CONNECTING $\top$ AND LATTICE-VALUED CONVERGENCES

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Abstract. $\top$-filters can be used to define $\top$-convergence spaces in the lattice-valued context. Connections between $\top$-convergence spaces and lattice-valued convergence spaces are given. Regularity of a $\top$-convergence space has recently been defined and studied by Fang and Yue. An equivalent characterization is given in the present work in terms of convergence of closures of $\top$-filters. Moreover, a compactification of a $\top$-convergence space is constructed whenever L is a complete Boolean algebra.

1. Introduction

Lowen [13] introduced the notion of prefilter for the study of lattice-valued spaces L^X, where $L = [0, 1]$. Höhle [9] defined $\top$-filters which are suited for the study of more general lattice-valued spaces. Further results on $\top$-filters can be found in the works by García and Vicente [7].

The study made here is a continuation of that initiated by Fang and Yu [3]. They used $\top$-filters to define a new category of convergence spaces in the lattice context. Kowalsky [12] introduced a diagonal condition which characterizes whenever a (classical) convergence space is topological. Cook and Fischer [1] showed that the dual of Kowalsky's diagonal condition characterizes whenever a convergence space is regular. Gähler [6] and Jäger [11] have extended these results to the lattice-valued setting.

Connections between $\top$-convergence spaces and the more familiar stratified L-convergence spaces are given. Fang and Yue [4] defined a diagonal filter in the category of $\top$-convergence spaces and used this to define regularity in terms of a diagonal condition. Here, a closure operator is defined and an equivalent definition of regularity is given in terms of convergence of closures of $\top$-filters. Further, whenever L is a complete Boolean algebra, each $\top$-convergence space is shown to possess a $\top$-compactification.

2. Preliminaries

A lattice $(L, \wedge, \vee)$ is called a complete Heyting algebra or frame provided it is complete and obeys $\alpha \wedge \left(\bigvee_{j \in J} \beta_j \right) = \bigvee_{j \in J} (\alpha \wedge \beta_j)$ for all $\alpha, \beta_j \in L$, $j \in J$. The implication operator $\longrightarrow: L \times L \longrightarrow L$ is defined by $\alpha \longrightarrow \beta = \vee\{\delta \in L : \alpha \wedge \delta \leq \beta\} = \max\{\delta \in L : \alpha \wedge \delta \leq \beta\}$ since meets distribute over infinite joins. Let $\bot(\top)$

Received: January 2017; Revised: August 2017; Accepted: November 2017
Key words and phrases: Lattice-valued convergence, $\top$-convergence spaces, Regularity, Compactification, Topological.

denote the bottom(top) member of the complete lattice L, respectively. If X is any set, then the lattice operations on L can be extended pointwise to L^X as follows: $\Big(\bigvee_{j\in J} a_j\Big)(x) = \bigvee_{j\in J} a_j(x)$, $\Big(\bigwedge_{j\in J} a_j\Big)(x) = \bigwedge_{j\in J} a_j(x)$, and $(a \longrightarrow b)(x) = a(x) \longrightarrow b(x)$ for each $x \in X$. Then $(L^X, \wedge, \vee)$ is also a frame, and $\mathbf{1}_\varnothing(\mathbf{1}_X)$ are bottom(top) members of L^X, respectively. Here $\mathbf{1}_A$ denotes the characteristic function of the set A. For $\alpha \in L$, let $\alpha\mathbf{1}_X \in L^X$ denote the constant function with value α.

Definition 2.1. Let L be a frame. A map $\nu : L^X \longrightarrow L$ is called a stratified L-filter provided:

(F1) $\nu(\mathbf{1}_\varnothing) = \bot$, and $\nu(\alpha\mathbf{1}_X) \geq \alpha$, each $\alpha \in L$,
(F2) $a \leq b$ implies $\nu(a) \leq \nu(b)$, and
(F3) $\nu(a) \wedge \nu(b) \leq \nu(a \wedge b)$, for each $a, b \in L^X$.

Let $\mathfrak{F}^S_L(X)$ be the set of all stratified L-filters on X. If ν_1 and ν_2 are two stratified L-filters on X, denote $\nu_1 \leq \nu_2$ whenever $\nu_1(a) \leq \nu_2(a)$ for each $a \in L^X$.

Let $f : X \longrightarrow Y$ be a map. In [3], $f^{\rightarrow} : L^X \longrightarrow L^Y$ and $f^{\leftarrow} : L^Y \longrightarrow L^X$ are defined respectively by $f^{\rightarrow}(a)(y) = \vee\{a(x) : f(x) = y\}$ for each $a \in L^X$, $y \in Y$, and $f^{\leftarrow}(b) = b \circ f$ for all $b \in L^Y$. Next, if $\mu \in \mathfrak{F}^S_L(X)$, then the image of μ under f is defined by $f^{\Uparrow}(\mu)(b) = \mu(f^{\leftarrow}(b))$, for each $b \in L^X$. Further, if $\nu \in \mathfrak{F}^S_L(Y)$, then the inverse image of ν under f is defined as $f^{\Downarrow}(\nu)(a) = \vee\{\nu(b) : f^{\leftarrow}(b) \leq a\}$ whenever it exists. It is straightforward to check that $\mathfrak{F}^S_L(X)$ has a smallest element $\nu_\bot$ defined by $\nu_\bot(a) = \bigwedge_{x\in X} a(x)$. Moreover, for $x \in X$, define $\langle x\rangle \in \mathfrak{F}^S_L(X)$ by $\langle x\rangle(a) = a(x)$, for each $a \in L^X$. A Zorn's Lemma argument easily shows that each stratified L-filter on X is contained in a maximal stratified L-filter, called a stratified L-ultrafilter. Höhle [9] proved the following fundamental results:

Theorem 2.2. *(See* [9]*.) Suppose that L is a frame and ν is a stratified L-filter on X. Then for all $a \in L^X$,*

(i) ν is a stratified L-ultrafilter on X if and only if $\nu(a) = \nu(a \longrightarrow \mathbf{1}_\varnothing) \longrightarrow \bot$,
(ii) if ν is a stratified L-ultrafilter, then $\nu(a \longrightarrow \mathbf{1}_\varnothing) = \nu(a) \longrightarrow \bot$, and
(iii) $\bigwedge_{x\in X} a(x) \leq \nu(a) \leq \Big(\bigvee_{x\in X} a(x) \longrightarrow \bot\Big) \longrightarrow \bot$.

If L is a frame, X a set, and $a, b \in L^X$, define $[a, b] := \bigwedge_{x\in X} \big(a(x) \longrightarrow b(x)\big)$. Note that if $a \leq b$, then $a(x) \longrightarrow b(x) = \top$ for each $x \in X$ and thus $[a, b] = \top$. It follows that $[a, b]$ is a measure of the degree for which $a \leq b$.

The following lemma is a collection of properties of the implication operator.

Lemma 2.3. *Let L be a frame and X a set. Then,*

(i) $\alpha \leq \beta$ if and only if $\alpha \longrightarrow \beta = \top$,
(ii) $\alpha \wedge \beta = \alpha \wedge (\alpha \longrightarrow \beta)$,
(iii) $(\alpha \longrightarrow \gamma) \wedge (\beta \longrightarrow \delta) \leq (\alpha \wedge \beta) \longrightarrow (\gamma \wedge \delta)$,
(iv) $\bigwedge_{j\in J}(\alpha \longrightarrow \beta_j) = \alpha \longrightarrow \big(\bigwedge_{j\in J} \beta_j\big)$,

(v) $\bigwedge_{j\in J}(\alpha_j \longrightarrow \beta) = \big(\bigvee_{j\in J}\alpha_j\big) \longrightarrow \beta$,
(vi) $a \le b$ *if and only if* $[a,b] = \top$,
(vii) $[a, b\wedge c] = [a,b]\wedge[a,c]$,
(viii) $[a,b] \le [b,c] \longrightarrow [a,c]$ *and* $[b,c] \le [a,b] \longrightarrow [a,c]$,
(ix) $[b\vee c, a] = [b,a]\wedge[c,a]$,
(x) $[c,a] \le [b,a]$ *whenever* $b \le c$, *and*
(xi) $[a,b] \le [f^{\rightarrow}(a), f^{\rightarrow}(b)]$ *and* $[c,d] \le [f^{\leftarrow}(c), f^{\leftarrow}(d)]$ *whenever* $f : X \longrightarrow Y$ *is a map.*

The notion of a $\top$-filter is due to Höhle [8]. A particular version which follows here is due to Fang and Yu [3].

Definition 2.4. [3] Suppose that L is a frame and X a set. A non-empty subset $\mathfrak{F} \subseteq L^X$ is called a $\top$-filter provided:

($\top$F1) $\bigvee_{x\in X} b(x) = \top$ for each $b \in \mathfrak{F}$,
($\top$F2) if $a, b \in \mathfrak{F}$, then $a\wedge b \in \mathfrak{F}$, and
($\top$F3) if $\bigvee_{b\in\mathfrak{F}} [b,d] = \top$, then $d \in \mathfrak{F}$.

Let $\mathfrak{F}_L^{\top}(X)$ denote the set of all $\top$-filters on X. Let $x \in X$, define $[x] := \{a \in L^X : a(x) = \top\}$. One can show that $[x]$ is a $\top$-filter on X.

It is often convenient to work with $\top$-filter bases as defined below.

Definition 2.5. A non-empty subset $\mathfrak{B} \subseteq L^X$ is said to be a $\top$-filter base whenever:

($\top$B1) for each $b \in \mathfrak{B}$, $\bigvee_{x\in X} b(x) = \top$, and
($\top$B2) if $a_1, a_2 \in \mathfrak{B}$ then $\bigvee_{b\in\mathfrak{B}} [b, a_1\wedge a_2] = \top$.

According to [3], a $\top$-filter base $\mathfrak{B}$ generates the $\top$-filter $\mathfrak{F} = \{a \in L^X : \bigvee_{b\in\mathfrak{B}} [b,a] = \top\}$; that is, $\mathfrak{F}$ is the smallest $\top$-filter containing $\mathfrak{B}$. Moreover, if $f : X \longrightarrow Y$ is a map, then the image $f^{\Rightarrow}(\mathfrak{B}) = \{f^{\rightarrow}(b) : b \in \mathfrak{B}\}$ is a $\top$-filter base, and the image of a $\top$-filter $\mathfrak{F}$, denoted by $f^{\Rightarrow}(\mathfrak{F})$, is defined to be the $\top$-filter on Y having the $\top$-filter base $\{f^{\rightarrow}(a) : a \in \mathfrak{F}\}$. Further, if $\mathfrak{G}$ is a $\top$-filter on Y, then the inverse image of $\mathfrak{G}$, denoted by $f^{\Leftarrow}(\mathfrak{G})$, exists if and only if $\bigvee_{x\in X} a(f(x)) = \top$ for each $a \in \mathfrak{G}$. In this case, $f^{\Leftarrow}(\mathfrak{G})$ is defined to be the $\top$-filter on X whose $\top$-filter base is $\{f^{\leftarrow}(a) : a \in \mathfrak{G}\}$.

Definition 2.6. Assume that L is a frame and X a set. A function $q : \mathfrak{F}_L^{\top}(X) \longrightarrow 2^X$ is called a $\top$-convergence structure on X provided:

($\top$CS1) $[x] \xrightarrow{q} x$ for all $x \in X$, and
($\top$CS2) if $\mathfrak{F} \xrightarrow{q} x$ and $\mathfrak{F} \subseteq \mathfrak{G}$, then $\mathfrak{G} \xrightarrow{q} x$.

Note that $\mathfrak{F} \xrightarrow{q} x$ is shorthand for $x \in q(\mathfrak{F})$. The pair (X, q) is called a $\top$-convergence space.

A map $f : (X, q) \longrightarrow (Y, p)$ between two $\top$-convergence spaces is continuous if $f^{\Rightarrow}(\mathfrak{F}) \xrightarrow{p} f(x)$ whenever $\mathfrak{F} \xrightarrow{q} x$. Let $\top$-**Conv** denote the category whose objects are all the $\top$-convergence spaces and whose morphisms are all the continuous maps between objects.

Throughout this paper, if Cat is a category, we let $|\mathsf{Cat}|$ denote the objects of the category Cat and will write $A \in |\mathsf{Cat}|$ to mean A is an object of Cat.

Definition 2.7. Suppose that L is a frame and X a set. The pair $(X, \overline{q})$, where $\overline{q} = (q_\alpha)_{\alpha \in L}$ and $q_\alpha : \mathfrak{F}_L^S(X) \longrightarrow 2^X$, is called a stratified L-convergence space provided it satisfies:

(SL1) $\langle x \rangle \xrightarrow{q_\alpha} x$ and $\nu_\perp \xrightarrow{q_\perp} x$, for each $x \in X$ and $\alpha \in L$,

(SL2) $\mu \geq \nu \xrightarrow{q_\alpha} x$ implies $\mu \xrightarrow{q_\alpha} x$, and

(SL3) if $\mu \xrightarrow{q_\beta} x$ and $\alpha \leq \beta$, then $\mu \xrightarrow{q_\alpha} x$.

