ASSOCIATED PROBABILITY INTUITIONISTIC FUZZY WEIGHTED OPERATORS IN BUSINESS START-UP DECISION MAKING

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Abstract. In the study, we propose the Associated Probability Intuitionistic Fuzzy Weighted Averaging (As-P-IFWA) and the Associated Probability Intuitionistic Fuzzy Weighted Geometric (As-P-IFWG) aggregation operators with associated probabilities of a fuzzy measure presenting an uncertainty. Decision makers' evaluations are given as intuitionistic fuzzy values and are used as the arguments of the aggregation operators. In the paper, we prove correctness of extensions and show the conjugate connections between the constructed operators. Several versions of the new operators are successfully used in the business start-up decision making problem.

1. Introduction

Probability aggregation operators are applied as a stochastic aggregation tool in many decision making models, particularly, in multi-criteria decision making (MCDM) ([6, 20, 21, 22, 23, 24, 26, 30, 56] and others). The stochastic aggregation process is based on the assumption that the states of nature (criteria, factors, attributes) in MCDM model are independent, and probability aggregation operators have a linear behavior. They are developed using additive measures, which are presented by the axioms of independence [8]. However, in real MCDM models (such as in the example presented in Section 4), we have certain degree of interdependence or interactivity between combinations of states of nature [8, 28]. Decision making persons (DMPs), which are invited to solve decision problem, have almost similar knowledge and preferences. The subjective preferences of DMPs are usually nonlinear. Thus, the conditions of independence between these states of nature and preferential independence of DMPs are not satisfied.

Our main goal in this paper is to create fuzzy-probabilistic construction in fuzzy decision-making models. A bibliometric analysis of recent researches on fuzzy decision making is presented by Liu and Liao in [18]. An instrument of fuzzy set and fuzzy relation theories are intensively used in the different problems of fuzzy decision making ([7, 11, 27, 37, 38, 39, 40, 41, 42] and others). In [25], Merigo et al. presented a general overview of research in the fuzzy sciences using bibliometric indicators. These indicators provide a general picture, identifying some of the most

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influential studies in the fuzzy research areas. Intuitionistic fuzzy set (IFS) was introduced by Atanassov [1], as a generalization of a Zadeh's fuzzy set (FS) [58]. Because each element of IFS is assigned a membership degree, a non-membership degree and a hesitancy degree, IFS is more powerful in dealing with uncertainty and imprecision than FS. IFS theory has been widely studied and applied to a variety of areas ([2, 52] and others). Some basic operations on IFSs are defined by Atanassov [1, 2] and others. Being proposed in 1983, the IFS theory has immensely grown in popularity during the past decades and has been widely used in the decision making and other fields of researches. Considering the large volume of available materials, the authors of the work [57] intend to make a scientometric review on IFS studies to reveal the most cited papers, influential authors and important journals in the field. Below we will briefly describe the concept of intuitionistic fuzzy sets and probability intuitionistic fuzzy aggregation operators concerning extension of the Choquet integral [4] in intuitionistic fuzzy environment. Fuzzy multi-attribute decision making models and aggregations for intuitionistic fuzzy arguments are constructed by Wei et al. [36, 44, 45, 46, 47, 59] and others. Xu and Yager [54] presented some geometric, and Xu [51] defined some arithmetic intuitionistic fuzzy aggregation operators. In [15], Liao and Xu proposed a family of intuitionistic fuzzy hybrid weighted (averaging and geometric) aggregation operators, while in [55] they offered a comprehensive survey on decision making with intuitionistic fuzzy preference relations (IFPRs) with the aim of providing a clear perspective on the novelty, the consistency, the preference and the consensus of IFPRs. Liao et al. [16] proposed two novel consensus measures for intuitionistic fuzzy group decision making that allow to assess the degree of agreement among experts in the group. In addition, they considered all other existing measures of agreement, compared them with each other and with the proposed measures, and emphasized the advantages of the proposed consensus measures.

In [33], Sugeno introduced the definition of a non-additive (fuzzy) measure that only requires monotonicity, instead of additivity. This proved to be the most effective tool for modelling phenomena of interaction [8, 13] and dealing with decision problems, in which the probability is changed by the fuzzy measure. In [17], authors published a review on decision maker's behavior for such MCDM problems. There are several methods to determine or identify a fuzzy measure. For instance, linear methods [19], quadratic methods [8], Sugeno's λ-additive measures based methods [31] and others. In [9], the discussion is focused on the use of Choquet integral in some identification methods of a fuzzy measure. In [49], 2-order additive fuzzy measure identification problem is described. The recent publications on some of the methods that have been proposed to reduce the complexity of identifying some of the fuzzy measure values are reviewed in [14].

All previously mentioned and other intuitionistic fuzzy aggregation operators only consider situations when all elements in IFS are independent. That is, they only consider the importance of individual elements. Nevertheless, in many practical problems, there is a correlation or interaction between the criteria in the fuzzy multi-criteria decision making [8, 33]. Tan and Chen in [35] provided an algorithm for multi-criteria decision-making based on intuitionistic fuzzy Choquet integral operator. Xu in [53] used Choquet integral to propose some intuitionistic fuzzy aggregation operators. Xia and Xu [50] presented new extension of the Choquet integral - intuitionistic multiplicative Choquet ordered averaging operator, which reflects the correlations of the intuitionistic multiplicative preference information in the decision making procedure. Wu et al. [48] discussed in detail the aggregation properties of the intuitionistic fuzzy-valued Choquet integral. These operators not only consider the importance of the elements or their ordered position, but also *reflect the correlations or interactions among some combinations of criteria* (states of nature).

Though many aggregation operators based on Choquet aggregation operator have been developed under intuitionistic fuzzy environment, there is still an issue that needs to be addressed: Choquet integral *does not reflect the overall interactions among all combinations of states of nature. Hence, our goal was to create new operators based on the Probability Weighted Averaging operator* [20] for intuitionistic fuzzy environment, when probabilities are changed by the associated probabilities of a fuzzy measure. Developed operators take into account the overall interactions among all combinations of the states of nature.

In Section 2, we briefly review recent extensions of probability weighted averaging and geometric aggregation operators for intuitionistic fuzzy arguments. We also present main properties of associated probabilities of a fuzzy measure and its relation to the finite Choquet integral. Section 3 presents the Associated Probability Intuitionistic Fuzzy Weighted Averaging and Geometric operators, considers some properties of the constructed intuitionistic fuzzy operators, proves propositions on the correctness of extensions and shows the conjugate connection between constructed operators. In Section 4, we demonstrate the work of the constructed operators when applied to the fuzzy decision making problem regarding business start-up. In Section 5, we draw main conclusions of the paper.

2. **Preliminary Concepts**

We review several relevant concepts and highlight the correspondence between the notions in IFS theory [1, 2, 34, 51, 52].

Definition 2.1. [1] Let S be a fixed ordinary set. An intuitionistic fuzzy set (IFS) α on S is defined as follows:

$$\alpha = \{\langle s, \mu_\alpha(s), \nu_\alpha(s)\rangle / s \in S\},$$

where the functions $\mu_\alpha(s)$ and $\nu_\alpha(s)$ denote the degrees of membership and non-membership of the element $s \in S$ to the set α, respectively, with the conditions $0 \le \mu_\alpha(s) \le 1$, $0 \le \nu_\alpha(s) \le 1$, $0 \le \mu_\alpha(s) + \nu_\alpha(s) \le 1$, and $\pi_\alpha(s) = 1 - \mu_\alpha(s) - \nu_\alpha(s)$ is usually called the degree of indeterminacy of S to α. The pair $\alpha = (\mu_\alpha, \nu_\alpha)$ is called an intuitionistic fuzzy value (IFV) and $IFVs$ denotes the set of all IFVs. In $IFVs$ partial order relation $\le_{L_{IFVs}}$ is defined as $(\mu_1, \nu_1) \le_{L_{IFVs}} (\mu_2, \nu_2) \Leftrightarrow \mu_1 \le \mu_2$ and $\nu_1 \ge \nu_2$.

Definition 2.2. [1, 51] The following operations in $IFVs$ are defined. For $\alpha = (\mu_\alpha, \nu_\alpha)$, $\alpha_1, \alpha_2 \in IFVs$ we have: $\alpha^c = (\nu_\alpha, \mu_\alpha)$; $\alpha_1 \oplus \alpha_2 = (\mu_{\alpha_1} + \mu_{\alpha_2} - \mu_{\alpha_1} \cdot \mu_{\alpha_2},$

$\nu_{\alpha_1}\cdot\nu_{\alpha_2})$; $\alpha_1\otimes\alpha_2=(\mu_{\alpha_1}\cdot\mu_{\alpha_2},\ \nu_{\alpha_1}+\nu_{\alpha_2}-\nu_{\alpha_1}\cdot\nu_{\alpha_2})$; $\alpha_1\vee\alpha_2=(\max(\mu_{\alpha_1},\mu_{\alpha_2}),$ $\min(\nu_{\alpha_1},\nu_{\alpha_2}))$; $\alpha_1\wedge\alpha_2=(\min(\mu_{\alpha_1},\mu_{\alpha_2}),\ \max(\nu_{\alpha_1},\nu_{\alpha_2}))$; $\lambda\cdot\alpha=(1-(1-\mu_\alpha)^\lambda,$ $\alpha)$, $\alpha^\lambda=(\mu_\alpha^\lambda,1-(1-\alpha)^\lambda)$, $\lambda>0$.

The following total order relation $\leq_t$ in $IFVs$ is used: $\alpha\leq_t\beta$ ($\alpha,\beta\in IFVs$) if and only if a) $\alpha<_t\beta$ if (i) $S(\alpha)<S(\beta)$ or (ii) $S(\alpha)=S(\beta)$ and $h(\alpha)<h(\beta)$; b) $\alpha=_t\beta$ if $S(\alpha)=S(\beta)$ and $h(\alpha)=h(\beta)$, where $S(\alpha)$ is the Score of the IFV: $S(\alpha)=\mu_\alpha-\nu_\alpha$ and $h(\alpha)$ is the Accuracy of the IFV: $h(\alpha)=\mu_\alpha+\nu_\alpha$.

