta V_2(y(k), k) - (\mu - 1) V_2(y(k), k) \leq y^T (k) Q y(k) - \mu \tau_M y^T (k-\tau_M) Q y(k-\tau_M).$$

Moreover, it follows from Assumption (H2), that

$$\left[ f_i(y_i(k)) - \mathcal{F}_i^- y_i(k) \right]^T \left[ f_i(y_i(k)) - \mathcal{F}_i^+ y_i(k) \right] \leq 0, \quad i = 1, 2, \ldots, n,$$

which is equivalent to

$$\left[ \begin{array}{c} y(k) \\ f(y(k)) \end{array} \right]^T \left[ \begin{array}{cc} \mathcal{F}_i^- \mathcal{F}_i^+ y_i y_i^T - \frac{\mathcal{F}_i^- + \mathcal{F}_i^+}{2} y_i y_i^T \\ -\frac{\mathcal{F}_i^- + \mathcal{F}_i^+}{2} y_i y_i^T \end{array} \right] \left[ \begin{array}{c} y(k) \\ f(y(k)) \end{array} \right] \leq 0,$$

where $y_i$ denotes the units column vector having element 1 on its $i$th row and zeros elsewhere. Let $\mathcal{W}_i = \text{diag}\{w_{i1}, w_{i2}, \ldots, w_{in}\} > 0$, it is easy to see that

$$\sum_{i=1}^{n} w_{i1} \left[ \begin{array}{c} y(k) \\ f(y(k)) \end{array} \right]^T \left[ \begin{array}{cc} \mathcal{F}_i^- \mathcal{F}_i^+ y_i y_i^T - \frac{\mathcal{F}_i^- + \mathcal{F}_i^+}{2} y_i y_i^T \\ -\frac{\mathcal{F}_i^- + \mathcal{F}_i^+}{2} y_i y_i^T \end{array} \right] \left[ \begin{array}{c} y(k) \\ f(y(k)) \end{array} \right] \leq 0.$$

That is,

$$\left[ \begin{array}{c} y(k) \\ f(y(k)) \end{array} \right]^T \left[ \begin{array}{cc} -\mathcal{W}_i \mathcal{F}_i \mathcal{W}_i & \mathcal{F}_i \mathcal{W}_i \\ * & -\mathcal{W}_i \end{array} \right] \left[ \begin{array}{c} y(k) \\ f(y(k)) \end{array} \right] \geq 0. \quad (15)$$
Similar to above, for any positive diagonal matrices $\mathcal{W}_2$ and $\mathcal{W}_3$, we can obtain the following inequalities:

\[
\begin{bmatrix}
    y(t - \tau(k)) \\
    f(y(t - \tau(k)))
\end{bmatrix}^T \begin{bmatrix}
    \mathcal{F}_1 \mathcal{W}_2 & \mathcal{F}_2 \mathcal{W}_2 \\
    * & -\mathcal{W}_2
\end{bmatrix} \begin{bmatrix}
    y(t - \tau(k)) \\
    f(y(t - \tau(k)))
\end{bmatrix} \geq 0,
\]

(16)

and

\[
\begin{bmatrix}
    y(k) \\
    f(y(k)) \\
    y(k - \tau(k)) \\
    f(y(t - \tau(k)))
\end{bmatrix}^T \begin{bmatrix}
    \mathcal{F}_1 \mathcal{W}_3 & \mathcal{F}_2 \mathcal{W}_3 & \mathcal{F}_3 \mathcal{W}_3 & \mathcal{F}_4 \mathcal{W}_3 \\
    * & -\mathcal{W}_3 & \mathcal{F}_4 \mathcal{W}_3 & \mathcal{W}_3 \\
    * & * & \mathcal{F}_4 \mathcal{W}_3 & \mathcal{F}_3 \mathcal{W}_3 \\
    * & * & * & -\mathcal{W}_3
\end{bmatrix} \begin{bmatrix}
    y(k) \\
    f(y(k)) \\
    y(k - \tau(k)) \\
    f(y(t - \tau(k)))
\end{bmatrix} \geq 0.
\]