Again, note that $\mu \xrightarrow{q_\alpha} x$ is shorthand for $x \in q_\alpha(\mu)$. A map $f : (X, \overline{q}) \longrightarrow (Y, \overline{p})$ between two stratified L-convergence spaces is said to be continuous provided that $f^{\Uparrow}(\mu) \xrightarrow{p_\alpha} f(x)$ whenever $\mu \xrightarrow{q_\alpha} x$. Let S$L$-CS denote the category whose objects are all the stratified L-convergence spaces and whose morphisms are all the continuous maps between objects.

The following results due to Höhle ([9], [8]) are needed.

Theorem 2.8. *(See* [9], [8]*.)*

(i) Assume that L is a frame, $\mathfrak{F} \in \mathfrak{F}_L^\top(X)$, and define $\nu_{\mathfrak{F}}(a) = \bigvee_{b \in \mathfrak{F}} [b, a]$ for each $a \in L^X$. Then $\nu_{\mathfrak{F}} \in \mathfrak{F}_L^S(X)$ and $\mathfrak{F} = \{a \in L^X : \nu_{\mathfrak{F}}(a) = \top\}$.

(ii) Suppose that the frame L is also a Boolean algebra, $\nu \in \mathfrak{F}_L^S(X)$, and define $\mathfrak{F}_\nu = \{a \in L^X : \nu(a) = \top\}$. Then the map $\nu \mapsto \mathfrak{F}_\nu$ is an order preserving bijection from $\mathfrak{F}_L^S(X)$ onto $\mathfrak{F}_L^\top(X)$. In particular, a $\top$-filter is maximal if and only if $\nu_{\mathfrak{F}}$ is maximal.

3. Embedding $\top$-Conv in SL-CS

It is shown in this section that the category $\top$-**Conv** can be embedded in SL-CS.

Lemma 3.1. *Suppose that L is a frame, X a set and $\mathfrak{B}$ a $\top$-filter base for the $\top$-filter $\mathfrak{F}$. Then for $d \in L^X$, $\bigvee_{b \in \mathfrak{B}} [b, d] = \bigvee_{b_1, b_2 \in \mathfrak{B}} [b_1 \wedge b_2, d] = \bigvee_{a \in \mathfrak{F}} [a, d]$.*

Proof. Assume that $d \in L^X$, $c \in \mathfrak{B}$, $a \in \mathfrak{F}$, then according to Lemma 2.3 (viii), $[b, a] \leq [a, d] \longrightarrow [b, d]$. Since $a \in \mathfrak{F}$, $\top = \bigvee_{b \in \mathfrak{B}} [b, a] \leq \bigvee_{b \in \mathfrak{B}} \big([a, d] \longrightarrow [b, d]\big) \leq [a, d] \longrightarrow \bigvee_{b \in \mathfrak{B}} [b, d]$. Then since $\top = [a, d] \longrightarrow \bigvee_{b \in \mathfrak{B}} [b, d]$, by Lemma 2.3 (i), $[a, d] \leq \bigvee_{b \in \mathfrak{B}} [b, d]$. Consequently, $\bigvee_{a \in \mathfrak{F}} [a, d] \leq \bigvee_{b \in \mathfrak{B}} [b, d]$, and since $\mathfrak{B} \subseteq \mathfrak{F}$, $\bigvee_{a \in \mathfrak{F}} [a, d] = \bigvee_{b \in \mathfrak{B}} [b, d]$.

Next, fix $b_1, b_2 \in \mathfrak{B}$; then since $\mathfrak{B}$ is a $\top$ filter base, $\bigvee_{b \in \mathfrak{B}} [b, b_1 \wedge b_2] = \top$, and it follows from the definition of $\mathfrak{F}$ that $b_1 \wedge b_2 \in \mathfrak{F}$. Then $\mathfrak{B} \subseteq \{b_1 \wedge b_2 : b_1, b_2 \in \mathfrak{B}\} \subseteq \mathfrak{F}$, so that $\bigvee_{b\in\mathfrak{B}}[b,d] \le \bigvee_{b_1,b_2\in\mathfrak{B}}[b_1\wedge b_2,d] \le \bigvee_{a\in\mathfrak{F}}[a,d]$ and by the previous part of this proof, we have equality throughout this last expression. □

Lemma 3.2. *Let L be a frame, $f : X \longrightarrow Y$ a map, and $\mathfrak{F} \in \mathfrak{F}^{\top}_L(X)$; then $f^{\Uparrow}(\nu_{\mathfrak{F}}) = \nu_{f^{\Rightarrow}(\mathfrak{F})}$.*

Proof. Let $c \in L^X$. By definition, $f^{\Uparrow}(\nu_{\mathfrak{F}})(c) = \nu_{\mathfrak{F}}(f^{\leftarrow}(c)) = \bigvee_{d\in\mathfrak{F}}[d,f^{\leftarrow}(c)]$. According to Lemma 2.3 (xi), $[d,f^{\leftarrow}(c)] \le [f^{\rightarrow}(d), f^{\rightarrow}(f^{\leftarrow}(c))]$. Since $f^{\rightarrow}(f^{\leftarrow}(c)) \le c$ and $[\,\cdot\,,\cdot\,]$ is increasing in the second component, we have $f^{\Uparrow}(\mu_{\mathfrak{F}})(c) \le \bigvee_{d\in\mathfrak{F}}[f^{\rightarrow}(d),c]$. Since $\{f^{\rightarrow}(d) : d\in\mathfrak{F}\}$ is a $\top$-filter base for $f^{\Rightarrow}(\mathfrak{F})$, it follows from Lemma 3.1 that $\bigvee_{d\in\mathfrak{F}}[f^{\rightarrow}(d),c] = \bigvee_{e\in f^{\Rightarrow}(\mathfrak{F})}[e,c] = \nu_{f^{\Rightarrow}(\mathfrak{F})}(c)$. Hence $f^{\Uparrow}(\nu_{\mathfrak{F}}) \le \nu_{f^{\Rightarrow}(\mathfrak{F})}$.

Conversely, $\nu_{f^{\Rightarrow}(\mathfrak{F})}(c) = \bigvee_{b\in\mathfrak{F}}[f^{\rightarrow}(b),c]$ and by Lemma 2.3 (xi), $[f^{\rightarrow}(b),c] \le [f^{\leftarrow}(f^{\rightarrow}(b)), f^{\leftarrow}(c)]$. Since $b \le f^{\leftarrow}(f^{\rightarrow}(b))$, it follows that $\nu_{f^{\Rightarrow}(\mathfrak{F})}(c) \le \bigvee_{b\in\mathfrak{F}}[b,f^{\leftarrow}(c)] = \nu_{\mathfrak{F}}(f^{\leftarrow}(c))$, and thus $\nu_{f^{\Rightarrow}(\mathfrak{F})} \le f^{\Uparrow}(\nu_{\mathfrak{F}})$. Therefore $f^{\Uparrow}(\nu_{\mathfrak{F}}) = \nu_{f^{\Rightarrow}(\mathfrak{F})}$. □

Definition 3.3. Given $(X,q) \in |\top\text{-}\mathbf{Conv}|$, we define $(X,\overline{q}_*)$, $\overline{q}_* = (q_{*,\alpha})_{\alpha\in L}$ as follows:

(i) $\mu \xrightarrow{q_{*,\top}} x$ if and only if there exists $\mathfrak{F} \xrightarrow{q} x$ such that $\mu \ge \nu_{\mathfrak{F}}$, and

(ii) for $\alpha < \top$, $\mu \xrightarrow{q_{*,\alpha}} x$ if and only if $\mu \ge \nu_{\perp}$.

Note that $\langle x\rangle = \nu_{[x]}$, and since $[x] \xrightarrow{q} x$, $\langle x\rangle \xrightarrow{q_{*,\alpha}} x$ for each $x \in X$ and $\alpha \in L$. It follows that $(X,\overline{q}_*)$ is a stratified L-convergence space.

Definition 3.4. Further, given $(X,q) \in |\top\text{-}\mathbf{Conv}|$, define $\overline{q}^* = (q^*_{\alpha})_{\alpha\in L}$ as follows:

(a) $\mu \xrightarrow{q^*_{\perp}} x$ if and only if $\mu \ge \nu_{\perp}$, and

(b) for $\alpha > \perp$, $\mu \xrightarrow{q^*_{\alpha}} x$ if and only if there exists $\mathfrak{F} \xrightarrow{q} x$ such that $\mu \ge \nu_{\mathfrak{F}}$.

It follows that $(X,\overline{q}^*)$ is a stratified L-convergence space. Observe that for each $\alpha \in L$, $q^*_{\alpha} \ge q_{*,\alpha}$ and thus $\overline{q}^* \ge \overline{q}_*$. Let E_* and E^* denote the full subcategories of SL-CS whose objects are of the form $(X,\overline{q}_*)$ and $(X,\overline{q}^*)$, respectively, where $(X,q) \in |\top\text{-}\mathbf{Conv}|$.

Lemma 3.5. *The categories $\top$-**Conv**, E_* and E^* are isomorphic.*

Proof. Let $\theta : \top\text{-}\mathbf{Conv} \longrightarrow E_*$ be defined by $\theta(X,q) = (X,\overline{q}_*)$ and $\theta(f) = f$. Suppose that $f : (X,q) \longrightarrow (Y,p)$ is continuous in $\top$-**Conv**; it is shown that $f : (X,\overline{q}_*) \longrightarrow (Y,\overline{p}_*)$ is continuous in E_*. If $\mu \xrightarrow{q_{*,\top}} x$, then by definition there exists $\mathfrak{F} \xrightarrow{q} x$ with $\mu \ge \nu_{\mathfrak{F}}$. By Lemma 3.2, $f^{\Uparrow}(\mu) \ge f^{\Uparrow}(\nu_{\mathfrak{F}}) = \nu_{f^{\Rightarrow}(\mathfrak{F})}$, and since $f^{\Rightarrow}(\mathfrak{F}) \xrightarrow{p} f(x)$, it follows that $f^{\Uparrow}(\nu_{\mathfrak{F}}) \xrightarrow{p_{*,\top}} f(x)$ and hence $f^{\Uparrow}(\mu) \xrightarrow{p_{*,\top}} f(x)$. Next, if $\mu \xrightarrow{q_{*,\alpha}} x$, $\alpha < \top$, then $\mu \ge \nu_{\perp}$ on X and thus $f^{\Uparrow}(\mu) \ge f^{\Uparrow}(\nu_{\perp}) \xrightarrow{p_{*,\alpha}} f(x)$ and $f^{\Uparrow}(\mu) \xrightarrow{p_{*,\alpha}} f(x)$. Therefore $f : (X,\overline{q}_*) \longrightarrow (Y,\overline{p}_*)$ is continuous and θ is a functor.

By definition, θ is a surjection onto the objects of E_*; next we show θ is an injection. Assume that $\theta(X,q)=\theta(X,p)$ and $\mathfrak{F}\xrightarrow{q}x$. We must show that $\mathfrak{F}\xrightarrow{p}x$. We have that $\nu_{\mathfrak{F}}\xrightarrow{q_{*,\top}}x$ and thus $\nu_{\mathfrak{F}}\xrightarrow{p_{*,\top}}x$. Thus there exists $\mathfrak{G}\xrightarrow{p}x$ such that $\nu_{\mathfrak{F}}\geq\nu_{\mathfrak{G}}$. If $c\in\mathfrak{G}$, then $\nu_{\mathfrak{G}}(c)=\top$ and thus $\nu_{\mathfrak{F}}(c)=\top$. By Theorem 2.8 (i) this implies that $c\in\mathfrak{F}$. Hence $\mathfrak{F}\geq\mathfrak{G}$ and $\mathfrak{F}\xrightarrow{p}x$, and thus $p=q$.

Finally, suppose that $f:(X,\overline{q}_*)\longrightarrow(Y,\overline{p}_*)$ is continuous in E_*; it is shown that $f:(X,q)\longrightarrow(Y,p)$ is continuous in $\top$-**Conv**. Assume that $\mathfrak{F}\xrightarrow{q}x$; then $\nu_{\mathfrak{F}}\xrightarrow{q_{*,\top}}x$ and thus $f^{\Uparrow}(\nu_{\mathfrak{F}})=\nu_{f^{\Rightarrow}(\mathfrak{F})}\xrightarrow{p_{*,\top}}f(x)$ by the continuity of f. It follows that there exists $\mathfrak{G}\xrightarrow{p}f(x)$ such that $\nu_{f^{\Rightarrow}(\mathfrak{F})}\geq\nu_{\mathfrak{G}}$, and thus as before $f^{\Rightarrow}(\mathfrak{F})\geq\mathfrak{G}$. Then $f^{\Rightarrow}(\mathfrak{F})\xrightarrow{p}f(x)$ and thus $f:(X,q)\longrightarrow(Y,p)$ is continuous in $\top$-**Conv**. Therefore $\theta:\top$-**Conv**$\longrightarrow E_*$ is an isomorphism.