In stochastic or deterministic environment researchers usually use the probability averaging or geometric operators, or their combinations with associative weights of the states of nature. Merigó and others [20, 21, 24] have developed a probability weighted average and geometric operators. Now we define intuitionistic fuzzy extensions of these operators. In [43, 51, 52], some probability intuitionistic fuzzy aggregation operators are given:

Definition 2.3. [43] Let $\alpha_1,\dots,\alpha_n\in IFVs$ be some collection of IFVs as values on S. We define the following intuitionistic fuzzy aggregation operators:
a) The probability intuitionistic fuzzy weighted averaging (P-IFWA) operator is:

$$P-IFWA_P(\alpha_1,\dots,\alpha_n)=\mathop{\oplus}_{i=1}^{n}(\bar p_i\alpha_i)=\left(1-\prod_{i=1}^{n}(1-\mu_{\alpha_i})^{\bar p_i},\ \prod_{i=1}^{n}(\nu_{\alpha_i})^{\bar p_i}\right),$$

b) The probability intuitionistic fuzzy weighted geometric (P-IFWG) operator is:

$$P-IFWG_P(\alpha_1,\dots,\alpha_n)=\mathop{\otimes}_{i=1}^{n}(\alpha_i)^{\bar p_i}=\left(\prod_{i=1}^{n}(\mu_{\alpha_i})^{\bar p_i},\ 1-\prod_{i=1}^{n}(1-\nu_{\alpha_i})^{\bar p_i}\right),$$

where $\bar p_i=\beta p_i+(1-\beta)w_i$ with weighted parameter $\beta\in[0;1]$; $P=\{\ p_i\}_{i=1}^n$, $p_i\equiv prob(a_i)$, is a probability distribution $0<p_i<1$, $\sum\limits_{i=1}^{n}p_i=1$, and $w=(w_1,\dots,w_n)$ is an aggregation weighted vector, such that $0\leq w_j\leq 1$, $\sum\limits_{j=1}^{n}w_j=1$.

We adapted the definition of a fuzzy measure ([5, 29] and others) for the case of a finite referential as follows:

Definition 2.4. [5] Let $S=\{s_1,\ s_2,\dots,\ s_n\}$ be a finite set and g be a set function $g:\ 2^S\to[0,\ 1]$. We say g is a fuzzy measure on S if it satisfies

$$(i)\ \ g(\varnothing)=0;\ \ g(S)=1;\quad (ii)\ \ \forall\,A,\,B\subseteq S,\ \text{ if } A\subseteq B,\ \text{then } g(A)\leq g(B).$$

The possible orderings of the elements of S are given by the permutations of a set with elements, which form the group S_n. Now, we consider a definition of associated probabilities ([3, 32] and others) induced by a fuzzy measure on the group S_n.

Definition 2.5. [3] The probability functions P_σ defined by

$$\begin{aligned}&P_\sigma(s_{\sigma(1)})=g(\{s_{\sigma(1)}\}),\ \dots,\\&P_\sigma(s_{\sigma(i)})=g(\{s_{\sigma(1)},\dots,s_{\sigma(i)}\})-g(\{s_{\sigma(1)},\dots,s_{\sigma(i-1)}\}),\ \dots,\\&P_\sigma(s_{\sigma(n)})=1-g(\{s_{\sigma(1)},\dots,s_{\sigma(n-1)}\}),\ \ g(\{s_{\sigma(0)}\})\equiv 0\end{aligned}$$

for each $\sigma = (\sigma(1), \sigma(2), \ldots, \sigma(n)) \in S_n$, are called the associated probabilities and the Associated Probability Class (APC) - $\{P_\sigma\}_{\sigma \in S_n}$ of a fuzzy measure g.

We say that g_* and g^* fuzzy measures on S are dual iff $\forall A \in S$, $g_*(A) = 1 - g^*(\overline{A})$.

Definition 2.6. [3] Let (g_*, g^*) be a pair of dual fuzzy measures on S:
g_* is a lower capacity of order two if and only if $\forall\, A,\, B \subseteq S$, $g_*(A \cup B) + g_*(A \cap B) \geq g_*(A) + g_*(B)$;
g^* is an upper capacity of order two if and only if $\forall\, A,\, B \subseteq S$, $g^*(A \cup B) + g^*(A \cap B) \leq g^*(A) + g^*(B)$.

We consider some aspects of Choquet averaging and geometric operators [4] as most typical aggregation operators with a fuzzy measure.

Definition 2.7. [4] Assume $S = \{s_1, s_2, \ldots, s_n\}$ is a set of states of nature on which we have a fuzzy measure g on S and a variable of expert evaluations $a : S \Rightarrow R_0^+$, such that $a(s_i) \equiv a_i \geq 0$, $i = 1, 2, \ldots, n$. Then the aggregation

$$CA_g(a_1, a_2, \ldots, a_n) = \sum_{j=1}^{n} p_j a_{i(j)}$$

is called the discrete Choquet Averaging (CA) operator with respect to the fuzzy measure g and

$$CG_g(a_1, \ldots, a_n) = \prod_{j=1}^{n} a_{i(j)}^{p_j}$$

is called the discrete Choquet Geometric (CG) operator with respect to the fuzzy measure g, where in the proceeding $i(\cdot)$ is index function, such that $a_{i\,(j)}$ is the j-th largest of the $\{a_i\}_{i=1}^{n}$;

$$p_j = g(\{\, s_{i(1)}, \ldots, s_{i(j)}\}) - g(\{\, s_{i(1)}, \ldots, s_{i(j-1)}\}), \quad g(\{\, s_{i(0)}\}) \equiv 0\,.$$

Proposition 2.8. [3] *A necessary and sufficient condition for a pair of dual fuzzy measures (g_*, g^*) to be lower and upper capacities of order two, respectively, is that for every variable $a : S \Rightarrow R_0^+$, it holds*

$$CA_{g_*}(a) = \min_{\sigma \in S_n} PA_{P_\sigma}(a), \quad CA_{g^*}(a) = \max_{\sigma \in S_n} PA_{P_\sigma}(a) \quad \textit{or}$$
$$CG_{g_*}(a) = \min_{\sigma \in S_n} PG_{P_\sigma}(a), \quad CG_{g^*}(a) = \max_{\sigma \in S_n} PG_{P_\sigma}(a).$$

Now we consider extensions of the CA and CG operators on IFVs:

Definition 2.9. [35, 53] Assume $S = \{s_1, s_2, \ldots, s_n\}$ is a set of states of nature on which we have a fuzzy measure g and an intuitionistic fuzzy variable of expert evaluations $\alpha : S \Rightarrow IFVs$, such that $\alpha(s_i) \equiv \alpha_i = (\mu_{\alpha_i}, \nu_{\alpha_i})$, $i = 1, 2, \ldots, n$. Then the aggregation

$$IFCA_g(\alpha_1, \ldots, \alpha_n) = \mathop{\oplus}_{j=1}^{n} [p_j \alpha_{i(j)}] = \left(1 - \prod_{j=1}^{n} (1 - \mu_{\alpha_{i(j)}})^{p_j},\ \prod_{j=1}^{n} (\nu_{\alpha_{i(j)}})^{p_j}\right) \qquad (1)$$

is called the Intuitionistic Fuzzy Choquet Averaging (IFCA) operator with respect to the fuzzy measure g and the aggregation

$$IFCG_g(\alpha_1, \ldots, \alpha_n) = \underset{j=1}{\overset{n}{\otimes}} (\alpha_{i(j)})^{p_j} = \left(\prod_{j=1}^{n} (\mu_{\alpha_{i(j)}})^{p_j},\ 1 - \prod_{j=1}^{n} (1 - \nu_{\alpha_{i(j)}})^{p_j} \right) \tag{2}$$

is called the Intuitionistic Fuzzy Choquet Geometric (IFCG) operator with respect to the fuzzy measure g, where

$$p_j = g(\{ s_{i(1)}, \ldots, s_{i(j)} \}) - g(\{ s_{i(1)}, \ldots, s_{i(j-1)} \}), \quad g(\{s_{i(0)}\}) \equiv 0,$$

and $\alpha_{i(j)}$ is the j-th largest of α_i, $i = 1, 2, \ldots, n$ according to the total order relation $\geq_t$.

The main properties of the IFCA and IFCG operators are studied in [48, 53].

3. Associated Probabilities in the Probability Intuitionistic Fuzzy Weighted Operators

In previous section the IFCA or IFCG operators were defined along with their probability representations by associated probability class (APC) $\{P_\sigma\}_{\sigma \in S_m}$, where the number of probability distributions on S is equal to $k = n\,!$

However, in formulas (1)–(2) only one associated probability is used. That probability is defined by the permutation of indexes according to [3]. Therefore, in MCDM problems when states of nature are dependent or interactive in decision making procedure, IFCA or IFCG operators *reflect the overall interaction only among the consonant combinations of states of nature according to* (1)–(2):

$$\{s_{i(1)}\},\ \{s_{i(1)}, s_{i(2)}\},\ \ldots\ ,\{s_{i(1)}, s_{i(2)}, \ldots, s_{i(n)}\}\,.$$

As mentioned in the Introduction, *our main purpose in this study was to create new constructions of probabilistic aggregation operators for intuitionistic fuzzy environment, which reflect the overall interactions among all combinations of the states of nature.* To achieve the goal, we include an associated probability class (APC) $\{P_\sigma\}_{\sigma \in S_m}$ in the P-IFWA and P-IFWG operators. We consider some intuitionistic fuzzy operation M based on Definition 2.2.

Definition 3.1. The associated probability F operator for some intuitionistic fuzzy operation M , $M : IFV_s^k \to IFVs$, $k = n!$ and with respect to a fuzzy measure g on S is a mapping $As - F_M : IFVs^n \to IFVs$:

$$As - F_M(\alpha_1, \ldots, \alpha_n) = M(F_{P_{\sigma_1}}(\alpha_1, \ldots, \alpha_n), \ldots, F_{P_{\sigma_k}}(\alpha_1, \ldots, \alpha_n))$$

where $F \in \{P - IFWA, P - IFWG\}$; $F_{P_{\sigma_i}}(\alpha_1, \ldots, \alpha_n),\ i = 1, \ldots, n!$ is a value of the F operator calculated with respect to associated probability distribution P_{σ_i} of a fuzzy measure g.