(17)

Then from (12) and adding (13)–(17), gives

\[
\Delta V(y(k), k) - (\mu - 1)V(y(k), k) - u^T(k)\mathcal{W}u(k) \leq \sum_{i=1}^r h_i(w(k)) [\xi^T(k)\Omega_i \xi(k)],
\]

(18)

where

\[
\xi(k) = [y^T(k) y^T(k - \tau(k)) y^T(k - \tau_M) f^T(y(k)) f^T(y(k - \tau(k))) u^T(k)]^T.
\]

Hence if the LMIs (8) hold, it is easy to get

\[
\Delta V(y(k), k) - (\mu - 1)V(y(k), k) - u^T(k)\mathcal{W}u(k) \leq 0
\]

\[
V(k + 1) - V(k) \leq (\mu - 1)V(y(k), k) + u^T(k)\mathcal{W}u(k)
\]

(19)

\[
\leq (\mu - 1)V(y(k), k) + \lambda_{\max}(\mathcal{W}) u^T(k)u(k).
\]

Simple computation gives $V(k + 1) \leq \mu V(y(k), k) + \lambda_{\max}(\mathcal{W}) u^T(k)u(k)$.

Noticing $\mu \geq 1$, it follows that

\[
V(k) \leq \mu^k V(0) + \lambda_{\max}(\mathcal{W}) \sum_{i=0}^{k-1} \mu^{k-i-1} u^T(i)u(i),
\]

from the Assumption (H3), we have

\[
V(k) \leq \mu^k V(0) + \mu^k \lambda_{\max}(\mathcal{W}) \vartheta.
\]

(20)

Further, from (11), we get

\[
V(0) = y^T(0)\mathcal{P}y(0) + \sum_{i=-\tau_M}^{-1} \mu^{-i-1} y^T(i)\mathcal{Q}y(i).
\]

Letting $\overline{\mathcal{P}} = R^{-1/2} \mathcal{P} R^{-1/2}$, $\overline{\mathcal{Q}} = R^{-1/2} \mathcal{Q} R^{-1/2}$, we obtain

\[
V(0) = y^T(0) R^{1/2} \overline{\mathcal{P}} R^{1/2} y(0) + \sum_{i=-\tau_M}^{-1} \mu^{-i-1} y^T(i) R^{1/2} \overline{\mathcal{Q}} R^{1/2} y(i)
\]

\[
\leq \lambda_{\max}(\overline{\mathcal{P}}) y^T(0) R y(0) + \lambda_{\max}(\overline{\mathcal{Q}}) \sum_{i=-\tau_M}^{-1} \mu^{-i-1} y^T(i) R y(i)
\]

\[
\leq \left[\lambda_{\max}(\overline{\mathcal{P}}) + \tau_M \mu^{\tau - 1} \lambda_{\max}(\overline{\mathcal{Q}})\right] c_1.
\]
On the other hand, from (11), we can obtain that
\[ V(k) \geq y^T(k)P y(k) \geq y^T(k)R^{1/2}T^{1/2}y(k) \geq \lambda_{\min}(P)y^T(k)Ry(k) \tag{21} \]

From (20) and (21), we get
\[ y^T(k)Ry(k) < \frac{\mu^k \left( (\lambda_{\max}(P) + \tau_M \mu^{T_M - 1} \lambda_{\max}(Q))c_1 + \lambda_{\max}(W) \vartheta \right)}{\lambda_{\min}(P)} < c_2. \]

This implies \( y^T(k)Ry(k) < c_2, \forall k \in \{1, 2, ..., N\} \). Thus by Definition 2.1 the DNNs (7) is finite-time bounded with respect to \((c_1, c_2, R, N, \vartheta)\). This completes the proof. \( \square \)

4. Finite-time Passivity

In this section, we focus on the finite-time passivity of DNN (7) with output \( z(t) \).