Next, we show in a similar fashion that $\top$-**Conv** and E^* are isomorphic. Let $\phi:\top$-**Conv**$\longrightarrow E^*$ be defined by $\phi(X,q)=(X,\overline{q}^*)$ and $\phi(f)=f$. Suppose that $f:(X,q)\longrightarrow(Y,p)$ is continuous in $\top$-**Conv**; it is shown that $f:(X,\overline{q}^*)\longrightarrow(Y,\overline{p}^*)$ is continuous in E^*. If $\mu\xrightarrow{q^*_\alpha}x$, $\alpha>\bot$, then by definition there exists $\mathfrak{F}\xrightarrow{q}x$ such that $\mu\geq\nu_{\mathfrak{F}}$. Then by Lemma 3.2, $f^{\Uparrow}(\mu)\geq f^{\Uparrow}(\nu_{\mathfrak{F}})=\nu_{f^{\Rightarrow}(\mathfrak{F})}$, and since $f^{\Rightarrow}(\mathfrak{F})\xrightarrow{p}f(x)$, it follows that $f^{\Uparrow}(\mu)\xrightarrow{p^*_\alpha}f(x)$. Next, if $\mu\xrightarrow{q^*_\bot}x$ then $\mu\geq\nu_\bot$ on X and thus $f^{\Uparrow}(\mu)\geq f^{\Uparrow}(\nu_\bot)\xrightarrow{p^*_\bot}f(x)$ and thus $f^{\Uparrow}(\mu)\xrightarrow{p^*_\bot}f(x)$. Therefore $f:(X,\overline{q}^*)\longrightarrow(Y,\overline{p}^*)$ is continuous and ϕ is a functor.

By definition ϕ is a surjection onto the objects of E^*; next we show it is an injection. Assume that $\phi(X,q)=\phi(X,p)$ and $\mathfrak{F}\xrightarrow{q}x$. We must show that $\mathfrak{F}\xrightarrow{p}x$. We have that for each $\alpha>\bot$, $\nu_{\mathfrak{F}}\xrightarrow{q^*_\alpha}x$ and thus for each $\alpha>\bot$, we also have $\nu_{\mathfrak{F}}\xrightarrow{p^*_\alpha}x$. Thus there exists $\mathfrak{G}\xrightarrow{p}x$ such that $\nu_{\mathfrak{F}}\geq\nu_{\mathfrak{G}}$. If $c\in\mathfrak{G}$, then $\nu_{\mathfrak{G}}(c)=\top$ and thus $\nu_{\mathfrak{F}}(c)=\top$. By Theorem 2.8 (i), this implies that $c\in\mathfrak{F}$. Hence $\mathfrak{F}\geq\mathfrak{G}$ and $\mathfrak{F}\xrightarrow{p}x$, and thus $p=q$.

Finally, suppose that $f:(X,\overline{q}^*)\longrightarrow(Y,\overline{p}^*)$ is continuous; it is shown that $f:(X,q)\longrightarrow(Y,p)$ is continuous. Assume that $\mathfrak{F}\xrightarrow{q}x$; then $\nu_{\mathfrak{F}}\xrightarrow{q^*_\alpha}x$ and thus $f^{\Uparrow}(\nu_{\mathfrak{F}})=\nu_{f^{\Rightarrow}(\mathfrak{F})}\xrightarrow{p^*_\alpha}f(x)$ by the continuity of f. Therefore, there exists $\mathfrak{G}\xrightarrow{p}f(x)$ with $\nu_{f^{\Rightarrow}(\mathfrak{F})}\geq\nu_{\mathfrak{G}}$, and thus as before $f^{\Rightarrow}(\mathfrak{F})\geq\mathfrak{G}$. Then $f^{\Rightarrow}(\mathfrak{F})\xrightarrow{p}f(x)$ and so $f:(X,q)\longrightarrow(Y,p)$ is continuous. Therefore $\phi:\top$-**Conv**$\longrightarrow E^*$ is an isomorphism. □

Lemma 3.6. *Assume that the frame L is also a Boolean algebra, and let $(X,\overline{q})$ be a stratified L-convergence space. Then there exists $(X,Q)\in|\top$-**Conv**$|$ such that $(X,\overline{Q}_*)\in E_*$ with $q_\top=Q_{*,\top}$.*

Proof. Given $(X,\overline{q})$, where $\overline{q}=(q_\alpha)_{\alpha\in L}$, define Q as follows: $\mathfrak{F}\xrightarrow{Q}x$ if and only if $\nu_{\mathfrak{F}}\xrightarrow{q_\top}x$. Then $[x]\xrightarrow{Q}x$ since $\nu_{[x]}=\langle x\rangle$ and if $\mathfrak{G}\geq\mathfrak{F}\xrightarrow{Q}x$, it follows that $\nu_{\mathfrak{G}}\geq\nu_{\mathfrak{F}}$ and thus $\nu_{\mathfrak{G}}\xrightarrow{q_\top}x$. Hence $\mathfrak{G}\xrightarrow{Q}x$ and (X,Q) is a $\top$-convergence space.

As defined above, $(X, \overline{Q}_*) \in E_*$, where $\mu \xrightarrow{Q_{*,\top}} x$ if and only if there exists $\mathfrak{F} \xrightarrow{Q} x$ such that $\mu \geq \nu_{\mathfrak{F}}$ and for $\alpha < \top$, $\mu \xrightarrow{Q_{*,\alpha}} x$ if and only if $\mu \geq \nu_{\perp}$.

It remains to show that $q_{\top} = Q_{*,\top}$. Assume that $\nu \xrightarrow{q_{\top}} x$. Since L is a Boolean algebra, by Theorem 2.8 (ii), $\nu_{\mathfrak{F}_{\nu}} = \nu$. Since $\nu_{\mathfrak{F}_{\nu}} = \nu \xrightarrow{q_{\top}} x$, it follows that $\mathfrak{F}_{\nu} \xrightarrow{Q} x$ and thus $\nu = \nu_{\mathfrak{F}_{\nu}} \xrightarrow{Q_{*,\top}} x$. Hence $q_{\top} \geq Q_{*,\top}$.

Conversely, suppose that $\mu \xrightarrow{Q_{*,\top}} x$; then there exists $\mathfrak{F} \xrightarrow{Q} x$ such that $\mu \geq \nu_{\mathfrak{F}}$. It follows that $\nu_{\mathfrak{F}} \xrightarrow{q_{\top}} x$ and thus $\mu \xrightarrow{q_{\top}} x$. Hence $Q_{*,\top} \geq q_{\top}$ and $q_{\top} = Q_{*,\top}$. □

Theorem 3.7. *Assume that L is a frame. Then,*

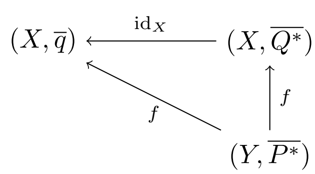
(i) $\top$-**Conv** *is embedded as a bicoreflective subcategory of SL-CS, and*

(ii) provided that L is also a Boolean algebra, $\top$-**Conv** *is embedded as a bireflective subcategory of SL-CS.*

Proof. (i) Using Lemma 3.5, it suffices to show that E^* is bicoreflective in SL-CS. Let $(X, \overline{q}) \in |\text{S}L\text{-CS}|$, where $\overline{q} = (q_{\alpha})_{\alpha \in L}$. Define Q as follows: $\mathfrak{F} \xrightarrow{Q} x$ if and only if $\nu_{\mathfrak{F}} \xrightarrow{q_{\top}} x$. Then $(X, Q) \in |\top\text{-}\mathbf{Conv}|$ and define $\overline{Q^*} = (Q^*_{\alpha})_{\alpha \in L}$ as in Definition 3.4. Then $(X, \overline{Q^*}) \in |E^*|$. It is shown that $\text{id}_X : (X, Q^*_{\alpha}) \longrightarrow (X, q_{\alpha})$ is continuous, $\perp < \alpha$.

Suppose $\mu \xrightarrow{Q^*_{\alpha}} x$; then there exists $\mathfrak{F} \xrightarrow{Q} x$ with $\mu \geq \nu_{\mathfrak{F}}$. Since $\nu_{\mathfrak{F}} \xrightarrow{q_{\top}} x$, $\nu_{\mathfrak{F}} \xrightarrow{q_{\alpha}} x$ and thus $\mu \xrightarrow{q_{\alpha}} x$. Hence $Q^*_{\alpha} \geq q_{\alpha}$, $\perp < \alpha$, and also $Q^*_{\perp} = q_{\perp}$. Then $\text{id}_X : (X, \overline{Q^*}) \longrightarrow (X, \overline{q})$ is continuous.

Let $\phi : \top\text{-}\mathbf{Conv} \longrightarrow E^*$ be as in Lemma 3.5. Consider the diagram below; where $f : (Y, \overline{P^*}) \longrightarrow (X, \overline{q})$ is continuous.



It is shown that $f : (Y, \overline{P^*}) \longrightarrow (X, \overline{Q^*})$ is continuous in E^*. Since $\phi : \top\text{-}\mathbf{Conv} \longrightarrow E^*$ is an isomorphism, it is sufficient to show that $f : (Y, P) \longrightarrow (X, Q)$ is continuous in $\top$-**Conv**.

Suppose that $\mathfrak{G} \xrightarrow{P} y$; then for $\perp < \alpha$, $\nu_{\mathfrak{G}} \xrightarrow{P^*_{\alpha}} y$ and thus by the continuity of $f : (Y, \overline{P^*}) \to (X, \overline{q})$ in SL-CS, $\nu_{f^{\Rightarrow}(\mathfrak{G})} = f^{\Uparrow}(\nu_{\mathfrak{G}}) \xrightarrow{q_{\alpha}} f(y)$. It follows from the definition of Q that $f^{\Rightarrow}(\mathfrak{G}) \xrightarrow{Q} f(y)$, and thus $f : (Y, P) \longrightarrow (X, Q)$ is continuous in $\top$-**Conv**. Hence $f : (Y, \overline{P^*}) \to (X, \overline{Q^*})$ is continuous in E^*, and thus E^* is bicoreflective in SL-CS.

(ii) Assume that $(X, \overline{q}) \in |\text{S}L\text{-CS}|$ and define Q as above; then $\mu \xrightarrow{Q_{*,\top}} x$ if and only if there exists $\mathfrak{F} \xrightarrow{Q} x$ such that $\mu \geq \nu_{\mathfrak{F}}$; otherwise $\mu \xrightarrow{Q_{*\alpha}} x$ if and only if $\mu \geq \nu_{\perp}$ for $\alpha < \top$. Then by Lemma 3.6, $q_{\top} = Q_{*,\top}$ and so $\text{id}_X : (X, \overline{q}) \to (X, \overline{Q}_*)$ is continuous. Suppose that $f : (X, \overline{q}) \to (Y, \overline{P}_*)$ is continuous and consider the diagram; where $f : (X, \overline{q}) \longrightarrow (Y, \overline{P_*})$ is continuous.

$$\begin{array}{ccc} (X,\overline{q}) & \xrightarrow{\ \mathrm{id}_X\ } & (X,\overline{Q_*}) \\ & \searrow^{f} & \downarrow f \\ & & (Y,\overline{P_*}) \end{array}$$

It remains to show that $f : (X, \overline{Q}_*) \longrightarrow (Y, \overline{P}_*)$ is continuous in E_*. Let $\theta : \top$-**Conv**$\rightarrow E_*$ be as in Lemma 3.5. Since θ is an isomorphism, it suffices to show that $f : (X, Q) \rightarrow (Y, P)$ is continuous in $\top$-**Conv**.

Assume that $\mathfrak{F} \xrightarrow{Q} x$; then $\nu_{\mathfrak{F}} \xrightarrow{q_\top} x$ and thus $\nu_{f^\Rightarrow(\mathfrak{F})} = f^\Uparrow(\nu_{\mathfrak{F}}) \xrightarrow{P_{*,\top}} f(x)$. Hence there exists $\mathfrak{G} \xrightarrow{P} f(x)$ such that $\nu_{f^\Rightarrow(\mathfrak{F})} \geq \nu_{\mathfrak{G}}$, and it follows that $f^\Rightarrow(\mathfrak{F}) \geq \mathfrak{G}$. Then $f^\Rightarrow(\mathfrak{F}) \xrightarrow{P} f(x)$ and thus $f : (X, \overline{Q}_*) \longrightarrow (Y, \overline{P}_*)$ is continuous. Therefore E_* is bireflective in SL-CS whenever L is a Boolean algebra. By Lemma 3.5, E_* and $\top$-**Conv** are isomorphic and so $\top$-**Conv** is bireflective in SL-CS. □

4. **Regularity**

Regularity for lattice-valued convergence spaces has been studied; for example, see Jäger [11] and Li and Jin [14]. The notion of closure of a $\top$-filter is defined in this section and related to regularity. The following connection between a $\top$-filter base and a $\top$-filter is useful. It will be assumed throughout this section that L is a frame.