In the paper we only consider certain cases, where $M = \vee$ (intuitionistic max) or $M = \wedge$ (intuitionistic min). Therefore, we list four intuitionistic fuzzy operators:

$$As - P - IFWA_M, As - P - IFWG_M,\ \ M \in \{\vee, \wedge\}\ .$$

The associated probability intuitionistic fuzzy weighted averaging operators:

$$As - P - IFWA_{\vee}(\alpha_1, \ldots, \alpha_n) = \bigvee_{\sigma \in S_n} \left[\bigoplus_{i=1}^{n} (\bar{p}_{\sigma(i)} \cdot \alpha_i) \right],$$

$$As - P - IFWA_{\wedge}(\alpha_1, \ldots, \alpha_n) = \bigwedge_{\sigma \in S_n} \left[\bigoplus_{i=1}^{n} (\bar{p}_{\sigma(i)} \cdot \alpha_i) \right],$$

The associated probability intuitionistic fuzzy weighted geometric operators:

$$As - P - IFWG_{\vee}(\alpha_1, \ldots, \alpha_n) = \bigvee_{\sigma \in S_n} \left[\bigotimes_{i=1}^{n} ({\alpha_i}^{\bar{p}_{\sigma(i)}}) \right],$$

$$As - P - IFWG_{\wedge}(\alpha_1, \ldots, \alpha_n) = \bigwedge_{\sigma \in S_n} \left[\bigotimes_{i=1}^{n} ({\alpha_i}^{\bar{p}_{\sigma(i)}}) \right],$$

$w = (w_1, \ldots, w_n)$ is the aggregation weighted vector, such that $w_j \in [0,1]$, $\sum_{j=1}^{n} w_j = 1$ and for each associated probability P_σ, $\sigma \in S_n$, $\bar{p}_{\sigma(i)} = \beta p_{\sigma(i)} + (1-\beta)\, w_i$, $p_{\sigma(i)} \equiv P_\sigma(\alpha_i)$ *is associated probability induced by the fuzzy measure g and weight.*

Now we consider some propositions on the correctness of extensions.

Proposition 3.2. *For dual fuzzy measures g_* and g^* on S we have equalities*

$$As - F_{*M}(\alpha_1, \ldots, \alpha_n) = As - F^*_M(\alpha_1, \ldots, \alpha_n), \tag{3}$$

where $As - F_{*M}$ and $As - F^*_M$ operators' values are calculated with respect to dual fuzzy measures g_* and g^*, respectively, and $F \in \{P - IFWA, P - IFWG\}$, $M \in \{\vee, \wedge\}$.

Proof. Any operator constructed in the Definition 3.1 is symmetric. By applying Proposition 2.8, we receive the equalities (3). □

Proposition 3.3. *The $As - P - IFWA_{\vee}$ operator's value is an intuitionistic fuzzy value and*

$$As - P - IFWA_{\vee}(\alpha_1, \ldots, \alpha_n) = \left(1 - \min_{\sigma \in S_n} \left(\prod_{i=1}^{n} (1-\mu_{\alpha_i})^{\bar{p}_{\sigma(i)}} \right), \min_{\sigma \in S_n} \left(\prod_{i=1}^{n} (\nu_{\alpha_i})^{\bar{p}_{\sigma(i)}} \right)\right). \tag{4}$$

Proof. The first condition is immediately derived from Definitions 2.2 and 3.1. Below we prove the equation (4) by using mathematical induction on n.

For $n = 2$, according to the operational laws of Definition 2.2 we have: For any permutation $\sigma \in \{(1,2),(2,1)\} = S_2$. Since $\bar{p}_{\sigma(i)} \cdot \alpha_i = (1-(1-\mu_{\alpha_i})^{\bar{p}_{\sigma(i)}}, (\nu_{\alpha_i})^{\bar{p}_{\sigma(i)}})$ and

$$(\bar{p}_{\sigma(1)} \cdot \alpha_1) \oplus (\bar{p}_{\sigma(2)} \cdot \alpha_2) = (1-(1-\mu_{\alpha_1})^{\bar{p}_{\sigma(1)}} \cdot (1-\mu_{\alpha_2})^{\bar{p}_{\sigma(2)}}, (\nu_{\alpha_1})^{\bar{p}_{\sigma(1)}} \cdot (\nu_{\alpha_2})^{\bar{p}_{\sigma(2)}}),$$

we receive

$$As - P - IFWA_{\vee}(\alpha_1, \alpha_2) = \bigcup_{\sigma \in S_2} [(\bar{p}_{\sigma(1)} \cdot \alpha_1) \oplus (\bar{p}_{\sigma(2)} \cdot \alpha_2)]$$
$$= \left(\max_{\sigma \in S_2} \langle 1-(1-\mu_{\alpha_1})^{\bar{p}_{\sigma(1)}} \cdot (1-\mu_{\alpha_2})^{\bar{p}_{\sigma(2)}} \rangle, \min_{\sigma \in S_2} \langle (\nu_{\alpha_1})^{\bar{p}_{\sigma(1)}} \cdot (\nu_{\alpha_2})^{\bar{p}_{\sigma(2)}} \rangle \right)$$

$$= \left(1 - \min_{\sigma \in S_2} \left(\prod_{i=1}^{2} (1-\mu_{\alpha_i})^{\bar{p}_{\sigma(i)}} \right), \min_{\sigma \in S_2} \left(\prod_{i=1}^{2} (\nu_{\alpha_i})^{\bar{p}_{\sigma(i)}} \right) \right).$$

That is, for $n = 2$ the equation (4) holds.

Suppose that for $n = k$ the equation (4) holds, i.e.

$$As - P - IFWA_{\vee}(\alpha_1, \ldots, \alpha_k)$$
$$= \left(1 - \min_{\sigma \in S_k} \left(\prod_{i=1}^{k} (1-\mu_{\alpha_i})^{\bar{p}_{\sigma(i)}} \right), \min_{\sigma \in S_k} \left(\prod_{i=1}^{k} (\nu_{\alpha_i})^{\bar{p}_{\sigma(i)}} \right) \right).$$

Then, for $n = k+1$, according to Definition 3.1, we have

$$As - P - IFWA_{\vee}(\alpha_1, \ldots, \alpha_k, \alpha_{k+1}) = \bigcup_{\sigma \in S_{k+1}} \left[\bigoplus_{i=1}^{n} (\bar{p}_{\sigma(i)} \cdot \alpha_i) \right]$$
$$= \bigcup_{\sigma \in S_{k+1}} \left[(\bigoplus_{i=1}^{k} (\bar{p}_{\sigma(i)} \cdot \alpha_i)) \otimes (\bar{p}_{\sigma(k+1)} \cdot \alpha_{k+1}) \right]$$
$$= \bigcup_{\sigma \in S_{k+1}} \left[1 - \prod_{i=1}^{k} (1-\mu_{\alpha_i})^{\bar{p}_{\sigma(i)}} + 1 - (1-\mu_{\alpha_{k+1}})^{\bar{p}_{\sigma(k+1)}} \left(1 - \prod_{i=1}^{k} (1-\mu_{\alpha_{\alpha_i}})^{\bar{p}_{\sigma(i)}} \right) \right.$$
$$\left. \times (1 - (1-\mu_{\alpha_{k+1}})^{\bar{p}_{\sigma(k+1)}}) \prod_{i=1}^{k} (\nu_{\alpha_i})^{\bar{p}_{\sigma(i)}} \cdot (\nu_{\alpha_{k+1}})^{\bar{p}_{\sigma(k+1)}} \right]$$
$$= \bigcup_{\sigma \in S_{k+1}} \left[1 - \prod_{i=1}^{k} (1-\mu_{\alpha_i})^{\bar{p}_{\sigma(i)}} \cdot (1-\mu_{\alpha_{k+1}})^{\bar{p}_{\sigma(k+1)}}, \prod_{i=1}^{k+1} (\nu_{\alpha_i})^{\bar{p}_{\sigma(i)}} \right]$$
$$= \bigcup_{\sigma \in S_{k+1}} \left(1 - \prod_{i=1}^{k+1} (1-\mu_{\alpha_i})^{\bar{p}_{\sigma(i)}}, \prod_{i=1}^{k+1} (\nu_{\alpha_i})^{\bar{p}_{\sigma(i)}} \right)$$
$$= \left(\max_{\sigma \in S_{k+1}} (1 - \prod_{i=1}^{k+1} (1-\mu_{\alpha_i})^{\bar{p}_{\sigma(i)}}), \min_{\sigma \in S_{k+1}} (\prod_{i=1}^{k+1} (\nu_{\alpha_i})^{\bar{p}_{\sigma(i)}}) \right)$$
$$= \left(1 - \min_{\sigma \in S_{k+1}} \left(\prod_{i=1}^{k+1} (1-\mu_{\alpha_i})^{\bar{p}_{\sigma(i)}} \right) \min_{\sigma \in S_{k+1}} \left(\prod_{i=1}^{k+1} (\nu_{\alpha_i})^{\bar{p}_{\sigma(i)}} \right) \right).$$

That is, for $n = k+1$, the equation (4) still holds. Therefore, for all n, the equation (4) always holds, which completes the proof of the proposition. □

The proof of the following proposition is omitted, as it is similar to the proof of Proposition 3.3.

Proposition 3.4. *The following $As - F_M$ operators' values are intuitionistic fuzzy values and*

$$As-P-IFWA_{\wedge}(\alpha_1,\dots,\alpha_n) = \left(1-\max_{\sigma\in S_n}\left(\prod_{i=1}^{n}(1-\mu_{\alpha_i})^{\bar{p}_{\sigma(i)}}\right), \max_{\sigma\in S_n}\left(\prod_{i=1}^{n}(\nu_{\alpha_i})^{\bar{p}_{\sigma(i)}}\right)\right), \quad (5)$$

$$As-P-IFWG_{\vee}(\alpha_1,\dots,\alpha_n) = \left(\max_{\sigma\in S_n}\left(\prod_{i=1}^{n}(\mu_{\alpha_i})^{\bar{p}_{\sigma(i)}}\right), 1-\max_{\sigma\in S_n}\left(\prod_{i=1}^{n}(1-\nu_{\alpha_i})^{\bar{p}_{\sigma(i)}}\right)\right), \quad (6)$$

$$As-P-IFWG_{\wedge}(\alpha_1,\dots,\alpha_n) = \left(\min_{\sigma\in S_n}\left(\prod_{i=1}^{n}(\mu_{\alpha_i})^{\bar{p}_{\sigma(i)}}\right), 1-\min_{\sigma\in S_n}\left(\prod_{i=1}^{n}(1-\nu_{\alpha_i})^{\bar{p}_{\sigma(i)}}\right)\right). \quad (7)$$

We continue presentation of propositions on the correctness of extensions:

Proposition 3.5. *Let g be a lower capacity of order two on S. Then the $As-P-IFWA_{\vee}$ ($As-P-IFWG_{\wedge}$) operator coincides with the IFCA (IFCG) operator:*

(a) $As-P-IFWA_{\vee}(\alpha_1,\dots,\alpha_n) = IFCA_{\bar{g}}(\alpha_1,\dots,\alpha_n)$,

(b) $As-P-IFWG_{\wedge}(\alpha_1,\dots,\alpha_n) = IFCG_{\bar{g}}(\alpha_1,\dots,\alpha_n)$

if $\alpha_1,\dots,\alpha_n \in IFVs$ are ordered with respect to the partial order relation $\geq_{L_{IFVs}}$; $\bar{g}$ is a lower capacity of order two, associated probabilities' values of which are defined as $\forall\sigma\in S_n$, $\bar{p}_{\sigma(i)} = \beta p_{\sigma(i)} + (1-\beta)w_i,\ 0<\beta\leq 1;\ i=1,\dots,n$, where $p_{\sigma(\cdot)}$ is an associated probability value of a fuzzy measure g and w_i is a weight of α_i.