**Theorem 4.1.** Under assumptions (H2) \& (H3) hold, for given scalars \( \mu \geq 1, \tau_M \), the system (7) is finite-time bounded with respect to \((c_1, c_2, R, N, \gamma, \vartheta)\), if there exist matrices \( P > 0 \), and \( Q > 0 \), diagonal matrices \( W_i > 0, i = 1, 2, 3 \), matrix \( W \) of appropriate dimension, and positive scalar \( \gamma \), such that the following inequalities hold:

\[
\begin{bmatrix}
(1, 1) & \mathcal{F}_2 W_3 & 0 & \mathcal{E}_T P \mathcal{A}_1 + \mathcal{F}_2 W_1 + \mathcal{F}_1 W_3 & \mathcal{E}_T P \mathcal{A}_1 - \mathcal{F}_2 W_3 & 0 \\
* & -\mathcal{F}_1 W_2 - W_3 & 0 & -\mathcal{F}_2 W_3 & W_3 & 0 \\
* & * & -\mu^T Q & 0 & 0 & 0 \\
* & * & * & \mathcal{E}_T P \mathcal{A}_1 - \mathcal{F}_1 W_3 - W_1 & \mathcal{E}_T P \mathcal{A}_1 + \mathcal{F}_2 W_3 - \vartheta^T & 0 \\
* & * & * & * & \mathcal{E}_T P \mathcal{A}_1 - W_2 - W_3 & 0 \\
* & * & * & * & * & -\gamma I \\
\end{bmatrix} \begin{bmatrix}
\mathcal{F}_1 W_1 - \mathcal{F}_3 W_3 + Q - \mu P \\
\mathcal{F}_1 W_1 - \mathcal{F}_3 W_3 + Q - \mu P \\
\mathcal{F}_1 W_1 - \mathcal{F}_3 W_3 + Q - \mu P \\
\mathcal{F}_1 W_1 - \mathcal{F}_3 W_3 + Q - \mu P \\
\mathcal{F}_1 W_1 - \mathcal{F}_3 W_3 + Q - \mu P \\
\mathcal{F}_1 W_1 - \mathcal{F}_3 W_3 + Q - \mu P \\
\end{bmatrix} < 0, \tag{22}
\]

\[ 0 \leq \mathcal{Q} \leq \lambda_{\max}(\mathcal{Q})I, \ i = 1, 2, ..., r, \tag{23} \]

\[
\frac{\mu^N \left( (\lambda_{\max}(P) + \tau_M \mu^{T_M - 1} \lambda_{\max}(\mathcal{Q}))c_1 + \mu^{-N} \gamma \vartheta \right)}{\lambda_{\min}(P)} < c_2, \tag{24}
\]

where

\[ (1, 1) = \mathcal{E}_T P \mathcal{E}_4 - \mathcal{F}_1 W_1 - \mathcal{F}_3 W_3 + Q - \mu P, \quad P = R^{-1/2}P R^{-1/2}, \quad \mathcal{Q} = R^{-1/2}QR^{-1/2}. \]

**Proof.** The proof is similar to that in Theorem 3.1. In Theorem 3.1 by choosing \( \mathcal{W} = \gamma \mu^{-N} \) in \( \mathcal{J} \) and using similar lines of (19), it follows that

\[
\Delta V(k) - (\mu - 1)V(k) - 2z^T(k)u(k) - \gamma \mu^{-N}u^T(k)u(k) \leq 0
\]

\[ V(k + 1) - V(k) \leq (\mu - 1)V(k) + 2z^T(k)u(k) + \gamma \mu^{-N}u^T(k)u(k) \]