Using Lemma 3.1 and Lemma 2.3 (xi), it is straightforward to verify that if $f : X \longrightarrow Y$ is a map and $\mathfrak{B}$ is a $\top$-filter base for $\mathfrak{F}$ on X, then $\{f^\rightarrow(b) : b \in \mathfrak{B}\}$ is a $\top$-filter base for $f^\Rightarrow(\mathfrak{F})$ on Y.

Definition 4.1. Assume that $(X, q) \in |\top$-**Conv**$|$ and $a \in L^X$. The closure of a is defined by $\overline{a}(x) = \vee\{\nu_{\mathfrak{G}}(a) : \mathfrak{G} \xrightarrow{q} x\}$, for each $x \in X$.

Some basic properties of the closure operator are listed below.

Lemma 4.2. *Let* $(X, q) \in |\top$-**Conv**$|$, $a, b \in L^X$ *and* $\alpha \in L$. *Then*

(i) $\overline{\mathbf{1}}_\varnothing = \mathbf{1}_\varnothing$,

(ii) $a \leq \overline{a}$,

(iii) $a \leq b$ *implies* $\overline{a} \leq \overline{b}$,

(iv) $\overline{a \wedge \alpha\mathbf{1}_X} = \overline{a} \wedge \alpha\mathbf{1}_X$, *and*

(v) if L *is a Boolean algebra, it follows that* $\overline{a \vee b} = \overline{a} \vee \overline{b}$.

Proof. (i)-(iv) These follow easily from the properties of stratified L-filters.

(v) Clearly $\overline{a} \vee \overline{b} \leq \overline{a \vee b}$. Employing Corollary 2.1.6 [8], $\mu(a \vee b) = \mu(a) \vee \mu(b)$ for each stratified L-ultrafilter μ on X. Since closures are determined by $\top$-ultrafilters, $\overline{a \vee b}(x) = \bigvee\{\nu_{\mathfrak{F}}(a \vee b : \mathfrak{F} \xrightarrow{q} x, \mathfrak{F}$ a $\top$-ultrafilter$\} = \bigvee\{\nu_{\mathfrak{F}}(a) \vee \nu_{\mathfrak{F}}(b) : \mathfrak{F} \xrightarrow{q} x, \mathfrak{F}$ a $\top$-ultrafilter$\} \leq \overline{a}(x) \vee \overline{b}(x)$ and thus $\overline{a \vee b} = \overline{a} \vee \overline{b}$. □

Definition 4.3. Given $(X, q) \in |\top$-**Conv**$|$ and $\mathfrak{F} \in \mathfrak{F}^\top_L(X)$, the closure of $\mathfrak{F}$, denoted by $\overline{\mathfrak{F}}$, is defined to be the $\top$-filter whose $\top$-filter base is $\{\overline{a} : a \in \mathfrak{F}\}$. Further, if $\mathfrak{B}$ is a $\top$-filter base, define $\overline{\mathfrak{B}} = \{\overline{b} : b \in \mathfrak{B}\}$.

Lemma 4.4. *Let* $(X, q) \in |\top\text{-}\mathbf{Conv}|$ *and* $b, c \in L^X$. *Then* $[b, c] \leq [\overline{b}, \overline{c}]$.

Proof. Since $[\overline{b}, \overline{c}] = \bigwedge_{x \in X} (\overline{b}(x) \longrightarrow \overline{c}(x))$, it suffices to show that for fixed $x \in X$, $[b, c] \leq \overline{b}(x) \longrightarrow \overline{c}(x)$. According to Lemma 2.3 (viii), $[b, c] \leq [a, b] \longrightarrow [a, c]$ for any $a \in L^X$. Recall that $\nu_{\mathfrak{G}}(c) = \bigvee_{h \in \mathfrak{G}} [h, c]$. Further fix $\mathfrak{G} \xrightarrow{q} x$ and let $g \in \mathfrak{G}$. Then

$$[g, c] \leq \bigvee_{h \in \mathfrak{G}} [h, c] = \nu_{\mathfrak{G}}(c) \leq \bigvee \{\nu_{\mathfrak{H}}(c) : \mathfrak{H} \xrightarrow{q} x\} = \overline{c}(x).$$

Now since $(\cdot \longrightarrow \cdot)$ is increasing in the second component, we have $[b, c] \leq [g, b] \longrightarrow [g, c] \leq [g, b] \longrightarrow \overline{c}(x)$. It follows from the distributive property in Lemma 2.3 (v) that

$$[b, c] \leq \bigwedge \{[g, b] \longrightarrow \overline{c}(x) : g \in \mathfrak{G}\} = \left(\bigvee_{g \in \mathfrak{G}} [g, b] \right) \longrightarrow \overline{c}(x) = \nu_{\mathfrak{G}}(b) \longrightarrow \overline{c}(x).$$

Thus we have,

$$[b, c] \leq \bigwedge_{\mathfrak{G} \xrightarrow{q} x} \big(\nu_{\mathfrak{G}}(b) \longrightarrow \overline{c}(x)\big) = \left(\bigvee_{\mathfrak{G} \xrightarrow{q} x} \nu_{\mathfrak{G}}(b) \right) \longrightarrow \overline{c}(x) = \overline{b}(x) \longrightarrow \overline{c}(x).$$

As this holds for any $x \in X$, $[b, c] \leq \bigwedge_{x \in X} \big(\overline{b}(x) \longrightarrow \overline{c}(x)\big) = [\overline{b}, \overline{c}]$. □

Lemma 4.5. *Let* $\mathfrak{B}$ *be a* $\top$*-filter base for the* $\top$*-filter* $\mathfrak{F}$ *on* (X, q). *Then* $\overline{\mathfrak{B}}$ *is a base for* $\overline{\mathfrak{F}}$.

Proof. Note that by Lemma 4.2 (iii), Lemma 4.4 and the fact that $[\cdot, \cdot]$ is increasing in the second component, if $b_1, b_2 \in \mathfrak{B}$, then $\bigvee_{b \in \mathfrak{B}} [\overline{b}, \overline{b_1} \wedge \overline{b_2}] \geq \bigvee_{b \in \mathfrak{B}} [\overline{b}, \overline{b_1 \wedge b_2}] \geq \bigvee_{b \in \mathfrak{B}} [b, b_1 \wedge b_2] = \top$, as $\mathfrak{B}$ is a $\top$-filter base for $\mathfrak{F}$. Also $\bigvee_{x \in X} \overline{b}(x) \geq \bigvee_{x \in X} b(x) = \top$ and thus $\overline{\mathfrak{B}}$ is a $\top$-filter base.

Let $c \in \mathfrak{F}$; then by Lemma 4.4, as $\mathfrak{B}$ is a base for $\mathfrak{F}$, $\bigvee_{b \in \mathfrak{B}} [\overline{b}, \overline{c}] \geq \bigvee_{b \in \mathfrak{B}} [b, c] = \top$. Thus $\overline{c}$ belongs to the $\top$-filter generated by $\overline{\mathfrak{B}}$; that is, the $\top$-filter generated by $\overline{\mathfrak{B}}$ includes $\{\overline{c} : c \in \mathfrak{F}\}$. Therefore $\overline{\mathfrak{B}}$ generates $\overline{\mathfrak{F}}$. □

Kowalsky [12] introduced a diagonal axiom which characterizes when a convergence space is topological. The dual of the diagonal axiom was shown by Cook and Fischer [1] to characterize when a convergence space, or topological space, is regular. An appropriate diagonal axiom is used by Fang and Yue [4] to define regularity in $\top$-**Conv**.

Let (X, q) be a $\top$-convergence space, J a non-empty set and let $\psi : J \longrightarrow X$ and $\sigma : J \longrightarrow \mathfrak{F}_L^\top(X)$ be maps such that $\sigma(j) \xrightarrow{q} \psi(j)$ for each $j \in J$. For each $b \in L^X$, define $e_b : \mathfrak{F}_L^\top(X) \longrightarrow L$ by $e_b(\mathfrak{G}) = \nu_{\mathfrak{G}}(b)$. Then, given $\mathfrak{H} \in \mathfrak{F}_L^\top(J)$ we define $\mathfrak{B}_{\mathfrak{H}} = \{b \in L^X : e_b \circ \sigma \in \mathfrak{H}\}$. We let $\underline{\kappa\sigma\mathfrak{H}}$ be the $\top$-filter generated by $\mathfrak{B}_{\mathfrak{H}}$.

The definition of $\kappa\sigma\mathfrak{H}$ is due to Fang and Yue [4]. It is shown in Lemma 3.6 of [4] that $\kappa\sigma\mathfrak{H}$ is a $\top$-filter on X, and it is referred to as the $\top$-diagonal filter of $\mathfrak{H}$. They use the diagonal filter to define regularity.

The following definition is given by Fang and Yue [4]. The notation "(TR)" is used in [4] to denote the diagonal condition.

Definition 4.6. Suppose that L is a frame and (X,q) is a $\top$-convergence space. We say that (X,q) is regular in $\top$-**Conv**, provided that for any non-empty set J and maps $\psi : J \longrightarrow X$ and $\sigma : J \longrightarrow \mathfrak{F}_L^{\top}(X)$ such that $\sigma(j) \xrightarrow{q} \psi(j)$ for all $j \in J$, $\psi^{\Rightarrow}(\mathfrak{H}) \xrightarrow{q} x$ whenever $\mathfrak{H} \in \mathfrak{F}_L^{\top}(J)$ and $\kappa\sigma\mathfrak{H} \xrightarrow{q} x$.

Lemma 4.7. *Assume that L is a frame and (X,q) is a $\top$-convergence space. Denote $J = \{(\mathfrak{G}, y) \in \mathfrak{F}_L^{\top}(X) \times X : \mathfrak{G} \xrightarrow{q} y\}$ and define $\psi : J \longrightarrow X$ by $\psi(\mathfrak{G}, y) = y$ and define $\sigma : J \longrightarrow \mathfrak{F}_L^{\top}(X)$ by $\sigma(\mathfrak{G}, y) = \mathfrak{G}$. Then for each $\mathfrak{F} \in \mathfrak{F}_L^{\top}(X)$ there exists $\hat{\mathfrak{F}} \in \mathfrak{F}_L^{\top}(J)$ such that $\mathfrak{F} \subseteq \kappa\sigma\hat{\mathfrak{F}}$.*

Proof. Suppose that $a \in L^X$; define $\hat{a} : J \longrightarrow L$ by $\hat{a}(\mathfrak{G}, y) = \nu_{\mathfrak{G}}(a)$. Then $\hat{a} \in L^J$ and note that if $a, b \in L^X$, $\widehat{(a \wedge b)}(\mathfrak{G}, y) = \nu_{\mathfrak{G}}(a \wedge b) = \nu_{\mathfrak{G}}(a) \wedge \nu_{\mathfrak{G}}(b) = \hat{a}(\mathfrak{G}, y) \wedge \hat{b}(\mathfrak{G}, y)$ and thus $\widehat{(a \wedge b)} = \hat{a} \wedge \hat{b}$. Observe that if $a \in \mathfrak{F}$, then $\bigvee_{x \in X} a(x) = \top$ and thus

$$\begin{aligned}\bigvee\{\hat{a}(\mathfrak{G}, y) : \mathfrak{G} \xrightarrow{q} y\} &\geq \bigvee_{x \in X} \hat{a}([x], x) = \bigvee_{x \in X} \nu_{[x]}(a) \\ &= \bigvee_{x \in X} \bigvee_{b \in [x]} [b, a] = \bigvee_{x \in X} [\mathbf{1}_{\{x\}}, a] \\ &= \bigvee_{x \in X} a(x) = \top.\end{aligned}$$

Thus $\bigvee\{\hat{a}(\mathfrak{G}, y) : \mathfrak{G} \xrightarrow{q} y\} = \top$. It follows that $\mathfrak{D} = \{\hat{a} : a \in \mathfrak{F}\}$ is a $\top$-filter base on J which is closed under finite infima. Let $\hat{\mathfrak{F}}$ be the $\top$-filter with base $\mathfrak{D}$.