Proof. Case a). Let g be a lower capacity of order two on S. In Proposition 3.3 (formula (4)) we consider function Min of the products with respect to associated probability distributions

$$\prod_{i=1}^{n}(1-\mu_{\alpha_i})^{\bar{p}_{\sigma(i)}},\ \ \sigma\in S_n. \quad (8)$$

This product takes its Min on the argument points same for the function $\ln(\prod_{i=1}^{n}(1-\mu_{\alpha_i})^{\bar{p}_{\sigma(i)}})$ because logarithm is a strictly monotone function. Therefore,

$$\ln\left(\prod_{i=1}^{n}(1-\mu_{\alpha_i})^{\bar{p}_{\sigma(i)}}\right) = \sum_{i=1}^{n}\bar{p}_{\sigma(i)}\cdot\ln(1-\mu_{\alpha_i})$$

and

$$\min_{\sigma\in S_n}\left[\ln\left(\prod_{i=1}^{n}(1-\mu_{\alpha_i})^{\bar{p}_{\sigma(i)}}\right)\right] = \min_{\sigma\in S_n}\left[\sum_{i=1}^{n}\bar{p}_{\sigma(i)}\cdot\ln(1-\mu_{\alpha_i})\right].$$

Using the results of the Proposition 2.8, the latest Min is a Choquet averaging operator for the values $\ln(1-\mu_{\alpha_1}),\dots,\ln(1-\mu_{\alpha_n})$ with respect to fuzzy measure - $\bar{g}:\ \forall A\subset S,\ \bar{g}(A) = \beta g(A) + (1-\beta)\cdot\sum_{s_i\in A} w_i$. It is clear that the fuzzy measure $\bar{g}$ is also a lower capacity of order two with an APC - $\{\bar{p}_\sigma\}_{\sigma\in S_n}$. If $\pi\in S_n$ is such a permutation, for which

$$\ln(1-\mu_{\alpha_{\pi(1)}}) \geq \ln(1-\mu_{\alpha_{\pi(2)}}) \geq \cdots \geq \ln(1-\mu_{\alpha_{\pi(n)}}) \quad (9)$$

then following the formulas in Proposition 2.8, we receive:

$$\min_{\sigma\in S_n}\left[\ln\left(\prod_{i=1}^{n}(1-\mu_{\alpha_i})^{\bar{p}_{\sigma(i)}}\right)\right]=CA_{\bar{g}}(\ln(1-\mu_{\alpha_{\pi(1)}}),\ldots,\ln(1-\mu_{\alpha_{\pi(n)}})).$$

From the inequalities (9) follows that

$$1-\mu_{\alpha_{\pi(1)}}\geq 1-\mu_{\alpha_{\pi(2)}}\geq\cdots\geq 1-\mu_{\alpha_{\pi(n)}}$$

and for the product function (8) we receive

$$\min_{\sigma\in S_n}\left[\left(\prod_{i=1}^{n}(1-\mu_{\alpha_i})^{\bar{p}_{\sigma(i)}}\right)\right]=\prod_{i=1}^{n}(1-\mu_{\alpha_i})^{\bar{p}_{\tau(i)}}. \tag{10}$$

where $\bar{p}_{\pi(i)}=\bar{g}(\{s_{\pi(1)},\ldots,s_{\pi(i)}\})-\bar{g}(\{s_{\pi(1)},\ldots,s_{\pi(i-1)}\})$.

Analogously to the product function (8), we receive the result for the product function $\prod\limits_{i=1}^{n}(\nu_{\alpha_i})^{\bar{p}_{\sigma(i)}}$:

$$\min_{\sigma\in S_n}\left[\prod_{i=1}^{n}(\nu_{\alpha_i})^{\bar{p}_{\sigma(i)}}\right]=\prod_{i=1}^{n}(\nu_{\alpha_{\pi'(i)}})^{\bar{p}_{\pi'(i)}}, \tag{11}$$

where $\pi'\in S_n$; $\bar{p}_{\pi'(i)}=\bar{g}(\{s_{\pi'(1)},\ldots,s_{\pi'(i)}\})-\bar{g}(\{s_{\pi'(1)},\ldots,s_{\pi'(i-1)}\})$ and $\nu_{\alpha_{\pi'(1)}}\geq\nu_{\alpha_{\pi'(2)}}\geq\cdots\geq\nu_{\alpha_{\pi'(n)}}$.

Because values $\alpha_1,\ldots,\alpha_n$ are ordered with respect to partial ordered relation $\leq_{L_{IFVs}}$, we will have the permutation $\tau\in S_n$, such that

$$\mu_{\alpha_{\tau(1)}}\leq\mu_{\alpha_{\tau(2)}}\leq\cdots\leq\mu_{\alpha_{\tau(n)}}\ \text{ and }\ \nu_{\alpha_{\tau(1)}}\geq\nu_{\alpha_{\tau(2)}}\geq\cdots\geq\nu_{\alpha_{\tau(n)}}.$$

Following the formulas (10) and (11) we can take the same permutation $\pi=\pi'=\tau$. We receive:

$$\begin{aligned}As-P-IFWA(\alpha_1,\ldots,\alpha_n)&=\left(1-\prod_{i=1}^{n}(1-\mu_{\alpha_{\tau(i)}})^{\bar{p}_{\tau(i)}},\prod_{i=1}^{n}(\nu_{\alpha_{\tau(i)}})^{\bar{p}_{\tau(i)}}\right)\\&=IFCA_{\bar{g}}(\alpha_1,\ldots,\alpha_n),\end{aligned}$$

where $\bar{p}_{\tau(i)=}\bar{g}(\{s_{\tau(1)},\ldots,s_{\tau(i)}\})-\bar{g}(\{s_{\tau(1)},\ldots,s_{\tau(i-1)}\})$. The proof of the case b) is similar and, thus, we omit it. □

Similar to Proposition 3.5, we present Proposition 3.6, proof of which is the same and, thus, is omitted here.

Proposition 3.6. *Let g be an upper capacity of order two on S. Then the $As-P-IFWA_{\wedge}$ $(As-P-IFWG_{\vee})$ operator coincides with the IFCA (IFCG) operator:*

$$As-P-IFWA_{\wedge}(\alpha_1,\ldots,\alpha_n)=IFCA_{\bar{g}}(\alpha_1,\ldots,\alpha_n),$$
$$(As-P-IFWG_{\vee}(\alpha_1,\ldots,\alpha_n)=IFCG_{\bar{g}}(\alpha_1,\ldots,\alpha_n)),$$

if $\alpha_1,\ldots,\alpha_n\in IFVs$ are ordered with respect to the partial order relation $\geq_{L_{IFVs}}$; $\bar{g}$ is an upper capacity of order two, associated probabilities' values of which are defined as $\forall\sigma\in S_n$, $\bar{p}_{\sigma(i)}=\beta p_{\sigma(i)}+(1-\beta)w_i$, $0<\beta\leq 1$, $i=1,\ldots,n$, where $p_{\sigma(\cdot)}$ is an associated probability value of a fuzzy measure g and w_i is a weight of α_i.

Proposition 3.7. *The functions* $As-P-IFWA_M$ *and* $As-P-IFWG_M$*, where* $M \in \{\vee, \wedge\}$*, are intuitionistic fuzzy aggregation operators on* L_{IFVs}*.*

Proof. Case 1. Let $M = \vee$ and we consider $As - P - IFWA_\vee : L^n_{IFVs} \to L_{IFVs}$. From Proposition 3.3 (formula (4)) we have:

$$As - P - IFWA_\vee(\alpha_1, \dots, \alpha_n) = \left(1 - \min_{\sigma\in S_n}\left(\prod_{i=1}^n (1-\mu_{\alpha_i})^{\bar{p}_{\sigma(i)}}\right), \min_{\sigma\in S_n}\left(\prod_{i=1}^n (\nu_{\alpha_i})^{\bar{p}_{\sigma(i)}}\right)\right).$$

Let us have two collections of IFVs: $(\alpha_1, \alpha_2, \dots, \alpha_n)$ and $(\beta_1, \beta_2, \dots, \beta_n)$ with the pair ordering $\leq_{L_{IFVs}}$: or $\forall i \in \{1, 2, \dots, n\}$, $\alpha_i \leq_{L_{IFVs}} \beta_i \Leftrightarrow \mu_{\alpha_i} \leq \mu_{\beta_i}$ and $\nu_{\alpha_i} \geq \nu_{\beta_i}$. We must prove that

$$As - P - IFWA_\vee(\alpha_1, \dots, \alpha_n) \leq_{L_{IFVs}} As - P - IFWA_\vee(\beta_1, \dots, \beta_n). \tag{12}$$

We have, $\forall\sigma \in S_n$:

$$1-\mu_{\alpha_i} \geq 1-\mu_{\beta_i} \geq 0 \;\Rightarrow\; \prod_{i=1}^n (1-\mu_{\alpha_i})^{\bar{p}_{\sigma(i)}} \geq \prod_{i=1}^n (1-\mu_{\beta_i})^{\bar{p}_{\sigma(i)}} \text{ and}$$

$$\min_{\sigma\in S_n}\left(\prod_{i=1}^n (1-\mu_{\alpha_i})^{\bar{p}_{\sigma(i)}}\right) \geq \min_{\sigma\in S_n}\left(\prod_{i=1}^n (1-\mu_{\beta_i})^{\bar{p}_{\sigma(i)}}\right)$$

$$\Rightarrow\; 1 - \min_{\sigma\in S_n}\left(\prod_{i=1}^n (1-\mu_{\alpha_i})^{\bar{p}_{\sigma(i)}}\right) \leq 1 - \min_{\sigma\in S_n}\left(\prod_{i=1}^n (1-\mu_{\beta_i})^{\bar{p}_{\sigma(i)}}\right). \tag{13}$$

This is the first inequality necessary for (12). Similarly, we receive:

$$\nu_{\alpha_{\sigma(i)}} \geq \nu_{\beta_{\sigma(i)}} \geq 0 \;\Rightarrow\; (\nu_{\alpha_i})^{\bar{p}_{\sigma(i)}} \geq (\nu_{\beta_i})^{\bar{p}_{\sigma(i)}} \;\Rightarrow\; \prod_{i=1}^n (\nu_{\alpha_i})^{\bar{p}_{\sigma(i)}} \geq \prod_{i=1}^n (\nu_{\beta_i})^{\bar{p}_{\sigma(i)}}$$

$$\Rightarrow\; \min_{\sigma\in S_n}\left(\prod_{i=1}^n (\nu_{\alpha_i})^{\bar{p}_{\sigma(i)}}\right) \geq \min_{\sigma\in S_n}\left(\prod_{i=1}^n (\nu_{\beta_i})^{\bar{p}_{\sigma(i)}}\right). \tag{14}$$

The inequality (12) is derived from the inequalities (13) and (14), therefore, the $As - P - IFWA_\vee$ operator is monotone with respect to $\leq_{L_{IFVs}}$.