By simple computation
\[ V(k) \leq \mu^k V(0) + 2 \sum_{i=0}^{k-1} \mu^{k-i-1} z^T(i)u(i) + \gamma \mu^{-N} \sum_{i=0}^{k-1} \mu^{k-i-1} u^T(i)u(i). \tag{25} \]
Under the zero initial condition and noticing $V(k) \geq 0, \forall k \in \{1, 2, ..., N\}$, we have
\[
2 \sum_{i=0}^{k-1} \mu^{k-i-1} z^T(i) u(i) \geq -\gamma \mu^{-N} \sum_{i=0}^{k-1} \mu^{k-i-1} u^T(i) u(i).
\]
Noticing that $\mu \geq 1$, we have
\[
2 \sum_{k=0}^{N} \mu^{N-k} z^T(k) u(k) \geq -\gamma \mu^{-N} \sum_{k=0}^{N} \mu^{N-k} u^T(k) u(k).
\]
By Definition 2.2, can be concluded that the nominal system (7) is finite-time passive. This completes the proof. □

Based on Theorem 4.1 and Lemma 2.4, we derive the finite-time passivity for discrete-time T-S fuzzy neural networks with uncertainties in the following Theorem(4.2).

**Theorem 4.2.** Under assumptions (H2) & (H3), for given scalars $\mu \geq 1$, $\tau M$, the system (1) is finite-time bounded with respect to $(c_1, c_2, R, N, \gamma, \vartheta)$, if there exist matrices $P > 0, Q > 0$, diagonal matrices $W_i > 0$, ($i = 1, 2, 3$), and matrix $W$ of appropriate dimension, and positive scalar $\gamma$, such that the following inequalities hold:
\[
\Omega_i^2 = \begin{bmatrix}
\Omega_i^1 + \varepsilon_i \mathcal{H}_i \mathcal{H}_i^T & \mathcal{G}_i^T \\
* & -\varepsilon_i I
\end{bmatrix} < 0, \quad i = 1, 2, ..., r
\]
\[
0 \leq \mathcal{Q} \leq \lambda_{max}(\mathcal{Q}) I, \quad i = 1, 2, ..., r,
\]
\[
\mu^{-N} \left(\frac{(\lambda_{max}(\mathcal{P}) + \tau M \mu^{\tau M - 1} \lambda_{max}(\mathcal{Q})) c_1 + \mu^{-N} \gamma \vartheta}{\lambda_{min}(\mathcal{P})}\right) < c_2,
\]
where $\Omega_i^1$ is given in Theorem 4.1 and
\[
\mathcal{H}_i = \begin{bmatrix}
\mathcal{H}_i & 0 & 0 & 0 & 0 & 0
\end{bmatrix}^T,
\]
\[
\mathcal{G}_i = \begin{bmatrix}
\mathcal{G}_i & 0 & 0 & \mathcal{G}_i & 0
\end{bmatrix}^T,
\]
\[
\mathcal{F}_i = \mathcal{F}_i(k) \mathcal{H}_i + \mathcal{G}_i^T \mathcal{F}_i^T(k) \mathcal{H}_i < 0, \quad i = 1, 2, ..., r.
\]
\[
\Omega_i^1 + \varepsilon_i \mathcal{H}_i \mathcal{H}_i^T + \varepsilon_i^{-1} \mathcal{G}_i \mathcal{G}_i^T < 0, \quad i = 1, 2, ..., r.
\]

Applying Schur compliment Lemma 2.3 in (30), the T-S fuzzy DNN model (5) is robustly finite-time passive. This completes the proof. □
Remark 4.3. The equation (1) is described by a discrete-time T-S fuzzy neural networks with time-varying delays modelled in (4). In this model the system dynamics are captured by a set of fuzzy IF-THEN rules that represent local linear input-output relations of a nonlinear system.

Remark 4.4. Theorem 4.1 develops a finite-time passivity criterion of discrete-time T-S fuzzy neural networks with time-varying delays. Theorem 4.1 makes full use of the information of the subsystems upper bounds of the time-varying delays, which also brings us the less conservativeness.

5. Numerical Examples

In this section, simulation examples are given to demonstrate the feasibility and efficiency of theoretic results.