Next it is shown that $\mathfrak{F} \subseteq \kappa\sigma\hat{\mathfrak{F}}$. Assume that $a \in \mathfrak{F}$; then $\hat{a} \in \mathfrak{D}$ and it remains to show that $e_a \circ \sigma \in \hat{\mathfrak{F}}$. Indeed, $(e_a \circ \sigma)(\mathfrak{G}, y) = e_a(\mathfrak{G}) = \nu_{\mathfrak{G}}(a) = \hat{a}(\mathfrak{G}, y)$ and so $e_a \circ \sigma = \hat{a} \in \mathfrak{D} \subseteq \hat{\mathfrak{F}}$. Thus according to the definition, $a \in \kappa\sigma\hat{\mathfrak{F}}$ and $\mathfrak{F} \subseteq \kappa\sigma\hat{\mathfrak{F}}$. $\square$

Theorem 4.8. *Suppose that L is a frame and (X,q) is a $\top$-convergence space. Then (X,q) is regular if and only if $\overline{\mathfrak{F}} \xrightarrow{q} x$ whenever $\mathfrak{F} \xrightarrow{q} x$.*

Proof. Assume that (X,q) is such that $\overline{\mathfrak{F}} \xrightarrow{q} x$ whenever $\mathfrak{F} \xrightarrow{q} x$. Suppose that $J \neq \varnothing$ is a set, $\psi : J \longrightarrow X$ and $\sigma : J \longrightarrow \mathfrak{F}_L^{\top}(X)$ are such that $\sigma(j) \xrightarrow{q} \psi(j)$ for each $j \in J$. Let $\mathfrak{H} \in \mathfrak{F}_L^{\top}(J)$ such that $\kappa\sigma\mathfrak{H} \xrightarrow{q} x$. It suffices to show that $\overline{\kappa\sigma\mathfrak{H}} \subseteq \psi^{\Rightarrow}(\mathfrak{H})$.

Recall that $\mathfrak{B}_{\mathfrak{H}} = \{b \in L^X : e_b \circ \sigma \in \mathfrak{H}\}$ is a $\top$-filter base for $\kappa\sigma\mathfrak{H}$ and $\mathfrak{B}_{\mathfrak{H}}$ is closed under finite infima. It follows from Lemma 4.5 that $\overline{\mathfrak{B}_{\mathfrak{H}}} = \{\overline{b} : b \in \mathfrak{B}_{\mathfrak{H}}\}$ is a $\top$-filter base for $\overline{\kappa\sigma\mathfrak{H}}$ on X and it suffices to show that $\overline{\mathfrak{B}_{\mathfrak{H}}} \subseteq \psi^{\Rightarrow}(\mathfrak{H})$.

Let $b \in \mathfrak{B}_{\mathfrak{H}}$; then $e_b \circ \sigma \in \mathfrak{H}$ and $\psi^{\rightarrow}(e_b \circ \sigma)(y) = \bigvee\{(e_b \circ \sigma)(j) : \psi(j) = y\} = \bigvee\{\nu_{\sigma(j)}(b) : \psi(j) = y\}$. Since $\sigma(j) \xrightarrow{q} \psi(j)$, it follows that $\psi^{\rightarrow}(e_b \circ \sigma)(y) =$

$\bigvee\{\nu_{\sigma(j)}(b) : \psi(j) = y\} \leq \bigvee\{\nu_{\mathfrak{G}}(b) : \mathfrak{G} \xrightarrow{q} y\} = \bar{b}(y)$ and thus $\psi^{\rightarrow}(e_b \circ \sigma) \leq \bar{b}$. It follows that $\bar{b} \in \psi^{\Rightarrow}(\mathfrak{H})$ and thus $\overline{\mathfrak{B}_{\mathfrak{H}}} \subseteq \psi^{\Rightarrow}(\mathfrak{H})$. Hence (X, q) is regular.

Conversely, suppose that (X, q) is regular and assume that $\mathfrak{F} \xrightarrow{q} x$. It must be shown that $\overline{\mathfrak{F}} \xrightarrow{q} x$. Let J, ψ, σ and $\hat{\mathfrak{F}} \in \mathfrak{F}_L^{\top}(J)$ be as in Lemma 4.7. According to Lemma 4.7, $\mathfrak{F} \subseteq \kappa\sigma\hat{\mathfrak{F}}$ and thus $\kappa\sigma\hat{\mathfrak{F}} \xrightarrow{q} x$ and since (X, q) is regular, $\psi^{\Rightarrow}(\hat{\mathfrak{F}}) \xrightarrow{q} x$. It remains to show that $\psi^{\Rightarrow}(\hat{\mathfrak{F}}) \subseteq \overline{\mathfrak{F}}$.

Recall that $\mathfrak{D} = \{\hat{a} : a \in \mathfrak{F}\}$ is a ⊤-filter base for $\hat{\mathfrak{F}}$ which is closed under finite infima. Then $\psi^{\rightarrow}(\hat{a})(y) = \bigvee\{\hat{a}(\mathfrak{K}, z) : \psi(\mathfrak{K}, z) = y\} = \bigvee\{\nu_{\mathfrak{G}}(a) : \mathfrak{G} \xrightarrow{q} y\} = \overline{a}(y)$, and thus $\psi^{\rightarrow}(\hat{a}) = \overline{a}$. Hence $\psi^{\Rightarrow}(\mathfrak{D}) \subseteq \overline{\mathfrak{F}}$ and it follows that $\overline{\mathfrak{F}} \xrightarrow{q} x$. □

Let $f : (X, q) \longrightarrow (Y, p)$ be a continuous map in ⊤-**Conv**. If $\mathfrak{F} \xrightarrow{q} x$, it easily follows that $\overline{\mathfrak{F}} \subseteq [x]$ and $\overline{f^{\Rightarrow}(\mathfrak{F})} \subseteq f^{\Rightarrow}(\overline{\mathfrak{F}})$.

Let ⊤-**RConv** denote the full subcategory of ⊤-**Conv** consisting of all of the regular objects in ⊤-**Conv**. Fang and Yu [3] have shown that ⊤-**Conv** is a topological construct that is also cartesian-closed. The proof of the next result uses a standard argument.

Theorem 4.9. *The category* ⊤-**RConv** *is a concretely reflective subcategory of* ⊤-**Conv**.

Proof. Note that the indiscrete ⊤-convergence structure ρ on X is regular. Since initial structures exist in ⊤-**Conv**, let σ denote the initial structure on X determined by $f_j : X \longrightarrow (Y_j, p_j)$, $j \in J$, where each $(Y_j, p_j) \in |$⊤-**RConv**$|$. Then $\mathfrak{F} \xrightarrow{\sigma} x$ if and only if $f_j^{\Rightarrow}(\mathfrak{F}) \xrightarrow{p_j} f_j(x)$, for each $j \in J$, and thus $f_j : (X, \sigma) \longrightarrow (Y_j, p_j)$ is continuous for each $j \in J$. Then for $a \in \mathfrak{F}$, $f_j^{\rightarrow}(\overline{a}^{\sigma}) \subseteq \overline{f_j^{\rightarrow}(a)}^{p_j}$ and thus $f_j^{\Rightarrow}(\overline{\mathfrak{F}}^{\sigma}) \supseteq \overline{f_j^{\Rightarrow}(\mathfrak{F})}^{p_j} \xrightarrow{p_j} f_j(x)$, for each $j \in J$. Hence $\overline{\mathfrak{F}}^{\sigma} \xrightarrow{\sigma} x$ and thus (X, σ) is regular.

Let rq denote the largest regular ⊤-convergence structure on X such that $rq \leq q$. Then $\mathrm{id}_X : (X, q) \longrightarrow (X, rq)$ is a continuous map.

Suppose that $f : (X, q) \longrightarrow (Y, p)$ is any continuous map and $(Y, p) \in |$⊤-**RConv**$|$. Let δ denote the initial ⊤-convergence structure on X defined by $f : X \longrightarrow (Y, p)$. Then $f : (X, \delta) \longrightarrow (Y, p)$ is continuous, $\delta \leq q$, and $(X, \delta) \in |$⊤-**RConv**$|$. It follows that $rq \geq \delta$ and thus $f : (X, rq) \longrightarrow (Y, p)$ is continuous. The following diagram commutes:

$$\begin{array}{ccc} (X, q) & \xrightarrow{\mathrm{id}_X} & (X, rq) \\ & \searrow^{f} & \downarrow f \\ & & (Y, p) \end{array}$$

and thus ⊤-**RConv** is concretely reflective in ⊤-**Conv**. □

Suppose that $(X, q), (Y, p) \in |$⊤-**Conv**$|$; let $C(X, Y)$ denote the set of all continuous maps from X to Y. A coarsest structure σ on $C(X, Y)$ is defined by Fang and Yu in[3], for which the evaluation map $\mathrm{ev} : (X, q) \times \big(C(X, Y), \sigma\big) \longrightarrow (Y, p)$ is jointly continuous. We mention without proof that if (Y, p) is regular, then $\big(C(X,Y),\sigma\big)$ is also regular.

Let $(X, \overline{q}) \in |\mathrm{S}L\text{-CS}|$, $\alpha \in L$, J an non-empty set and let $\psi : J \longrightarrow X$ and $\Sigma : J \longrightarrow \mathfrak{F}_L^S(X)$ be maps such that $\Sigma(j) \xrightarrow{q_\alpha} \psi(j)$ for each $j \in J$. Fix $b \in L^X$ and define $E_b : \mathfrak{F}_L^S(X) \longrightarrow L$ by $E_b(\nu) = \nu(b)$, for each $\nu \in \mathfrak{F}_L^S(X)$. Let $\mu \in \mathfrak{F}_L^S(J)$ and let $K\Sigma\mu \in \mathfrak{F}_L^S(X)$ be defined by $K\Sigma\mu(b) = \mu(E_b \circ \Sigma)$, for $b \in L^X$.

Definition 4.10. Assume that L is a frame and $(X, \overline{q}) \in |\mathrm{S}L\text{-CS}|$. Then $(X, \overline{q})$ is said to be regular in SL-CS provided that for each $\alpha \in L$, $\psi : J \longrightarrow X$, $\Sigma : J \longrightarrow \mathfrak{F}_L^S(X)$ such that $\Sigma(j) \xrightarrow{q_\alpha} \psi(j)$ and for $\mu \in \mathfrak{F}_L^S(J)$, we have that $\psi^{\Uparrow}\mu \xrightarrow{q_\alpha} x$ whenever $K\Sigma\mu \xrightarrow{q_\alpha} x$.

Lemma 4.11. *Let L be a frame, X a set, $\mathfrak{G} \in \mathfrak{F}_L^\top(X)$ and $\mu \in \mathfrak{F}_L^S(X)$. Then*
(i) $\mathfrak{F}_{\nu_\mathfrak{G}} = \mathfrak{G}$, and
(ii) $\mu = \nu_{\mathfrak{F}_\mu}$ whenever L is a Boolean algebra.

Proof. Parts (i) and (ii) follow from Theorem 2.8 (i) and (ii), respectively. □

Theorem 4.12. *Assume that the frame L is a Boolean algebra, $(X, q) \in |\top\text{-}\mathbf{Conv}|$, and let $(X, \overline{q}_*) \in |SL\text{-}CS|$ be as given in Definition 3.3. Then $(X, \overline{q}_*)$ is regular in SL-CS if and only if (X, q) is regular in $\top$-***Conv**.

Proof. Suppose that $(X, \overline{q}_*)$ is regular in SL-CS and assume that $\psi : J \longrightarrow X$, $\sigma : J \longrightarrow \mathfrak{F}_L^\top(X)$ is such that $\sigma(j) \xrightarrow{q} \psi(j)$ for each $j \in J$. Let $\mathfrak{G} \in \mathfrak{F}_L^\top(J)$ be such that $\kappa\sigma\mathfrak{G} \xrightarrow{q} x$; it is shown that $\psi^{\Rightarrow}\mathfrak{G} \xrightarrow{q} x$. Define $\Sigma(j) = \nu_{\sigma(j)}$ and since $\sigma(j) \xrightarrow{q} \psi(j)$, it follows that $\Sigma(j) \xrightarrow{q_{*,\top}} \psi(j)$ for each $j \in J$.

First it is shown that $\kappa\sigma\mathfrak{G} = \mathfrak{F}_{K\Sigma\nu_\mathfrak{G}}$. Assume that $b \in \mathfrak{F}_{K\Sigma\nu_\mathfrak{G}}$; observe that $(E_b \circ \Sigma)(j) = E_b(\nu_{\sigma(j)}) = \nu_{\sigma(j)}(b) = (e_b \circ \sigma)(j)$ and thus $E_b \circ \Sigma = e_b \circ \sigma$. Moreover, using Theorem 2.8 we have $\top = K\Sigma\nu_\mathfrak{G}(b) = \nu_\mathfrak{G}(E_b \circ \Sigma) = \nu_\mathfrak{G}(e_b \circ \sigma)$ and hence $e_b \circ \sigma \in \mathfrak{G}$. It follows that $b \in \kappa\sigma\mathfrak{G}$ and thus $\mathfrak{F}_{K\Sigma\nu_\mathfrak{G}} \subseteq \kappa\sigma\mathfrak{G}$.