It is easy to show that $As-P-IFWA_\vee(0_{L_{IFVs}}, \dots, 0_{L_{IFVs}}) = 0_{L_{IFVs}}$ and $As-IFWA_\vee(1_{L_{IFVs}}, \dots, 1_{L_{IFVs}}) = 1_{L_{IFVs}}$. Therefore, the function $As - P - IFWA_\vee$ is an intuitionistic fuzzy aggregation operator.

The cases when $M = \wedge$ for the $As - P - IFWA_M$ operator or the cases when $M \in \{\vee, \wedge\}$ for the $As-P-IFWG_M$ operator can be proved analogously to the case 1 and are omitted here. □

Remark 3.8. Using the monotonicity of the $As - F_M$, $M \in \{\vee, \wedge\}$, $F \in \{P-IFWA, P-IFWG\}$ operators, it is easy to prove the following inequalities: let us have two collections of IFVs: $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ and $(\beta_1, \beta_2, \ldots, \beta_n)$, then

$$\begin{aligned} &As - F_M(\alpha_1 \wedge \beta_1, \ldots, \alpha_n \wedge \beta_n) \leq_{L_{IFVs}} \\ &As - F_M(\alpha_1, \ldots, \alpha_n) \wedge As - F_M(\beta_1, \ldots, \beta_n); \\ &As - F_M(\alpha_1 \vee \beta_1, \ldots, \alpha_n \vee \beta_n) \geq_{L_{IFVs}} \\ &As - F_M(\alpha_1, \ldots, \alpha_n) \vee As - F_M(\beta_1, \ldots, \beta_n). \end{aligned}$$

Remark 3.9. One can easily show that the aggregation functions $As - F_M$, $M \in \{\vee, \wedge\}$, $F \in \{P - IFWA, P - IFWG\}$ are idempotent.

Proposition 3.10. *The aggregation functions $As - F_M$ $M \in \{\vee, \wedge\}$, $F \in \{P - IFWA, P - IFWG\}$ are bounded if $\alpha_1, \alpha_2, \ldots, \alpha_n \in IFVs$ and*

$$A^- \leq_{L_{IFVs}} IF - As - PA_M(\alpha_1, \alpha_2, \ldots, \alpha_n) \leq_{L_{IFVs}} A^+, \tag{15}$$

where A^- and A^+ are minimal and maximal elements of the set $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ with respect to operations $\vee$ and $\wedge$ correspondingly, and

$$\begin{aligned} A^- &= \bigwedge_{i=1}^{n} \alpha_i = \left(\min_{i=1,\ldots,n} \mu_{\alpha_i}, \max_{i=1,\ldots,n}, \nu_{\alpha_i} \right) = (\mu^-,\ \nu^+); \\ A^+ &= \bigvee_{i=1}^{n} \alpha_i = \left(\max_{i=1,\ldots,n} \mu_{\alpha_i}, \min_{i=1,\ldots,n} \nu_{\alpha_i} \right) = (\mu^+,\ \nu^-). \end{aligned}$$

Proof. **Case** 1: Let $M = \vee$ and, for the example, we consider the $As - P - IFWA_\vee$ operator. We know that

$$\begin{aligned} &As - P - IFWA_\vee(\alpha_1, \ldots, \alpha_n) \\ &\qquad = \left(1 - \min_{\sigma \in S_n} \left(\prod_{i=1}^{n} (1 - \mu_{\alpha_i})^{\bar{p}_{\sigma(i)}} \right), \min_{\sigma \in S_n} \left(\prod_{i=1}^{n} (\nu_{\alpha_i})^{\bar{p}_{\sigma(i)}} \right) \right). \end{aligned}$$

For $\forall \sigma \in S_n$ we have: $1 - \mu_i \leq 1 - \mu^- \ \Rightarrow\ (1 - \mu_i)^{\bar{p}_{\sigma(i)}} \leq (1 - \mu^-)^{\bar{p}_{\sigma(i)}}$ and

$$\prod_{i=1}^{n} (1 - \mu_{\alpha_i})^{\bar{p}_{\sigma(i)}} \leq \prod_{i=1}^{n} (1 - \mu^-)^{\bar{p}_{\sigma(i)}} = (1 - \mu^-)^{\sum_{i=1}^{n} \bar{p}_{\sigma(i)}} = 1 - \mu^- .$$

Therefore,

$$1 - \min_{\sigma \in S_n} \left(\prod_{i=1}^{n} (1 - \mu_{\alpha_i})^{\bar{p}_{\sigma(i)}} \right) \geq 1 - (1 - \mu^-) = \mu^-.$$

Analogously, we may receive the inequality:

$$\min_{\sigma \in S_n} \left(\prod_{i=1}^{n} (\nu_{\alpha_i})^{\bar{p}_{\sigma(i)}} \right) \leq \nu^+.$$

Thus, we proved the left inequality of the (15). The right inequality in (15) is received similarly. The case for the other $As - F_M$ operators can be proved analogously. □

Remark 3.11. One can easily show that the aggregation functions $As - P - IFWA_M$ and $As - P - IFWG_M$, $M \in \{\vee, \wedge\}$ are symmetric.

Below, we received the additional result:

Remark 3.12. One can easily show that the $As-P-IFWA_M$ and $As-P-IFWG_M$, $M \in \{\vee, \wedge\}$, operators are averaging aggregation operators (monotonicity, symmetricity, boundedness and idempotency) on the IFVs with respect to partial order relation $\geq_{L_{IFVs}}$.

The following proposition proves some inequalities between new constructed operators.

Proposition 3.13. *Let be given* $\alpha_1, \alpha_2, \ldots, \alpha_n \in IFVs$ *and let there exist ordering* $\leq_{L_{IFVs}}$ *on* $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$. *Then*

$$As - F_{\vee} \geq_{L_{IFVs}} As - F_{\wedge}$$

and if for $\alpha_i = (\mu_{\alpha_i}, \nu_{\alpha_i})$, $\mu_{\alpha_i} \geq \nu_{\alpha_i}$, $i = 1, 2, \ldots, n$, *then*

$$(As - IFWA_{\wedge})^c \geq_{L_{IFVs}} As - IFWG_{\vee},$$
$$(As - IFWA_{\vee})^c \leq_{L_{IFVs}} As - IFWG_{\wedge}.$$

Proof. On the basis of Propositions 3.3, 3.4 and Definitions 2.1, 2.2, it is easy to prove this proposition. □

Proposition 3.14. *Let be given a collection of IFVs -* $\alpha_1, \ldots, \alpha_n$ *and a fuzzy measure* g *on* S. *If* $\beta = (\mu_\beta, \nu_\beta)$ *is an IFV then for the aggregation function* $As - F_M$ $M \in \{\vee, \wedge\}$ $F \in \{P - IFWA, P - IFWG\}$ *we have*

$$As - F_M(\alpha_1 \oplus \beta, \ldots, \alpha_n \oplus \beta) = As - F_M(\alpha_1, \ldots, \alpha_n) \oplus \beta.$$

Proof. We consider the As-P-IFWA operator. Other cases will be analogous and are omitted. Since for any $\alpha_i = (\mu_{\alpha_i}, \nu_{\alpha_i})$, $i = 1, \ldots, n$, $\alpha_i \oplus \beta = (\mu_{\alpha_i} + \mu_\beta - \mu_{\alpha_i} \cdot \mu_\beta, \nu_{\alpha_i} \cdot \nu_\beta) = (1 - (1 - \mu_{\alpha_i})(1 - \mu_\beta), \nu_{\alpha_i} \cdot \nu_\beta)$. According to Proposition 3.3, we have

$$\begin{aligned}
&As - P - IFWA_{\bigcup}(\alpha_1 \oplus \beta, \ldots, \alpha_n \oplus \beta) \\
&= \begin{pmatrix} 1 - \min\limits_{\sigma \in S_n} \left(\prod\limits_{i=1}^{n} ((1 - \mu_{\alpha_i})(1 - \mu_\beta))^{\bar{p}_{\sigma(i)}} \right), \\ \min\limits_{\sigma \in S_n} \left(\prod\limits_{i=1}^{n} (\nu_{\alpha_i} \nu_\beta)^{\bar{p}_{\sigma(i)}} \right) \end{pmatrix} \\
&= \Bigg(1 - (1 - \mu_\beta)^{\sum_{i=1}^n \bar{p}_{\sigma(i)}} \cdot \min_{\sigma \in S_n} \left(\prod_{i=1}^{n} ((1 - \mu_{\alpha_i}))^{\bar{p}_{\sigma(i)}} \right), \\
&\qquad \nu_\beta^{\sum_{i=1}^n \bar{p}_{\sigma(i)}} \cdot \min_{\sigma \in S_n} \left(\prod_{i=1}^{n} (\nu_{\alpha_i})^{\bar{p}_{\sigma(i)}} \right) \Bigg) \\
&= \left(1 - (1 - \mu_\beta) \cdot \min_{\sigma \in S_n} \left(\prod_{i=1}^{n} ((1 - \mu_{\alpha_i}))^{\bar{p}_{\sigma(i)}} \right), \nu_\beta \cdot \min_{\sigma \in S_n} \left(\prod_{i=1}^{n} (\nu_{\alpha_i})^{\bar{p}_{\sigma(i)}} \right) \right).
\end{aligned}$$