Example 5.1. Consider the following discrete-time T-S fuzzy neural networks:

\[
y(k+1) = \sum_{i=1}^{r} h_i(w(k))\{-E_iy(k) + A_i f(y(k)) + B_i f(y(k - \tau(k))) + u(k)\},
\]

\[
y(k) = \phi(k), \quad k \in [-\tau_M, 0], \quad i = 1, \ldots, r,
\]

Mode Rule 1: IF \( w(k) = y_1(k) \) is “about 0”, THEN

\[
y(k+1) = -E_1y(k) + A_1 f(y(k)) + B_1 f(y(k - \tau(k))) + u(k)
\]

where

\[
E_1 = \begin{bmatrix} 0.6 & 0 \\ 0 & 0.4 \end{bmatrix}, \quad A_1 = \begin{bmatrix} -0.1 & 0.8 \\ 0.1 & 0.2 \end{bmatrix}, \quad B_1 = \begin{bmatrix} -0.2 & -0.9 \\ 0.1 & 0.2 \end{bmatrix},
\]

Mode Rule 2: IF \( w(k) = y_2(k) \) is “about 1”, THEN

\[
y(k+1) = -E_2y(k) + A_2 f(y(k)) + B_2 f(y(k - \tau(k))) + u(k)
\]

where

\[
E_2 = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.4 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -0.1 & 0.12 \\ 0.23 & 0.3 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0.4 & 0.5 \\ 0.15 & 0.25 \end{bmatrix},
\]

the activation functions are \( F_1 = 0, \quad F_2 = 0.3I \), the membership functions are given as \( h_1 = (1 - \sin(y_1(k)))/2 \), and \( h_2 = (1 + \sin(y_1(k)))/2 \). Take \( R = I, \quad c_1 = 1, \quad \vartheta = 5, \quad N = 30, \quad \tau_M = 0.9, \) and \( \mu = 2 \), by solving the LMI based finite-time bounded conditions in Theorem 3.1 using Matlab LMI toolbox, we obtain the feasible solutions as follows:

\[
P = 10^7 \times \begin{bmatrix} 1.0616 & -0.1867 \\ -0.1867 & 0.9422 \end{bmatrix}, \quad Q = 10^6 \times \begin{bmatrix} 6.7889 & -3.0254 \\ -3.0254 & 6.1954 \end{bmatrix},
\]

\[
W_1 = 10^7 \times \begin{bmatrix} 3.0026 & 0 \\ 0 & 3.0026 \end{bmatrix}, \quad W_2 = 10^7 \times \begin{bmatrix} 3.0466 & 0 \\ 0 & 3.0466 \end{bmatrix},
\]

\[
W_3 = 10^6 \times \begin{bmatrix} 8.5088 & 0 \\ 0 & 8.5088 \end{bmatrix}, \quad W = 10^7 \times \begin{bmatrix} 4.1306 & 0.1567 \\ 0.1567 & 3.9937 \end{bmatrix},
\]

\[
c_2 = 1.9005c + 0.10.
\]
Example 5.2. Consider the following discrete-time T-S fuzzy neural networks:

\[
y(k + 1) = \sum_{i=1}^{r} h_i(w(k))\{-\mathcal{E}_i y(k) + \mathcal{A}_i f(y(k)) + \mathcal{B}_i f(y(k - \tau(k))) + u(k)\},
\]

\[
z(k) = \sum_{i=1}^{r} h_i(w(k))\mathcal{D}_i f(y(k)),
\]

\[
y(k) = \phi(k), \quad k \in [-\tau_M, 0], \quad i = 1, \ldots, r,
\]

(32)

Mode Rule 1: IF \(w(k) = y_1(k)\) is “about 0”, THEN

\[
y(k + 1) = -\mathcal{E}_1 y(k) + \mathcal{A}_1 f(y(k)) + \mathcal{B}_1 f(y(k - \tau(k))) + u(k)
\]

\[
z(k) = \mathcal{D}_1 f(y(k))
\]

where

\[
\mathcal{E}_1 = \begin{bmatrix} 0.4 & 0 \\ 0 & 0.2 \end{bmatrix}, \quad \mathcal{A}_1 = \begin{bmatrix} 0.2 & 0.4 \\ 0.3 & 0.1 \end{bmatrix}, \quad \mathcal{B}_1 = \begin{bmatrix} -0.5 & -0.3 \\ 0.4 & 0.2 \end{bmatrix}, \quad \mathcal{D}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},
\]