Conversely, if $b \in \mathfrak{B}$, where $\mathfrak{B}$ is a base for $\kappa\sigma\mathfrak{G}$, then $e_b \circ \sigma \in \mathfrak{G}$. If follows that $K\Sigma\nu_\mathfrak{G}(b) = \nu_\mathfrak{G}(E_b \circ \Sigma) = \nu_\mathfrak{G}(e_b \circ \sigma) = \top$ since $e_b \circ \sigma \in \mathfrak{G}$. Then using Theorem 2.8, $b \in \mathfrak{F}_{K\Sigma\nu_\mathfrak{G}}$ and thus $\kappa\sigma\mathfrak{G} = \mathfrak{F}_{K\Sigma\nu_\mathfrak{G}}$.

According to Lemma 4.11, since $\kappa\sigma\mathfrak{G} \xrightarrow{q} x$, $K\Sigma\nu_\mathfrak{G} = \nu_{\mathfrak{F}_{K\Sigma\nu_\mathfrak{G}}} \xrightarrow{q_{*,\top}} x$. However, $(X, \overline{q}_*)$ is regular in SL-CS and it follows that $\psi^{\Uparrow}\nu_\mathfrak{G} \xrightarrow{q_{*,\top}} x$ and by Lemma 3.2, $\nu_{\psi^{\Rightarrow}\mathfrak{G}} = \psi^{\Uparrow}(\nu_\mathfrak{G}) \xrightarrow{q_{*,\top}} x$. Then $\psi^{\Rightarrow}\mathfrak{G} \xrightarrow{q} x$ and hence (X, q) is regular in $\top$-**Conv**.

Conversely, assume that (X, q) is regular in $\top$-**Conv**; it is shown that $(X, \overline{q}_*)$ is regular in SL-CS. Suppose that $\psi : J \longrightarrow X$ and $\Sigma : J \longrightarrow \mathfrak{F}_L^S(X)$ are such that $\Sigma(j) \xrightarrow{q_{*,\top}} \psi(j)$ for each $j \in J$, and $\mu \in \mathfrak{F}_L^S(X)$ for which $K\Sigma\mu \xrightarrow{q_{*,\top}} x$. Define $\sigma(j) = \mathfrak{F}_{\Sigma(j)}$; then $\Sigma(j) \xrightarrow{q_{*,\top}} \psi(j)$ implies there exists $\mathfrak{G} \xrightarrow{q} \psi(j)$ such that $\Sigma(j) \geq \nu_\mathfrak{G}$ and thus by Lemma 4.11 (i), $\mathfrak{F}_{\Sigma(j)} \geq \mathfrak{F}_{\nu_\mathfrak{G}} = \mathfrak{G}$. Hence $\sigma(j) \xrightarrow{q} \psi(j)$ for each $j \in J$. It is shown that $\kappa\sigma\mathfrak{F}_\mu = \mathfrak{F}_{K\Sigma\mu}$.

Suppose that $b \in \mathfrak{B}$, where $\mathfrak{B}$ is the base for $\kappa\sigma\mathfrak{F}_\mu$; then $e_b \circ \sigma \in \mathfrak{F}_\mu$. Hence $K\Sigma\mu(b) = \mu(E_b \circ \Sigma) = \mu(e_b \circ \sigma) = \top$, and thus $b \in \mathfrak{F}_{K\Sigma\mu}$ implies that $\kappa\sigma\mathfrak{F}_\mu \subseteq \mathfrak{F}_{K\Sigma\mu}$.

Conversely, if $b \in \mathfrak{F}_{K\Sigma\mu}$, then $\top = K\Sigma\mu(b) = \mu(e_b \circ \sigma)$ and thus $e_b \circ \sigma \in \mathfrak{F}_\mu$. Therefore $b \in \kappa\sigma\mathfrak{F}_\mu$ and hence $\kappa\sigma\mathfrak{F}_\mu = \mathfrak{F}_{K\Sigma\mu}$. Since $K\Sigma\mu \xrightarrow{q_*,\top} x$, it follows $\kappa\sigma\mathfrak{F}_\mu \xrightarrow{q} x$ and thus $\psi^{\rightarrow}\mathfrak{F}_\mu \xrightarrow{q} x$. However, $\mu = \nu_{\mathfrak{F}_\mu}$, implying $\psi^{\Uparrow}\mu = \psi^{\Uparrow}(\nu_{\mathfrak{F}_\mu}) = \nu_{\psi^{\Rightarrow}\mathfrak{F}_\mu} \xrightarrow{q_*,\top} x$, and thus $(X, \overline{q}_*)$ is regular in SL-CS. □

The next definition is the dual of Definition 4.6 and is given in [4]. The notation "(TF)" is used in [4] to denote the diagonal condition.

Definition 4.13. Suppose that L is a frame. Then $(X, q) \in |\top\text{-}\mathbf{Conv}|$ is called topological in $\top$-**Conv** provided that for each $\psi : J \longrightarrow X$, $\sigma : J \longrightarrow \mathfrak{F}_L^T(X)$ such that $\sigma(j) \xrightarrow{q} \psi(j)$ for each $j \in J$, we have $\kappa\sigma\mathfrak{H} \xrightarrow{q} x$ whenever $\psi^{\Rightarrow}\mathfrak{H} \xrightarrow{q} x$, $\mathfrak{H} \in \mathfrak{F}_L^\top(J)$.

The definition of a strong L-topological space used here can be found in Fang and Yue [3].

Definition 4.14. Let L be a frame and $\tau \subseteq L^X$. Then the pair (X, τ) is called a strong L-topological space provided it satisfies:

(ST1) $\alpha\mathbf{1}_X \in \tau$ for each $\alpha \in L$,
(ST2) $a, b \in \tau$ implies $a \wedge b \in \tau$,
(ST3) $a_j \in \tau$ for each $j \in J$ implies $\bigvee_{j\in J} a_j \in \tau$, and
(ST4) $a \in \tau$ implies $\alpha\mathbf{1}_X \longrightarrow a \in \tau$ for each $\alpha \in L$.

The following result appears as Theorem 3.11[4].

Theorem 4.15 (See [4])**.** *Suppose that* $(X, q) \in |\top\text{-}\mathbf{Conv}|$. *Then* (X, q) *is topological if and only if it is a strong L-topological space.*

Theorem 4.16. *Assume that the frame L is a Boolean algebra,* $(X, q) \in |\top\text{-}\mathbf{Conv}|$, *and let* $(X, \overline{q}_*) \in |SL\text{-}CS|$ *be as given in Definition 3.3. Then* (X, q) *is topological in* $\top$-**Conv** *if and only if* $(X, \overline{q}_*)$ *is topological in SL-CS.*

Proof. Suppose that $(X, \overline{q}_*)$ is topological in SL-CS. Assume that $\psi : J \longrightarrow X$ and $\sigma : J \longrightarrow \mathfrak{F}_L^\top(X)$ are such that $\sigma(j) \xrightarrow{q} \psi(j)$ for each $j \in J$, and $\mathfrak{G} \in \mathfrak{F}_L^\top(J)$ is such that $\psi^{\Rightarrow}\mathfrak{G} \xrightarrow{q} x$. Define $\Sigma(j) = \nu_{\sigma(j)}$ for each $j \in J$, and note that $\Sigma(j) \xrightarrow{q_*,\top} \psi(j)$. According to Lemma 3.2, $\psi^{\Uparrow}(\nu_\mathfrak{G}) = \nu_{\psi^{\Rightarrow}\mathfrak{G}} \xrightarrow{q_*,\top} x$ and it follows that $K\Sigma\nu_\mathfrak{G} \xrightarrow{q_*,\top} x$. It is shown in the proof of Theorem 4.12 that $\kappa\sigma\mathfrak{G} = \mathfrak{F}_{K\Sigma\nu_\mathfrak{G}}$. Since $K\Sigma\nu_\mathfrak{G} \xrightarrow{q_*,\top} x$, according to Definition 3.3 we have that $K\Sigma\nu_\mathfrak{G} \geq \nu_\mathfrak{H}$ for some $\mathfrak{H} \xrightarrow{q} x$, and thus $\mathfrak{F}_{K\Sigma\nu_\mathfrak{G}} \geq \mathfrak{F}_{\nu_\mathfrak{H}} = \mathfrak{H}$. Then $\mathfrak{F}_{K\Sigma\nu_\mathfrak{G}} \xrightarrow{q} x$ and thus $\kappa\sigma\mathfrak{G} \xrightarrow{q} x$. It follows that (X, q) is topological in $\top$-**Conv**.

Conversely, assume that (X, q) is topological in $\top$-**Conv**, $\psi : J \longrightarrow X$ and $\Sigma : X \longrightarrow \mathfrak{F}_L^S(X)$ are such that $\Sigma(j) \xrightarrow{q_*,\top} \psi(j)$ for each $j \in J$, and that $\mu \in \mathfrak{F}_L^S(J)$ is such that $\psi^{\Uparrow}\mu \xrightarrow{q_*,\top} x$. Define $\sigma(j) = \mathfrak{F}_{\Sigma(j)}$; then $\sigma(j) \xrightarrow{q} \psi(j)$ for each $j \in J$. It is straightforward to show that $\psi^{\Rightarrow}\mathfrak{F}_\mu = \mathfrak{F}_{\psi^{\Uparrow}\mu}$ and hence $\psi^{\Rightarrow}\mathfrak{F}_\mu \xrightarrow{q} x$. Since (X, q) is topological in $\top$-**Conv**, $\kappa\sigma\mathfrak{F}_\mu \xrightarrow{q} x$. It is shown in Theorem 4.12

that $\kappa\sigma\mathfrak{F}_\mu = \mathfrak{F}_{K\Sigma\mu}$. It follows from Lemma 4.11 that $K\Sigma\mu = \nu_{\mathfrak{F}_{K\Sigma\mu}}$ and thus $K\Sigma\mu \xrightarrow{q_*,\top} x$. Hence $(X, \overline{q}_*)$ is topological in SL-CS. □

Remark 4.17. Theorems 4.12 and 4.16 remain valid whenever $(X, \overline{q_*})$ is replaced by $(X, \overline{q^*})$.

Remark 4.18. Some comments concerning related works in [14] and [15] are in order. Let $(X, \overline{q}) \in |\text{S}L\text{-CS}|$, when $\overline{q} = (q_\alpha)_{\alpha \in L}$, and $a \in L^X$. According to Definition 4.1 [14], $\overline{a}^{q_\alpha}(x) = \bigvee\{\mu(a) : \mu \xrightarrow{q_\alpha} x\}$, for each $x \in X$ and $\alpha \in L$. Our definition of closure in $\top$-**Conv** coincides with this definition whenever L is a complete Boolean algebra. Further, whenever L is complete Boolean algebra and $\mu \in \mathfrak{F}_L^S(X)$, it is shown in Theorem 4.4 [14] that the q_α-closure of μ in SL-CS is $\overline{\mu}^{q_\alpha}(a) = \bigvee\{\mu(b) : \overline{b}^{q_\alpha} \leq a\}$, for each $a \in L^X$. This definition seems to be coarser than our corresponding definition in $\top$-**Conv**.

5. **Compactification**

Whenever L is a frame, Jäger [10] showed that every lattice-valued convergence space possesses a compactification. The same ideas used by Jäger are employed in our construction. In order to show that our extension space is compact, the assumption that L is a Boolean algebra is needed. Hence the bijection between the stratified L-ultrafilters and $\top$-ultrafilters is used to show compactness of our extension space. The object $(X, q) \in |\top\text{-}\mathbf{Conv}|$ is said to be compact if every maximal $\top$-filter, or $\top$-ultrafilter, converges.

Definition 5.1. Assume that $(X, q) \in |\top\text{-}\mathbf{Conv}|$ is not compact. Then $\big((Y, p), f\big)$ is called a compactification of (X, q) provided:

(i) (Y, p) is compact,

(ii) $f : (X, q) \longrightarrow \Big(f(X), p\,\big|_{f(X)}\Big)$ and f^{-1} are continuous, and

(iii) for each $y \in Y$, there exists $\mathfrak{F} \in \mathfrak{F}_L^\top(X)$ such that $f^{\Rightarrow}\mathfrak{F} \xrightarrow{p} y$.

Whenever L is a complete Boolean algebra, a compactification of each non-compact $(X, q) \in |\top\text{-}\mathbf{Conv}|$ is constructed. Further, each continuous map from (X, q) into a compact regular object in $\top$-**Conv** has a continuous extension to the compactification.