According to Definition 2.2

$$As - P - IFWA_{\bigcup}(\alpha_1, \ldots, \alpha_n) \oplus \beta$$

$$= \left(1-(1-\mu_\beta)\cdot\min_{\sigma\in S_n}\left(\prod_{i=1}^{n}((1-\mu_{\alpha_i}))^{\bar{p}_{\sigma(i)}}\right), \nu_\beta\cdot\min_{\sigma\in S_n}\left(\prod_{i=1}^{n}(\nu_{\alpha_i})^{\bar{p}_{\sigma(i)}}\right)\right). \qquad \square$$

Proposition 3.15. *Let be given a collection of IFVs - $\alpha_1,\dots,\alpha_n$ and a fuzzy measure on S. If $\beta=(\mu_\beta,\nu_\beta)$ is an IFV then for $r>0$*

$$As-F_M(r\alpha_1,\dots,r\alpha_n)=r\cdot As-F_M(\alpha_1,\dots,\alpha_n).$$

Proof. We consider the As-P-IFWA operator. Other cases will be analogous and are omitted. According to the Definition 2.2, for any $\alpha_i=(\mu_{\alpha_i},\nu_{\alpha_i}),\ i=1,\dots,n$ and $r>0$

$$r\alpha_i=\left(1-(1-\mu_{\alpha_i})^r,{\nu_{\alpha_i}}^r\right).$$

Following Proposition 3.3, we have

$$\begin{aligned}&As-P-IFWA_\vee(r\alpha_1,\dots,r\alpha_n)\\&= \left(1-\min_{\sigma\in S_n}\left(\prod_{i=1}^{n}((1-\mu_{\alpha_i})^r)^{\bar{p}_{\sigma(i)}}\right), \min_{\sigma\in S_n}\left(\prod_{i=1}^{n}({\nu_{\alpha_i}}^r)^{\bar{p}_{\sigma(i)}}\right)\right)\\&= \left(1-\min_{\sigma\in S_n}\left(\prod_{i=1}^{n}((1-\mu_{\alpha_i}))^{r\bar{p}_{\sigma(i)}}\right), \min_{\sigma\in S_n}\left(\prod_{i=1}^{n}(\nu_{\alpha_i})^{r\bar{p}_{\sigma(i)}}\right)\right).\end{aligned}$$

According to Definition 2.2,

$$\begin{aligned}&r\cdot As-P-IFWA_\vee(\alpha_1,\dots,\alpha_n)\\&= r\cdot\left(1-\min_{\sigma\in S_n}\left(\prod_{i=1}^{n}((1-\mu_{\alpha_i}))^{\bar{p}_{\sigma(i)}}\right)^r, \min_{\sigma\in S_n}\left(\prod_{i=1}^{n}(\nu_{\alpha_i})^{\bar{p}_{\sigma(i)}}\right)\right)^r\\&= \left(1-\min_{\sigma\in S_n}\left(\prod_{i=1}^{n}((1-\mu_{\alpha_i}))^{r\bar{p}_{\sigma(i)}}\right), \min_{\sigma\in S_n}\left(\prod_{i=1}^{n}(\nu_{\alpha_i})^{r\bar{p}_{\sigma(i)}}\right)\right). \qquad \square\end{aligned}$$

Remark 3.16. Using Propositions 3.14, 3.15 we receive the following formula:

$$As-F_M(r\alpha_1\oplus\beta,\dots,r\alpha_n\oplus\beta)=r\cdot As-F_M(\alpha_1,\dots,\alpha_n)\oplus\beta.$$

Remark 3.17. Let in formulas (4)–(7) a fuzzy measure be a probability one ($g=P$). It is known that [3] associated probabilities of a fuzzy measure (as a probability) coincide and values in every operator (4)–(7) also coincide. Using an idempotency of these operators we receive: for $\forall M\in\{\vee,\wedge\}$ and $\forall F\in\{P-IFWA,P-IFWG\}$

$$As-F_M=F.$$

Remark 3.18. If in formulas (4)–(7) $\beta=0$, then for $\forall M\in\{\vee,\ \wedge\}$

$$As-P-IFWA_M=IFWA,\ As-P-IFWG_M=IFWG.$$

Remark 3.19. If in formulas (4)–(7) $\beta=1$ and a fuzzy measure g is a probability one, then for $\forall M\in\{\vee,\ \wedge\}$

$$As-P-IFWA_M=P-IFWA,\ As-P-IFWG_M=P-IFWG.$$

Remark 3.20. If in formulas (4)–(7) $\beta=0,\ \ w_s=1,$ and $w_j=0,\ \ j\neq s$, then $As-F_M(\alpha_1,\dots,\alpha_n)=\alpha_s$, and these operators are step-type operators.

Now we will consider connections between the constructed operators. In [48], Wu et al. in detail discussed the aggregation properties of the intuitionistic fuzzy Choquet averaging (IFCA) and intuitionistic fuzzy conjugate Choquet averaging operators.

Definition 3.21. [48] Assume $S = \{s_1, s_2, \ldots, s_n\}$ be a set of states of nature on which we have a fuzzy measure g and an intuitionistic fuzzy variable of expert evaluations $\alpha : S \Rightarrow IFVs$ such that $\alpha(s_i) \equiv \alpha_i = (\mu_{\alpha_i}, \nu_{\alpha_i}), \ \ i = 1, 2, \ldots, n$. Then the aggregation

$$(\bar{c})IFCA_g(\alpha_1, \ldots, \alpha_n) \equiv IFCCA_g(\alpha_1, \ldots, \alpha_n) = (IFCA_g((\alpha_1)^c, \ldots, (\alpha_n)^c))^c$$
$$= \Big(\overset{n}{\underset{j=1}{\oplus}} [p_j(\alpha_{i(j)})^c]\Big)^c = \left(\prod_{j=1}^{n} (\mu_{\alpha_{i(j)}})^{p_j}, \ 1 - \prod_{j=1}^{n} (1 - \nu_{\alpha_{i(j)}})^{p_j}\right)$$

is called the Intuitionistic Fuzzy Conjugate Choquet Averaging (IFCCA) operator with respect to the fuzzy measure g.

From Definitions 2.9, 3.21 we see that the IFCCA operator coincides with the intuitionistic fuzzy Choquet geometric operator

$$(\bar{c})IFCA_g(\alpha_1, \ldots, \alpha_n) = IFCG_g(\alpha_1, \ldots, \alpha_n). \tag{16}$$

If we extend Definition 3.1 on the intuitionistic fuzzy aggregation operators, then it is easy to show the following connection between the aggregation operators $As - F_M$, $M \in \{\vee, \wedge\}$, $F \in \{P - IFWA, P - IFWG\}$:

$$(\bar{c})As - P - IFWA_{\vee} = As - P - IFWG_{\wedge}\,,$$
$$(\bar{c})As - P - IFWA_{\wedge} = As - P - IFWG_{\vee}\,.$$

In the MCDM these new two pairs and Choquet mutual conjugate operators express the DMP's dependences on decision making risks in a possible spectrum from optimistic to pessimistic risk.

4. Application of the Operators in Group Decision Making of Business Start-up

4.1. **Business Start-up Decision Making Problem Formation.** G-FOOD Ltd. operates the Georgian restaurant "TIFLISO" located in picturesque and historic place (Tbilisi, Georgia), on Shardin Str. There are various Café-Bars and Restaurants deployed on the same street. "TIFLISO" has two floors with 350 m^2 space. There is 40 m^2 space adjacent to the restaurant building, which also belongs to "TIFLISO" restaurant, but the space is not yet occupied. Ltd G-FOOD plans to use the space for some business.

There are the following alternatives:

d_1 - *Wine Shop*; d_2 - *Fast-food Café*; d_3 - *Ice-cream and Confectionery*; d_4 - *Hookah bar*; d_5 - *Grill-Café*.

It is hard to choose from all the alternatives, as they depend on various factors (criteria) related to the G-FOOD Ltd.'s interests.

These factors are the following:

s_1 - *Investment amount*; s_2 - *Project implementation time*; s_3 - *Revenue needed for 0-Profit*; s_4 - *The Relativity (Ratio) of Revenue needed for 0-profit and approximate industrial revenue*; s_5 - *The competitive persons point of view.*

The project developed for G-FOOD Ltd. around the above-mentioned problem supported decision making process and solved the problem of choosing between the alternatives.

4.2. **Problem Solving Scheme.** Problem Solving Scheme is as follows:

1. Gather the relevant information on the factors and alternatives using such sources as business-plan and survey of competitive persons. Based on the information, form the method to support the optimal decision making.
2. Develop the project implementation matrix of data, where the elements of the matrix are real numbers or intervals.
3. Create the appraisal decision making matrix using experts' evaluations (presented in intuitionistic fuzzy values) based on the project implementation matrix of data.
4. Generate the new decision making aggregation operators that scale alternative's data into decision intuitionistic fuzzy values - as ranking levels.
5. Make a decision by ranking the alternatives from the high level to the low, provided that the final decision is the alternative with the highest level.

4.3. **Description of the Factors Influencing the Alternatives.** Let us consider the alternative d_1 - *Wine-Shop.* For this alternative, the values of the factors are:

Factor s_1 - *Investment amount: 38 200 Gel.* Based on the business-plan.

Factor s_2 - *Project implementation time: 35 Days.* Provided by the construction company.

Factor s_3 - *Revenue needed for 0-Profit: 6785.7 Gel.* Provided by the economics calculations of BEP (Break Even Point) based on the data from the business-plan.

Factor s_4 - *The Relativity (Ratio) of Revenue needed for 0-profit and approximate industrial revenue:* $\frac{R_0}{R_{ind}} = \frac{6\,785.7}{14\,000} \approx 0.485 \rightarrow 48.5\%$. Information is based on the financial documents from the similar existing projects.

Factor s_5 - *The competitive persons' point of view: 4 points.* This information is obtained through calculating the average of the data provided by 4 competitive independent experts, who assign points from 1 to 10 to each of the concrete alternative, based on their expert decision on whether the alternative is worthy to implement or not.

The same scheme is used for the description of other alternatives (d_2; d_3; d_4; d_5) by factors and is presented through the following *matrix of data* (Table 1).

The four persons who ordered the project (Decision making persons - DMPs) are making the appraisal of the data from the above provided matrix of data. The appraisal is done in the intuitionistic fuzzy values.