Mode Rule 2: IF \(w(k) = y_2(k)\) is “about 1”, THEN

\[
y(k + 1) = -\mathcal{E}_2 y(k) + \mathcal{A}_2 f(y(k)) + \mathcal{B}_2 f(y(k - \tau(k))) + u(k)
\]

\[
z(k) = \mathcal{D}_2 f(y(k))
\]

where

\[
\mathcal{E}_2 = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.1 \end{bmatrix}, \quad \mathcal{A}_2 = \begin{bmatrix} -0.6 & 0.32 \\ 0.43 & 0.15 \end{bmatrix}, \quad \mathcal{B}_2 = \begin{bmatrix} 0.24 & 0.25 \\ 0.45 & 0.15 \end{bmatrix}, \quad \mathcal{D}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},
\]

the activation functions are \(F_1 = 0, \ F_2 = 0.3I\), the membership functions are given as \(h_1 = (1 - \sin(y_1(k)))/2\), and \(h_2 = (1 + \sin(y_1(k)))/2\).

Let \(R = I, c_1 = 1, \vartheta = 5, N = 30, \tau_M = 0.5\), and \(\mu = 2\), by solving the LMI based finite-time passivity conditions in Theorem 4.1 using Matlab LMI toolbox, we obtain the feasible solutions as follows:

\[
\mathcal{P} = \begin{bmatrix} 30.7703 & 0.1543 \\ 0.1543 & 30.9746 \end{bmatrix}, \quad \mathcal{Q} = \begin{bmatrix} 26.6650 & 0.0752 \\ 0.0752 & 27.3428 \end{bmatrix}, \quad \mathcal{W}_1 = \begin{bmatrix} 35.5137 & 0 \\ 0 & 35.5137 \end{bmatrix},
\]

\[
\mathcal{W}_2 = \begin{bmatrix} 31.9156 & 0 \\ 0 & 31.9156 \end{bmatrix}, \quad \mathcal{W}_3 = \begin{bmatrix} 11.9073 & 0 \\ 0 & 11.9073 \end{bmatrix},
\]

\(\gamma = 34.7872, \ c_2 = 1.8330e + 009\).

Example 5.3. Consider the following T-S fuzzy neural networks with norm bounded uncertainties.

\[
\begin{cases}
y(k + 1) = \sum_{i=1}^{r} h_i(w(k))\{-\mathcal{E}_i + \Delta \mathcal{E}_i(k)\}y(k) + (\mathcal{A}_i + \Delta \mathcal{A}_i(k)) f(y(k)) \\
+ (\mathcal{B}_i + \Delta \mathcal{B}_i(k)) f(y(k - \tau(k))) + u(k)\},
z(k) = \sum_{i=1}^{r} h_i(w(k))\mathcal{D}_i f(y(k)), \quad i = 1, \ldots, r,
\end{cases}
\]

(33)
**Mode Rule 1:** IF \( w(k) = y_1(k) \) is “about 0”, THEN

\[
y(k + 1) = - (\mathcal{E}_1 + \mathcal{H}_1 \mathcal{F}(k) \mathcal{G}_{11}) y(k) + (\mathcal{A}_1 + \mathcal{H}_1 \mathcal{F}(k) \mathcal{G}_{12})
\]
\[+ (\mathcal{B}_1 + \mathcal{H}_1 \mathcal{F}(k) \mathcal{G}_{13}) f(y(k - \tau(k))) + u(k) \]
\[z(k) = \mathcal{D}_1 f(y(k))\]

where

\[
\mathcal{E}_1 = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}, \quad \mathcal{A}_1 = \begin{bmatrix} 0.2 & 0.4 \\ 0.4 & 0.1 \end{bmatrix}, \quad \mathcal{B}_1 = \begin{bmatrix} -0.6 & -0.5 \\ 0.5 & 0.4 \end{bmatrix}, \quad \mathcal{D}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},
\]

\[
\mathcal{H}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathcal{G}_{11} = \begin{bmatrix} 0.51 & 0.32 \\ 0.15 & 0.25 \end{bmatrix}, \quad \mathcal{G}_{12} = \begin{bmatrix} 0.2 & 0.6 \\ 0.5 & 0.4 \end{bmatrix}, \quad \mathcal{G}_{13} = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix},
\]