Assume that L is a complete Boolean algebra and let $(X, q) \in |\top\text{-}\mathbf{Conv}|$ which is not compact. Let $\mathcal{N}$ denote the set of all $\top$-ultrafilters on X which fail to converge. Define $X^* = X \cup \{\langle\mathfrak{G}\rangle : \mathfrak{G} \in \mathcal{N}\}$ and let $j : X \longrightarrow X^*$ denote the natural injection $j(x) = x$, $x \in X$. Recall from Theorem 2.8, that $\mathfrak{F} \mapsto \nu_{\mathfrak{F}}$ defines an order preserving bijection from the set of all $\top-$filters on X onto the set of all stratified L-filters on X, where $\nu_{\mathfrak{F}}(a) = \bigvee_{f \in \mathfrak{F}} [f, a]$, $a \in L^X$, and $\mathfrak{F} = \{b \in L^X : \nu_{\mathfrak{F}}(b) = \top\}$.

Given $a \in L^X$, define $a^* \in L^{X^*}$ as $a^*(z) = \begin{cases} a(x), & z = j(x) \\ \nu_{\mathfrak{G}}(a), & z = \langle\mathfrak{G}\rangle \end{cases}$. Observe that $(\perp\mathbf{1}_X)^* = \perp\mathbf{1}_{X^*}$ and $(\alpha\mathbf{1}_X)^* \geq \alpha\mathbf{1}_{X^*}$ since $(\alpha\mathbf{1}_X)^*(\langle\mathfrak{G}\rangle) = \nu_{\mathfrak{G}}(\alpha\mathbf{1}_X) \geq \alpha = (\alpha\mathbf{1}_{X^*})(\langle\mathfrak{G}\rangle)$. Moreover, $(a \wedge b)^*(\langle\mathfrak{G}\rangle) = \nu_{\mathfrak{G}}(a \wedge b) = \nu_{\mathfrak{G}}(a) \wedge \nu_{\mathfrak{G}}(b) = a^*(\langle\mathfrak{G}\rangle) \wedge$

$b^*(\langle\mathfrak{G}\rangle) = (a^* \wedge b^*)(\langle\mathfrak{G}\rangle)$ and thus $(a \wedge b)^* = a^* \wedge b^*$. Observe that if $\mathfrak{B}$ is a $\top$-filter base on X that is closed under finite infima, then $\mathfrak{B}^* = \{b^* : b \in \mathfrak{B}\}$ is a $\top$-filter base on X^* that is also closed under finite infima. Note that if $b \in \mathfrak{B}$, then $\bigvee_{z\in X^*} b^*(z) \geq \bigvee_{z\in X} b(x) = \top$. In particular, if $\mathfrak{F}$ is a $\top$-filter on X, then $\{f^* : f \in \mathfrak{F}\}$ is a $\top$-filter base on X^*; let $\mathfrak{F}^*$ denote the $\top$-filter on X^* that it generates.

Using the notation above, define a structure q^* on X^* as follows:

$\mathfrak{H} \xrightarrow{q^*} j(x)$ if and only if $\mathfrak{H} \geq \mathfrak{F}^*$ for some $\mathfrak{F} \xrightarrow{q} x$,

$\mathfrak{H} \xrightarrow{q^*} \langle\mathfrak{G}\rangle$ if and only if $\mathfrak{H} \geq \mathfrak{G}^*$.

Note that $[j(x)] \geq [x]^* \xrightarrow{q^*} x$ and thus $[j(x)] \xrightarrow{q^*} j(x)$ for each $x \in X$. Also, observe that $[\langle\mathfrak{G}\rangle] \geq \mathfrak{G}^*$. Indeed, if $g \in \mathfrak{G}$, then $g^*(\langle\mathfrak{G}\rangle) = \nu_{\mathfrak{G}}(g) = \top$ and thus $g^* \in [\langle\mathfrak{G}\rangle]$. It follows that the $\top$-filter base $\{g^* : g \in \mathfrak{G}\} \subseteq [\langle\mathfrak{G}\rangle]$ and thus the $\top$-filter $\mathfrak{G}^* \subseteq [\langle\mathfrak{G}\rangle]$. Clearly, if $\mathfrak{H} \xrightarrow{q^*} z$ and $\mathfrak{K} \geq \mathfrak{H}$, then $\mathfrak{K} \xrightarrow{q^*} z$ and hence $(X^*, q^*) \in |\top\text{-}\mathbf{Conv}|$.

Theorem 5.2. *Assume that the frame L is a Boolean algebra and suppose that $(X, q) \in |\top\text{-}\mathbf{Conv}|$ is not compact. Then $\big((X^*, q^*), j\big)$, as defined above, is a compactification of (X, q) in $\top\text{-}\mathbf{Conv}$. Moreover, if $\theta : (X, q) \longrightarrow (Y, p)$ is continuous and (Y, p) is compact and regular, then θ has a continuous extension $\theta^* : (X^*, q^*) \longrightarrow (Y, p)$ such that $\theta^* \circ j = \theta$.*

Proof. It was shown in the above that $(X^*, q^*) \in |\top\text{-}\mathbf{Conv}|$. We show that j is continuous. Observe that if $\mathfrak{F} \xrightarrow{q} x$, then $j^{\Rightarrow}(\mathfrak{F}) \supseteq \mathfrak{F}^*$. Indeed, if $f \in \mathfrak{F}$, then $j^{\rightarrow}(f)(z) = \begin{cases} f(x), & z = j(x) \\ \bot, & z = \langle\mathfrak{G}\rangle \end{cases}$ and thus $j^{\rightarrow}(f) \leq f^*$. Then $f^* \in j^{\Rightarrow}(\mathfrak{F})$ for each $f \in \mathfrak{F}$ and thus $j^{\Rightarrow}(\mathfrak{F}) \supseteq \mathfrak{F}^*$, and this implies that $j^{\Rightarrow}(\mathfrak{F}) \xrightarrow{q^*} j(x)$. Hence j is continuous.

Conversely, if $\mathfrak{F}$ is any $\top$-filter on X such that $j^{\Rightarrow}(\mathfrak{F}) \xrightarrow{q^*} j(x)$, then $j^{\Rightarrow}(\mathfrak{F}) \geq \mathfrak{K}^*$ for some $\mathfrak{K} \xrightarrow{q} x$. Hence $\mathfrak{F} = j^{\Leftarrow}(j^{\Rightarrow}(\mathfrak{F})) \geq j^{\Leftarrow}(\mathfrak{K}^*) = \mathfrak{K}$ and thus $\mathfrak{F} \xrightarrow{q} x$. It follows that $j : (X, q) \longrightarrow (X^*, q^*)$ is an embedding. Further, if $\mathfrak{G} \in \mathcal{N}$, then $j^{\Rightarrow}(\mathfrak{G}) \supseteq \mathfrak{G}^*$ implies that $j^{\Rightarrow}(\mathfrak{G}) \xrightarrow{q^*} \langle\mathfrak{G}\rangle$ and thus $j : (X, q) \longrightarrow (X^*, q^*)$ is a dense embedding.

It is shown that (X^*, q^*) is compact. Assume that $\mathfrak{H}$ is a $\top$-ultrafilter on X^*. According to Theorem 2.8, $\nu_{\mathfrak{H}}$ is a stratified L-ultrafilter on X^* and, moreover, $d \in \mathfrak{H}$ if and only if $\nu_{\mathfrak{H}}(d) = \top$. Define $\mu_{\mathfrak{H}} : L^X \longrightarrow L$ by $\mu_{\mathfrak{H}}(a) = \nu_{\mathfrak{H}}(a^*)$ for each $a \in L^X$.

Observe that $\mu_{\mathfrak{H}}(\bot\mathbf{1}_X) = \nu_{\mathfrak{H}}(\bot\mathbf{1}_{X^*}) = \bot$, $\mu_{\mathfrak{H}}(\alpha\mathbf{1}_X) = \nu_{\mathfrak{H}}((\alpha\mathbf{1}_X)^*) \geq \nu_{\mathfrak{H}}(\alpha\mathbf{1}_{X^*}) \geq \alpha$ and $\mu_{\mathfrak{H}}(a \wedge b) = \nu_{\mathfrak{H}}((a \wedge b)^*) = \nu_{\mathfrak{H}}(a^*) \wedge \nu_{\mathfrak{H}}(b^*) = \mu_{\mathfrak{H}}(a) \wedge \mu_{\mathfrak{H}}(b)$, for $a, b \in L^X$ and $\alpha \in L$. It follows that $\mu_{\mathfrak{H}}$ is a stratified L-filter on X. According to Theorem 2.2 (i), $\mu_{\mathfrak{H}}$ is a stratified L-ultrafilter if and only if for each $a \in L^X$, $\mu_{\mathfrak{H}}(a) = \mu_{\mathfrak{H}}(a \longrightarrow \mathbf{1}_{\varnothing}) \longrightarrow \bot$. Employing Theorem 2.2 (ii), for any $\mathfrak{G} \in \mathcal{N}$, $(a \longrightarrow \mathbf{1}_{\varnothing})^*(\langle\mathfrak{G}\rangle) = \nu_{\mathfrak{G}}(a \longrightarrow \mathbf{1}_{\varnothing}) = \nu_{\mathfrak{G}}(a) \longrightarrow \bot = a^*(\langle\mathfrak{G}\rangle) \longrightarrow \bot = (a^* \longrightarrow \mathbf{1}_{\varnothing})(\langle\mathfrak{G}\rangle)$. Hence $(a \longrightarrow \mathbf{1}_{\varnothing})^* = a^* \longrightarrow \mathbf{1}_{\varnothing}$.

Then $\mu_{\mathfrak{H}}(a) = \nu_{\mathfrak{H}}(a^*) = \nu_{\mathfrak{H}}(a^* \longrightarrow \mathbf{1}_{\varnothing}) \longrightarrow \bot = \nu_{\mathfrak{H}}((a \longrightarrow \mathbf{1}_{\varnothing})^*) \longrightarrow \bot = \mu_{\mathfrak{H}}(a \longrightarrow \mathbf{1}_{\varnothing}) \longrightarrow \bot$, and thus $\mu_{\mathfrak{H}}$ is a stratified L-ultrafilter on X. It follows from Theorem 2.8 (ii) that $\mathfrak{F}_{\mathfrak{H}} = \{a \in L^X : \mu_{\mathfrak{H}}(a) = \top\}$ is a $\top$-ultrafilter on X. Moreover, $a \in \mathfrak{F}_{\mathfrak{H}}$ if and only if $\nu_{\mathfrak{H}}(a^*) = \top$ if and only if $a^* \in \mathfrak{H}$. That is, $a \in \mathfrak{F}_{\mathfrak{H}}$ if and only if $a^* \in \mathfrak{H}$. Observe that $\mathfrak{B} = \{a^* : a \in \mathfrak{F}_{\mathfrak{H}}\}$ is a $\top$-filter base on X^* which is closed under finite infima. Let $\mathfrak{F}^*_{\mathfrak{H}}$ denote the $\top$-filter on X^* that it generates; then $\mathfrak{F}^*_{\mathfrak{H}} \subseteq \mathfrak{H}$.

Assume that $\mathfrak{F}_{\mathfrak{H}} \xrightarrow{q} x$; then $\mathfrak{H} \xrightarrow{q^*} j(x)$. If $\mathfrak{F}_{\mathfrak{H}} \in \mathcal{N}$, then $\mathfrak{H} \xrightarrow{q^*} \langle \mathfrak{F}_{\mathfrak{H}} \rangle$ and it follows that (X^*, q^*) is compact and therefore $\big((X^*, q^*), j\big)$ is a compactification of (X, q) in $\top$-**Conv**.

Next, suppose that $\theta : (X, q) \longrightarrow (Y, p)$ is a continuous map. Define $\theta^* : (X^*, q^*) \longrightarrow (Y, p)$ by $\theta^*(j(x)) = \theta(x)$ for $x \in X$, and $\theta^*(\langle \mathfrak{G} \rangle) = y$ where y is one of the limits of $\theta^{\Rightarrow}(\mathfrak{G})$ in (Y, p). First, for $a \in L^X$, $x \in X$ and $\mathfrak{G} \in \mathcal{N}$, it is shown that $\nu_{[\theta(x)]}(\theta^{\rightarrow}(a)) \geq a(x)$ and $\nu_{\theta^{\Rightarrow}(\mathfrak{G})}(\theta^{\rightarrow}(a)) \geq a^*(\langle \mathfrak{G} \rangle)$.

Note that

$$\nu_{[\theta(x)]}(\theta^{\rightarrow}(a)) = \bigvee_{b \in [\theta(x)]} [b, \theta^{\rightarrow}(a)] \geq [\mathbf{1}_{\{\theta(x)\}}, \theta^{\rightarrow}(a)] \geq \top \longrightarrow a(x) = a(x),$$

and thus $\nu_{[\theta(x)]}(\theta^{\rightarrow}(a)) \geq a(x)$. Further, if $\mathfrak{G} \in \mathcal{N}$, then using Lemma 2.3 (xi),

$$\nu_{\theta^{\Rightarrow}(\mathfrak{G})}(\theta^{\rightarrow}(a)) \geq \bigvee_{g \in \mathfrak{G}} [\theta^{\rightarrow}(g), \theta^{\rightarrow}(a)] \geq \bigvee_{g \in \mathfrak{G}} [g, a] = \nu_{\mathfrak{G}}(a) = a^*(\langle \mathfrak{G} \rangle),$$

and hence $\nu_{\theta^{\Rightarrow}(\mathfrak{G})}(\theta^{\rightarrow}(a)) \geq a^*(\langle \mathfrak{G} \rangle)$.