DMPs reflect factors' influence and not influence to possible alternatives by pair levels of IFV. Their evaluations are shown in Tables 2–5.

	s_1	s_2	s_3	s_4	s_5
d_1 - *Wine Shop*	38 200	35	6785.7	48.5%	4
d_2 - *Fast-food Café*	56 000	50	17 885	44.7%	4.4
d_3 - *Ice-cream and Confectionery*	41 200	45	8750	35%	5.3
d_4 - *Hookah bar*	52 500	60	8625	23%	7.5
d_5 - *Grill-Café*	66 500	60	10 885	24%	6.1

Table 1. The Matrix of Data

	s_1	s_2	s_3	s_4	s_5
d_1	(0.5, 0.3)	(0.8, 0.1)	(0.9, 0.0)	(0.9, 0.0)	(0.7, 0.3)
d_2	(0.5, 0.3)	(0.8, 0.1)	(0.7, 0.2)	(0.7, 0.3)	(0.5, 0.4)
d_3	(0.6, 0.4)	(0.6, 0.1)	(0.7, 0.2)	(0.9, 0.0)	(0.5, 0.4)
d_4	(0.7, 0.2)	(0.6, 0.1)	(0.5, 0.3)	(0.8, 0.2)	(0.7, 0.2)
d_5	(0.3, 0.4)	(0.7, 0.2)	(0.9, 0.1)	(0.7, 0.2)	(0.6, 0.3)

Table 2. Appraisal Matrix by DMP1

	s_1	s_2	s_3	s_4	s_5
d_1	(0.7, 0.2)	(0.8, 0.1)	(0.7, 0.3)	(0.6, 0.3)	(0.5, 0.4)
d_2	(0.5, 0.3)	(0.6, 0.3)	(0.5, 0.4)	(0.3, 0.6)	(0.4, 0.6)
d_3	(0.9, 0.1)	(0.7, 0.1)	(0.7, 0.2)	(0.3, 0.6)	(0.3,0.5)
d_4	(0.6, 0.3)	(0.3, 0.5)	(0.4, 0.2)	(0.8, 0.2)	(0.7, 0.1)
d_5	(0.4, 0.2)	(0.4, 0.6)	(0.4, 0.4)	(0.7, 0.1)	(0.6, 0.3)

Table 3. Appraisal Matrix by DMP2

	s_1	s_2	s_3	s_4	s_5
d_1	(0.6, 0.4)	(0.7, 0.1)	(0.8, 0.1)	(0.6, 0.3)	(0.3, 0.5)
d_2	(0.8, 0.1)	(0.9, 0.1)	(0.6, 0.4)	(0.7,0.2)	(0.6, 0.3)
d_3	(0.5, 0.3)	(0.7, 0.2)	(0.6, 0.3)	(0.3, 0.5)	(0.5, 0.6)
d_4	(0.4, 0.3)	(0.4, 0.6)	(0.8, 0.2)	(0.4, 0.3)	(0.4, 0.2)
d_5	(0.3, 0.6)	(0.4, 0.6)	(0.7, 0.1)	(0.6, 0.2)	(0.5, 0.2)

Table 4. Appraisal Matrix by DMP3

The Expertons Method [12] was modified for processing and synthesizing experts' intuitionistic fuzzy information into resulting decision matrix in order to apply it to the problem of Business Start-up decision making:

4.4. **Expertons Method.** In this study, the use of the Expertons Method [12] allows the decision making persons to evaluate each factor of each decision independently, condense data and obtain optimal joint evaluations. An experton is the generalization of probability, when cumulative probabilities are replaced by monotonically decreasing intervals. A group of experts - DMPs, statistically defines these intervals. We extended expertons method for intuitionistic fuzzy environment. The concept of the expertons method for intuitionistic fuzzy data can be briefly described as follows:

Let D be a set of all possible decisions (possible alternatives) and S be a set of possible factors (states of nature) in the decision making model. The group

	s_1	s_2	s_3	s_4	s_5
d_1	(0.7, 0.2)	(0.8, 0.0)	(0.5, 0.2)	(0.7, 0.2)	(0.5, 0.4)
d_2	(0.6, 0.3)	(0.7, 0.1)	(0.5, 0.3)	(0.7, 0.1)	(0.6, 0.2)
d_3	(0.6, 0.2)	(0.9, 0.1)	(0.7, 0.1)	(0.7, 0.2)	(0.3, 0.4)
d_4	(0.7, 0.1)	(0.6, 0.1)	(0.8, 0.1)	(0.9, 0.0)	(0.7, 0.3)
d_5	(0.3, 0.6)	(0.6, 0.1)	(0.7, 0.1)	(0.9, 0.1)	(0.6, 0.1)

Table 5. Appraisal Matrix by DMP4

	s_1	s_2	s_3	s_4	s_5
d_1	(0.66, 0.34)	(0.80, 0.16)	(0.75, 0.23)	(0.73, 0.27)	(0.55, 0.45)
d_2	(0.64, 0.32)	(0.77, 0.23)	(0.61, 0.39)	(0.64, 0.36)	(0.57, 0.43)
d_3	(0.68, 0.30)	(0.75, 0.20)	(0.70, 0.27)	(0.59, 0.39)	(0.45, 0.52)
d_4	(0.64, 0.30)	(0.52, 0.39)	(0.66, 0.27)	(0.75, 0.25)	(0.66, 0.27)
d_5	(0.39, 0.50)	(0.57, 0.43)	(0.70, 0.25)	(0.75, 0.23)	(0.61, 0.30)

Table 6. The Resulting Decision Matrix - averaging Intuitionistic Fuzzy Expertons

of r experts is requested to express their subjective opinion for every factor from S on each possible alternative from D in the form of intuitionistic fuzzy values: $\forall\, d \in D,\, s \in S : \alpha^j(d,s) = (\mu_\alpha^j(d,s), \nu_\alpha^j(d,s))$, where j is the number of an expert. Suppose we have some set of monotone compatibility levels: $0 \le e_1 < e_2 < \cdots < e_\ell \le 1$. For fixed (d,s) we consider the statistics - cumulative distribution laws $F_{\mu_\alpha}(e;d,s)$ and $F_{\nu_\alpha}(e;d,s)$. Thus, we obtain

$$\forall\, d \in D,\, s \in S, \quad \forall e \in \{e_1,\dots,e_l\} : \quad \tilde{A}(e;d,s) = (F_{\mu_\alpha}(e;d,s),\, F_{\nu_\alpha}(e;d,s)),$$

where

$$F_{\mu_\alpha}(e;d,s) = \Bigg(\sum_{j\,:\,\mu_\alpha^j(d,s)\ge e} 1 \Bigg) \Big/ r \quad F_{\nu_\alpha}(e;d,s) = \Bigg(\sum_{j\,:\,\nu_\alpha^j(d,s)\ge 1-e} 1 \Bigg) \Big/ r\,.$$

$\tilde{A}$ denotes an intuitionistic fuzzy experton of level e for decision d and state of nature s. An intuitionistic fuzzy experton of the pair $d \in D$, $s \in S$ is defined by averaging the intuitionistic fuzzy expertons of level:

$$\alpha(d,s) = (\mu_\alpha(d,s),\, \nu_\alpha(d,s)) = \left(\sum_{i=1}^{\ell} F_{\mu_\alpha}(e_i;d,s)/l,\; \sum_{i=1}^{\ell} F_{\nu_\alpha}(e_i;d,s)/l \right).$$

Resulting decision matrix is created by averaging intuitionistic fuzzy expertons $\{\alpha(d,s)\},\ d \in D,\ s \in S$.

4.5. **Results of the Realization and Decision Making.** The numerical calculations provide the resulting decision matrix (see Table 6).

In our example, we used the 2-order additive fuzzy measure [10] as a fuzzy measure. Therefore, it is required to introduce some definitions. The main feature of a fuzzy measure is the non-additivity (non-additive set function [5, 33]). In the MCDM framework, the value $g(A),\ A \subseteq S$ of a fuzzy measure g on the criteria set S can be considered as the importance of the criteria subset A, while the monotonicity of a fuzzy measure means that the weight of a criteria subset cannot decrease when new criteria are added to it [9]. The fuzzy measure can flexibly represent a certain kind of an interaction among the decision criteria and can vary from redundancy (negative interaction) to synergy (positive interaction) [10].

Definition 4.1. [10] Let g be a set function (not necessarily a fuzzy measure) on $S = \{s_1, \ldots, s_n\}$. The Mobius representation of g is a set function $m_g : 2^S \to R$ defined by

$$m_g(A) = \sum_{B \subset A} (-1)^{|A \setminus B|} g(B), \quad \forall A \subset S,$$

where $|A|$ is the cardinality of the set A, $R = (-\infty, +\infty)$.

Any fuzzy measure (any set function) g can be uniquely expressed in terms of its Mobius representation by

$$g(A) = \sum_{B \subset A} m_g(B), \quad \forall A \subset S.$$

Definition 4.2. [10] Let $k \in \{1, 2, \ldots, n\}$, a fuzzy measure g on S is said to be $k-$order additive, if its Mobius representation satisfies $m_g(A) = 0$ for all $A \subset S$ such that $|A| > k$ and there exists at least one subset $A \subset S$ such that $m_g(A) \neq 0$.

It should be noted, that 1-order additive fuzzy measure coincides with the additive measure. From Definition 4.2, a fuzzy measure is defined by $\sum_{l=1}^{k} (C_n^l)$ coefficients. For instance, we only need $n(n+1)/2$ coefficients to determine a 2-order additive fuzzy measure.