**Mode Rule 2:** IF \( w(k) = y_2(k) \) is “1”, THEN

\[
y(k + 1) = - (\mathcal{E}_2 + \mathcal{H}_2 \mathcal{F}(k) \mathcal{G}_{21}) y(k) + (\mathcal{A}_2 + \mathcal{H}_2 \mathcal{F}(k) \mathcal{G}_{22})
\]
\[+ (\mathcal{B}_2 + \mathcal{H}_2 \mathcal{F}(k) \mathcal{G}_{23}) f(y(k - \tau(k))) + u(k) \]
\[z(k) = \mathcal{D}_2 f(y(k))\]

where

\[
\mathcal{E}_2 = \begin{bmatrix} 0.6 & 0 \\ 0 & 0.6 \end{bmatrix}, \quad \mathcal{A}_2 = \begin{bmatrix} -0.4 & 0.22 \\ 0.2 & 0.25 \end{bmatrix}, \quad \mathcal{B}_2 = \begin{bmatrix} 0.34 & 0.33 \\ 0.4 & 0.5 \end{bmatrix}, \quad \mathcal{D}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},
\]

\[
\mathcal{H}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathcal{G}_{21} = \begin{bmatrix} 0.3 & 0.2 \\ 0.2 & 0.1 \end{bmatrix}, \quad \mathcal{G}_{22} = \begin{bmatrix} 0.5 & 0.6 \\ 0.4 & 0.2 \end{bmatrix}, \quad \mathcal{G}_{23} = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix},
\]

and \( \mathcal{F}(k) = diag\{\sin(k), \cos(k)\} \) the activation functions are \( F_1 = 0, \ F_2 = 0.3I \), the membership functions are given as \( h_1 = (1 - \sin(y_1(k))) / 2 \), and \( h_2 = (1 + \sin(y_1(k))) / 2 \).

Let \( R = I \), \( c_1 = 1 \), \( \vartheta = 5 \), \( N = 30 \), \( \tau_M = 0.5 \), and \( \mu = 2 \), by solving the LMI based robustly finite-time passivity conditions in Theorem 4.2 using Matlab LMI toolbox, we can obtain the feasible solutions as follows:

\[
\mathcal{P} = \begin{bmatrix} 15.3083 & 0.6592 \\ 0.6592 & 12.0839 \end{bmatrix}, \quad \mathcal{Q} = \begin{bmatrix} 8.1040 & -0.3226 \\ -0.3226 & 4.8648 \end{bmatrix}, \quad \mathcal{W}_1 = \begin{bmatrix} 17.8159 & 0 \\ 0 & 17.8159 \end{bmatrix},
\]

\[
\mathcal{W}_2 = \begin{bmatrix} 22.2845 & 0 \\ 0 & 22.2845 \end{bmatrix}, \quad \mathcal{W}_3 = \begin{bmatrix} 6.9637 & 0 \\ 0 & 6.9637 \end{bmatrix},
\]

\( \gamma = 12.6626, \ \varepsilon_1 = 6.4903, \ \varepsilon_2 = 4.7010, \ \varepsilon_2 = 1.8330e + 009. \)

6. Conclusion

This paper investigated the problem of discrete-time T-S fuzzy neural networks with time-varying delays. By constructing suitable Lyapunov-Krasovskii functional and using passivity theory sufficient conditions are derived to guarantee stability of concerned neural networks. Thus conditions for finite-time boundedness and passivity of T-S fuzzy neural networks are given in terms of LMIs which can be easily verified via the LMI toolbox. Finally, the effectiveness and superiority has been shown through numerical examples.
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References


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