If $a \in L^X$, it is shown that $\theta^{*\rightarrow}(a^*) \leq \overline{\theta^{\rightarrow}(a)}$. First, assume that $y = \theta^*(j(z)) = \theta(z) \in Y$. Then $\overline{\theta^{\rightarrow}(a)}(y) = \bigvee\{\nu_{\mathfrak{K}}(\theta^{\rightarrow}(a) : \mathfrak{K} \xrightarrow{p} y\} \geq \nu_{[\theta(z)]}(\theta^{\rightarrow}(a)) \geq a(z) = a^*(j(z))$. Next, suppose that $\theta^*(\langle \mathfrak{G} \rangle) = y$, where $\theta^{\Rightarrow}(\mathfrak{G}) \xrightarrow{p} y$. Then $\overline{\theta^{\rightarrow}(a)}(\theta^*(\langle \mathfrak{G} \rangle)$ $= \bigvee\{\nu_{\mathfrak{K}}(\theta^{\rightarrow}(a)) : \mathfrak{K} \xrightarrow{p} y\} \geq \nu_{\theta^{\Rightarrow}(\mathfrak{G})}(\theta^{\rightarrow}(a)) \geq a^*(\langle \mathfrak{G} \rangle)$. Combining these two results, $\overline{\theta^{\rightarrow}(a)}(y) \geq \bigvee\{a^*(z) : \theta^*(z) = y\} = \theta^{*\rightarrow}(a^*)(y)$ and thus $\theta^{*\rightarrow}(a^*) \leq \overline{\theta^{\rightarrow}(a)}$.

Assume $\mathfrak{F}$ is a $\top$-filter on X; then $\mathfrak{B}_1 = \{\overline{\theta^{\rightarrow}(a)} : a \in \mathfrak{F}\}$ and $\mathfrak{B}_2 = \{\theta^{*\rightarrow}(a^*) : a \in \mathfrak{F}\}$ are $\top$-filter bases on Y. Let $\overline{\theta^{\Rightarrow}(\mathfrak{F})}$ and $\theta^{*\Rightarrow}(\mathfrak{F}^*)$ denote the $\top$-filters generated by $\mathfrak{B}_1$ and $\mathfrak{B}_2$, respectively. Since for each $a \in \mathfrak{F}$, $\theta^{*\rightarrow}(a^*) \leq \overline{\theta^{\rightarrow}(a)}$, it follows that $\theta^{*\Rightarrow}(\mathfrak{F}^*) \geq \overline{\theta^{\Rightarrow}(\mathfrak{F})}$.

Finally, suppose that $\mathfrak{H} \xrightarrow{q^*} j(x)$. Then $\mathfrak{H} \geq \mathfrak{F}^*$ for some $\mathfrak{F} \xrightarrow{q} x$, and thus $\theta^{*\Rightarrow}(\mathfrak{H}) \geq \theta^{*\Rightarrow}(\mathfrak{F}^*) \geq \overline{\theta^{\Rightarrow}(\mathfrak{F})}$. Since (Y, q) is regular, it follows that $\theta^{*\Rightarrow}(\mathfrak{H}) \xrightarrow{p} \theta(x) = \theta^*(j(x))$. Similarly, if $\mathfrak{H} \xrightarrow{q^*} \langle \mathfrak{G} \rangle$, then $\mathfrak{H} \geq \mathfrak{G}^*$ and thus $\theta^{*\Rightarrow}(\mathfrak{H}) \geq \theta^{*\Rightarrow}(\mathfrak{G}^*) \geq \overline{\theta^{\Rightarrow}(\mathfrak{G})}$ and $\theta^{*\Rightarrow}(\mathfrak{H}) \xrightarrow{p} y = \theta^*(\langle \mathfrak{G} \rangle)$, where $\theta^{\Rightarrow}(\mathfrak{G}) \xrightarrow{p} y$. Hence $\theta^* : (X^*, q^*) \longrightarrow (Y, p)$ is continuous and $j \circ \theta^* = \theta$. □

Connections between the compactification constructed in Theorem 5.2 and that given by Jäger [10] are made below. Assume that $(X, q) \in |\top\text{-}\mathbf{Conv}|$ is not compact. In order to simplify the notation, let $\big((X^*, s), j\big)$ denote the compactification of (X, q) given in Theorem 5.2. According to Theorem 4.1 [5], there is an isomorphism between the full subcategory SL-LC-CS of "left-continuous" objects in SL-CS and the category SL-GCS of stratified L-generalized convergence spaces.

Let $(X, \overline{q}^*) \in |\mathrm{S}L\text{-CS}|$ denote the object given in Definition 3.4; it easily follows that $(X, \overline{q}^*) \in |\mathrm{S}L\text{-LC-CS}|$ but, in general, $(X, \overline{q}_*)$ is not left-continuous.

Jäger's [10] compactification $\big((X^*, \overline{p}), j\big)$ of $(X, \overline{q}^*)$ in SL-CS is described below. If $\mu \in \mathfrak{F}_L^S(X^*)$, define $\tilde{\mu} \in \mathfrak{F}_L^S(X)$ by $\tilde{\mu}(a) = \mu(a^*)$, for each $a \in L^X$. Then $\overline{p} = (p_\alpha)_{\alpha \in L}$ is defined as follows: for $\alpha > \bot$

$$\mu \xrightarrow{p_\alpha} j(x) \iff \tilde{\mu} \xrightarrow{q_\alpha^*} x$$

$$\mu \xrightarrow{p_\alpha} \langle \mathfrak{G} \rangle \iff \tilde{\mu} = \nu_{\mathfrak{G}}$$

$$\mu \xrightarrow{p_\bot} z \iff \mu \geq \nu_\bot,\ z \in X^*.$$

It is shown in Theorem 5.4 below that $\overline{p} = \overline{s}$.

Lemma 5.3. *Suppose that $(X, q) \in |\top\text{-}\mathbf{Conv}|$ is not compact and let $X^* = X \cup \{\langle \mathfrak{G} \rangle : \mathfrak{G} \in \mathcal{N}\}$. Then for $a, b \in L^X$, $\mathfrak{K} \in \mathfrak{F}_L^\top(X)$, and $\mathfrak{J} \in \mathfrak{F}_L^\top(X^*)$:*

(i) $[b^*, a^*] = [b, a]$

(ii) $\tilde{\nu}_{\mathfrak{K}^*} = \nu_{\mathfrak{K}}$

(iii) $\tilde{\nu_{\mathfrak{J}}} \geq \nu_{\mathfrak{K}}$ *implies* $\mathfrak{J} \geq \mathfrak{K}^*$.

Proof. (i) Observe that $[b^*, a^*] = \bigwedge_{x \in X} \big(b^*(j(x)) \longrightarrow a^*(j(x))\big) \wedge \bigwedge_{\mathfrak{G} \in \mathcal{N}} \big(b^*(\langle \mathfrak{G} \rangle) \longrightarrow a^*(\langle \mathfrak{G} \rangle)\big) = [b, a] \wedge \bigwedge_{\mathfrak{G} \in \mathcal{N}} \big(\nu_{\mathfrak{G}}(b) \longrightarrow \nu_{\mathfrak{G}}(a)\big)$. According to Corollary 3.3 [2], $\nu_{\mathfrak{G}}(b) \longrightarrow \nu_{\mathfrak{G}}(a) \geq [b, a]$ and it follows that $[b^*, a^*] = [b, a]$.

(ii) Fix $a \in L^X$; then using (i), $\tilde{\nu}_{\mathfrak{K}^*}(a) = \nu_{\mathfrak{K}^*}(a^*) = \bigvee_{b \in \mathfrak{K}} [b^* a^*] = \bigvee_{b \in \mathfrak{K}} [b, a] = \nu_{\mathfrak{K}}(a)$. Hence $\tilde{\nu}_{\mathfrak{K}^*} = \nu_{\mathfrak{K}}$.

(iii) Assume that $a \in \mathfrak{K}$; then $\top = \nu_{\mathfrak{K}}(a) \leq \tilde{\nu_{\mathfrak{J}}}(a) = \nu_{\mathfrak{J}}(a^*)$ and thus $a^* \in \mathfrak{J}$. Hence $\mathfrak{K}^* \subseteq \mathfrak{J}$. □

Theorem 5.4. *Assume that L is a complete Boolean algebra, $(X, q) \in |\top\text{-}\mathbf{Conv}|$ is not compact, $\big((X^*, s), j\big)$ is the compactification of (X, q) given in Theorem 5.2, $(X, \overline{q}^*)$ and (X^*, s^*) are as defined in Definition 3.4. If $\big((X^*, \overline{p}), j\big)$ denotes the compactification of $(X, \overline{q}^*)$ given by Jäger* [10], *then $\overline{s}^* = \overline{p}$.*

Proof. Fix $\alpha > \bot$. First, suppose that $\mu \xrightarrow{p_\alpha} j(x)$; then $\tilde{\mu} \xrightarrow{q_\alpha^*} x$ and thus $\tilde{\mu} \geq \nu_{\mathfrak{F}}$ for some $\mathfrak{F} \xrightarrow{q} x$. Since $\mu = \nu_{\mathfrak{H}}$ for some $\mathfrak{H} \in \mathfrak{F}_L^\top(X^*)$, $\tilde{\nu}_{\mathfrak{H}} \geq \nu_{\mathfrak{F}}$, and by Lemma 5.3 (iii), $\mathfrak{H} \geq \mathfrak{F}^*$. Then $\mu = \nu_{\mathfrak{H}} \geq \nu_{\mathfrak{F}^*}$ and $\mathfrak{F}^* \xrightarrow{s} j(x)$ implies that $\mu \xrightarrow{s_\alpha^*} j(x)$. Next, assume that $\mu \xrightarrow{p_\alpha} \langle \mathfrak{G} \rangle$; then $\mu = \nu_{\mathfrak{H}}$ and $\tilde{\mu} = \nu_{\mathfrak{G}}$. Since $\tilde{\nu}_{\mathfrak{H}} = \nu_{\mathfrak{G}}$, it follows by Lemma 5.3 (iii) that $\mathfrak{H} \geq \mathfrak{G}^*$ and thus $\mu = \nu_{\mathfrak{H}} \geq \nu_{\mathfrak{G}^*}$. Hence $\mu \xrightarrow{s_\alpha^*} \langle \mathfrak{G} \rangle$ and thus $\overline{p} \geq \overline{s}^*$.

Conversely, suppose that $\alpha > \bot$ and $\mu \xrightarrow{s_\alpha^*} j(x)$; then $\mu \geq \nu_{\mathfrak{F}^*}$ for some $\mathfrak{F} \xrightarrow{q} x$. It follows from Lemma 5.3 (ii) that $\tilde{\mu} \geq \tilde{\nu}_{\mathfrak{F}^*} = \nu_{\mathfrak{F}}$ and thus $\tilde{\mu} \xrightarrow{q_\alpha^*} x$.

Hence $\mu \xrightarrow{p_\alpha} j(x)$. Next, assume that $\mu \xrightarrow{s^*_\alpha} \langle \mathfrak{G} \rangle$; then $\mu \geq \nu_{\mathfrak{G}^*}$ implies that $\tilde{\mu} \geq \nu_{\mathfrak{G}}$. Since $\nu_{\mathfrak{G}}$ is a stratified L-ultrafilter on X, $\tilde{\mu} = \nu_{\mathfrak{G}}$ and thus $\mu \xrightarrow{p_\alpha} \langle \mathfrak{G} \rangle$. Then $\overline{s}^* \geq \overline{p}$ and thus $\overline{s}^* = \overline{p}$. □

6. **Conclusion**

Whenever L is a Heyting algebra, or a frame, a connection is made between the category of $\top$-convergence spaces and the category of stratified L-convergence spaces. In particular, $\top$-**Conv** is embedded as a bicoreflective subcategory of SL-CS. Regularity in $\top$-**Conv** is defined by Fang and Yue [4]; a characterization is given here in terms of convergence of $\top$-filters. A compactification of each object in $\top$-**Conv** is constructed whenever L is a complete Boolean algebra. Moreover, connections are made to the compactification given by Jäger [10]. An open problem is to characterize the objects in $\top$-**Conv** which have a regular compactification.

Acknowledgements. The authors express their appreciation to the referees for making numerous suggestions leading to significant improvements of the original manuscript.

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