Definition 4.3. [10] Let g be a fuzzy measure on S. 1) The overall importance of a criterion $s_i \in S$ is called its Shapley value, defined by

$$I_i = \sum_{A \subset S \setminus \{s_i\}} [(|S| - |A| - 1)!/(|S|!)] \cdot [g(A \cup \{s_i\}) - g(A)], \tag{17}$$

2) The interactive index of two criteria $s_i, s_j \in S$ is defined by

$$I_{ij} = \sum_{A \subset S \setminus \{s_i, s_j\}} [(|S| - |A| - 2)!/(|S|! - 1)] \cdot [g(A \cup \{s_i, s_j\}) - g(A \cup \{s_i\}) - g(A \cup \{s_j\}) + g(A)].$$

By the equation (17), it is easy to verify that $\sum_{i=1}^{n} I_i = 1$. It is also obvious that for a 2-order additive fuzzy measure we have

$$I_i = m_g(\{s_i\}) + (1/2) \cdot \sum_{s_j \in S \setminus \{s_i\}} m_g(\{s_i, s_j\}), \quad I_{ij} = m_g(\{s_i, s_j\}),$$

and $I_A = 0$ for $|A| > 2$. Using mathematical induction for a 2-order additive measure we can easily prove that $\forall \sigma = \{\sigma(1), \ldots, \sigma(n)\} \in S_n$ and $i = 1, \ldots, n$

$$g(\{s_{\sigma(1)}, \ldots, s_{\sigma(i)}\}) = \sum_{l=1}^{i} I_{\sigma(l)} - (1/2) \cdot \sum_{k \in N_{\sigma(i)}} \sum_{l=1}^{i} I_{\sigma(i)l} \ ,$$

	s_1	s_2	s_3	s_4	s_5	I_i
s_1	-	0.05	0.12	0.12	0.10	0.25
s_2	0.05	-	0.08	0.08	0.08	0.15
s_3	0.12	0.08	-	0.12	0.05	0.25
s_4	0.12	0.08	0.12	-	0.07	0.20
s_5	0.10	0.08	0.05	0.07	-	0.15

Table 7. The Matrix of Criteria Interaction Indexes

where $N_{\sigma(i)}$ denotes an index subset $N_{\sigma(i)} = \{1, \ldots, n\} \backslash \{\sigma(1), \ldots, \sigma(n)\}$. Using definition of the associated probabilities of a fuzzy measure (see Definition 2.5), we obtain connections between associated probabilities, overall importance and interactive index of criteria:

$$P_\sigma(s_{\sigma(i)}) = I_i + (1/2) \cdot \sum_{l=1}^{i-1} I_{\sigma(i)\sigma(l)} - (1/2) \cdot \sum_{l=i+1}^{n} I_{\sigma(i)\sigma(l)}, \tag{18}$$

where, if in (18) $i = 1$, then the second addend is zero, and if $i = n$, then the third addend is zero. Representation of the associated probability (18) has an interesting interpretation in terms of the representation of interaction between criteria. From Definition 2.5 it is obvious that associate probabilities of 2-order additive fuzzy measure are defined by the following monotone consonant structure:

$$\{s_{\sigma(1)}\},\ \{s_{\sigma(1)}, s_{\sigma(2)}\},\ \ldots\ , \{s_{\sigma(1)}, s_{\sigma(2)}, \ldots, s_{\sigma(n)}\}\,.$$

Then, in (18), we have corresponding interaction indexes of $\{s_{\sigma(1)}\}$, $\{s_{\sigma(1)}, s_{\sigma(2)}\}$, $\ldots$, $\{s_{\sigma(1)}, s_{\sigma(2)}, \ldots, s_{\sigma(i-1)}\}$ structure in the positive role and relevant interaction indexes of $\{s_{\sigma(i+1)}\}$, $\ldots$, $\{s_{\sigma(i+1)}, \ldots, s_{\sigma(n)}\}$, structure in the negative role. So are constructed $n \times n!$ probabilities with $n(n+1)/2$ interaction indexes $J = \{I_{ij}\}$, $i \neq j$, $I_{ij} = I_{ji}$ and n values of overall importance ($I = \{I_i\}$, $i = 1, \ldots, n$).

To construct associate probabilities the experts (DMPs) assessed values of overall importance and values of interaction indexes. Then they generated the criteria weights as values of overall importance based on the consensus:

$$I_1 = w_1 = 0.25;\ I_2 = w_2 = 0.15;\ I_3 = w_3 = 0.25;\ I_4 = w_4 = 0.20;\ I_5 = w_5 = 0.15.$$

Also based on the consensus the experts presented the interaction indexes matrix (see Table 7).

We used the Associated Probability Intuitionistic Fuzzy Weighted Averaging (As-P-IFWA) and the Associated Probability Intuitionistic Fuzzy Weighted Geometric (As-P-IFWG) operators to rank possible alternatives. Based on (18), we calculated an APC of a 2-order additive fuzzy measure (number of associated probability distributions is 5! =120). By formulas (4)–(7) we calculated values of the $As - F_M$ ($\beta = 0.7$) operators. For comparison, we also calculated values of known Intuitionistic Fuzzy Weighted Averaging (IFWA) and Intuitionistic Fuzzy Weighted Geometric (IFWG) operators [51, 52]. It should be noted that the 2-order additive fuzzy measure received is not a capacity of order two. Based on Proposition 3.6 there should be a difference between values of IFCA (IFCG) and As-P-IFWA (As-P-IFWG) operators (see Table 8).

Aggregation Operators	d_1	d_2	d_3	d_4	d_5
$IFWA$	$(0.71, 0.27)$	$(0.65, 0.34)$	$(0.65, 0.32)$	$(0.66, 0.29)$	$(0.62, 0.33)$
$IFWG$	$(0.70, 0.29)$	$(0.64, 0.35)$	$(0.64, 0.34)$	$(0.65, 0.29)$	$(0.58, 0.35)$
$IFCA$	$(0.69, 0.30)$	$(0.62, 0.38)$	$(0.60, 0.37)$	$(0.69, 0.27)$	$(0.68, 0.28)$
$IFCG$	$(0.68, 0.32)$	$(0.62, 0.38)$	$(0.58, 0.39)$	$(0.68, 0.28)$	$(0.66, 0.29)$
$As{-}P{-}IFWA_{\vee}$	$(0.70, 0.29)$	$(0.63, 0.37)$	$(0.62, 0.35)$	$(0.68, 0.28)$	$(0.67, 0.29)$
$As{-}P{-}IFWA_{\wedge}$	$(0.69, 0.29)$	$(0.63, 0.37)$	$(0.61, 0.36)$	$(0.67, 0.28)$	$(0.65, 0.30)$
$As{-}P{-}IFWG_{\vee}$	$(0.69, 0.31)$	$(0.63, 0.37)$	$(0.60, 0.37)$	$(0.67, 0.28)$	$(0.65, 0.30)$
$As{-}P{-}IFWG_{\wedge}$	$(0.68, 0.32)$	$(0.62, 0.38)$	$(0.59, 0.38)$	$(0.66, 0.28)$	$(0.63, 0.31)$

Table 8. The Overall Evaluations Aggregated by the Four New $As - F_M$ Operators and IFWA, IFWG, IFCA, IFCG Operators

Aggregation Operators	Ranking order
$IFWA$	$\mathrm{d}_1 \succ \mathrm{d}_4 \succ \mathrm{d}_3 \succ \mathrm{d}_2 \succ \mathrm{d}_5$
$IFWG$	$\mathrm{d}_1 \succ \mathrm{d}_4 \succ \mathrm{d}_3 \succ \mathrm{d}_2 \succ \mathrm{d}_5$
$IFCA$	$\mathrm{d}_4 \succ \mathrm{d}_5 \succ \mathrm{d}_1 \succ \mathrm{d}_2 \succ \mathrm{d}_3$
$IFCG$	$\mathrm{d}_4 \succ \mathrm{d}_5 \succ \mathrm{d}_1 \succ \mathrm{d}_2 \succ \mathrm{d}_3$
$As - P - IFWA_{\vee}$	$\mathrm{d}_1 \succ \mathrm{d}_4 \succ \mathrm{d}_5 \succ \mathrm{d}_3 \succ \mathrm{d}_2$
$As - P - IFWA_{\wedge}$	$\mathrm{d}_1 \succ \mathrm{d}_4 \succ \mathrm{d}_5 \succ \mathrm{d}_2 \succ \mathrm{d}_3$
$As - P - IFWG_{\vee}$	$\mathrm{d}_4 \succ \mathrm{d}_1 \succ \mathrm{d}_5 \succ \mathrm{d}_2 \succ \mathrm{d}_3$
$As - P - IFWG_{\wedge}$	$\mathrm{d}_4 \succ \mathrm{d}_1 \succ \mathrm{d}_5 \succ \mathrm{d}_2 \succ \mathrm{d}_3$

Table 9. The Ranking Orders Determined by the Aggregation Operators Based on the Relation $\succeq$

Total order relation $\geq_t$ (Section 2) induces ranking order relation on all possible alternatives:

$$d_i \succeq d_j \ \Leftrightarrow As - F_M(d_i) \geq_t As - F_M(d_j) \ \text{(see Table 9)}.$$

Table 9 shows that the well-known IFWA, IFWG, IFCA and IFCG operators' results are the particular cases of the $As - F_M$ operators' results. The optimal decisions can be d_4 (Hookah bar) or d_1 (Wine Shop). As expected, the different results are received by the conjugate $As - F_M$ operators. However, we cannot claim this for the conjugate operators IFWA and IFWG, since interaction of factors is not taken into account. The same applies to the conjugate operators IFCA and IFCG, because interaction of factors is considered only partially. In conclusion, we can note that the results indicate on the opportunity of the $As - F_M$ operators to consider the DMP's particular interests concerning decision risks.

5. **Conclusions**

In this work we created new extensions of the Probability Intuitionistic Fuzzy Weighted Averaging (P-IFWA) and the Probability Intuitionistic Fuzzy Weighted Geometric (P-IFWG) operators - $As - F_M$, $F \in \{P - IFWA, P - IFWG\}$, $M \in$

$\{\vee, \wedge\}$, where an uncertainty is presented by a fuzzy measure. New operators are defined with respect to associated probability class (APC) of a fuzzy measure. There are four versions of expressions of the $As - F_M$ operators for concrete intuitionistic operations: intuitionistic min and max, and two versions with respect to a concrete fuzzy measure.

We proved propositions on the correctness of the extensions. 1. The $As - F_M$ operators for the capacity of order 2 coincide with the Intuitionistic Fuzzy Choquet Averaging (IFCA) or Intuitionistic Fuzzy Choquet Geometric (IFCG) operators; 2. The $As - F_M$ operators coincide with the Intuitionistic Fuzzy Probabilistic Averaging (IFPA) or Intuitionistic Fuzzy Probabilistic Geometric (IFPG) operators when a probability measure is used in the role of a fuzzy measure.

In contrast to the Choquet integral aggregation, the constructed operators and their conjugate operators allow us to deal with the interactions among all the factors (states of nature) in intuitionistic fuzzy MCDM problems.

We also provided illustrative example of the applicability of the new aggregation operators - $As - F_M$ and considered the fuzzy group decision making problem regarding the business start-up. Risk management plays an important role in the business start-up operations. Risk assessment is typically used in the development process to identify and evaluate the risks inherent to the business start-up problems. We also compared the values of the IFWA, IFWG, IFCA and IFCG operators and those of the $As - F_M$ operators. Results indicate that the $As - F_M$ operators give an opportunity to consider the DMP's particular viewpoint on decision risks for the concrete problem, which can vary in the spectrum from pessimistic to optimistic expectations.

Other presentations of the newly developed operators, including interval valued intuitionistic, hesitant fuzzy and immediate fuzzy probability aggregations, will be considered in our following works.

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