PROPERTY ANALYSIS OF TRIPLE IMPLICATION METHOD FOR APPROXIMATE REASONING ON ATANASSOVS INTUITIONISTIC FUZZY SETS

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Abstract. Firstly, two kinds of natural distances between intuitionistic fuzzy sets are generated by the classical natural distance between fuzzy sets under a unified framework of residual intuitionistic implication operators. Secondly, the continuity and approximation property of a method for solving intuitionistic fuzzy reasoning are defined. It is proved that the triple implication method for intuitionistic fuzzy modus ponens has both continuity and approximation property with respect to these two kinds of natural distances based on Łukasiewicz implication, meanwhile the triple implication method for intuitionistic fuzzy modus tollens possesses unconditional continuity and conditional approximation property. Finally, some robustness results about the triple implication method of intuitionistic fuzzy reasoning are given.

1. Introduction

Since fuzzy set was firstly proposed by cybernetics expert Zadeh in 1965 (see [51]), it has provided a powerful tool to deal with fuzzy and uncertain information. It is well known that the fuzzy reasoning is a significant part of the theory of fuzzy sets. A flourishing achievement in fuzzy reasoning spreads out from the seminal paper of Zadeh [53] (see [8, 16, 18, 20, 22, 34, 35, 40, 43, 44, 46, 47, 48, 52, 54, 55]). Generally, two basic models of fuzzy reasoning are Fuzzy Modus Ponens and Fuzzy Modus Tollens. The basic reasoning principle underlying fuzzy modus ponens and Fuzzy Modus Tollens was Zadeh's Composition Rule of Inference (CRI). With the CRI method, fuzzy reasoning has been successfully applied to many fields (see [23, 32, 49]).

As an alternative for the CRI method, Wang [45] proposed a new method, called the full implication triple implication method, or simply the triple implication method, for fuzzy reasoning. This method may bring fuzzy reasoning within the framework of logical semantic implication, and it may be considered as a reasonable alternative or complement for the CRI method. Recently, many results of the triple implication method have been reported. Pei [36, 37] systematically discussed the method by using a class residual implications. Wang and Fu [42] gave the unified forms of the triple implication method by using the regular implication. Liu and Wang [29] obtained a class of restriction triple implication solution for

Received: March 2017; Revised: July 2017; Accepted: December 2017

Key words and phrases: Intuitionistic fuzzy reasoning, Triple implication method, Continuity, Approximation property, Robustness.

fuzzy reasoning. Pei [33] discussed the formalization problem of the triple implication fuzzy reasoning based on some first order logic systems and gave a sound logic deduce system for the unified triple implication method.

At the same time, the fuzzy control system requires the reasoning method to satisfy the approximation principle. Similar to approximation property, the continuity, perturbation and robustness can be used as a judge to decide whether the property of the existing reasoning method is good or bad. So analysis of properties of the fuzzy reasoning methods is a very active area of fuzzy research. Whether CRI method and triple implication method hold the continuity and approximation property was deeply discussed by Xu (see [50]), and how the approximation errors are propagated by them was given at last. Liu [27] carefully investigated the continuity and uniform continuity of the triple implication method. Dai discussed perturbation of fuzzy sets and fuzzy reasoning based on normalized Minkowski distances in [11].

While robustness analysis of fuzzy reasoning methods appears vital in particular. The class of t-norms and implications which are copulas and 1-l_∞-Lipschitz can guarantee good robustness of rule-based fuzzy reasoning (see [7]). In the literature [10], minimum similarity was used as a tool to analyse the robustness of triple implication method for fuzzy reasoning. However, Wang and Duan revised the concept of (T, δ) equality as a measure to evaluate robustness of fuzzy reasoning and proved the continuity of the triple implication method with respect to the revised measures (see [41]). Li et al. [26] extended the concept of perturbations of fuzzy sets and studied the perturbations of fuzzy connectives and fuzzy reasoning based on the proposed divergence measures composed by DF-metric. He et al. [21] discussed the robustness of fuzzy connectives and fuzzy reasoning by using the new defined robustness index. Duan and Li [15] analyzed whether there are isolated points for the four specific logic metric spaces on $F(X)$, and they concluded that the logic metrics induced by Łukasiewicz implication and Goguen implication are more beneficial to the robustness analysis of fuzzy reasoning. From a knowledge representation point of view, fuzzy sets should be handled in some kinds of possible world semantics, however, the role of fuzziness is not always to capture uncertainty (see [17, 19]).

For instance, people make choices by checking the good sides and the bad sides of alternatives separately. Then they choose according to whether the good or the bad sides are stronger. The intuitionistic fuzzy sets introduced by Atanassov is a pair of fuzzy sets, namely a membership and a non-membership function (see [1, 2, 4, 5]), which can represent positive and negative aspects of the given information respectively, therefore it constitutes an appropriate knowledge representation framework, and has been widely applied in many fields such as pattern recognition, machine learning, decision making, image processing and so on (see [6, 12, 25, 28, 30]). But since Atanassovs intuitionistic fuzzy reasoning is not as natural as most of the fuzzy reasoning, especially restrictions on the intuitionistic fuzzy implication operators are much more complicated than that of fuzzy implication operators, so it seems that approximate reasoning based on Atanassovs intuitionistic fuzzy sets have not been given enough attention to.

In recent years, some scholars try to construct a logic foundation for intuitionistic fuzzy reasoning. Deschrijver et al. [13, 14] have made fruitful pioneering work, they presented the intuitionistic fuzzy t-norms and intuitionistic fuzzy implication operator theory. Begum and Srinivasan [39] defined the basic operations on the Intuitionistic Fuzzy Sets of Third Type (IFSTT) and established the relation between the modal operators like Necessity and Possibility. Cornelis and Deschrijver [9] put forward the ICRI method for intuitionistic fuzzy reasoning problems (i.e., IFMP and IFMT) by extending the CRI method to the intuitionistic fuzzy setting. Zheng et al. [56, 57] presented the representation of intuitionistic fuzzy implications by fuzzy operators, they extended the triple implication method of fuzzy reasoning on intuitionistic fuzzy reasoning and showed that the triple implication method of Intuitionistic Fuzzy Modus Ponens possessed virtue of reductivity and the triple implication method of Intuitionistic Fuzzy Modus Ponens was local reductive.

But so far, whether the existing triple implication method of intuitionistic fuzzy reasoning satisfies the approximation property? How about the continuity and robustness of it? These issues have not yet been studied. The present paper aims to investigate the properties of continuity, approximation and robustness of the triple implication method of intuitionistic fuzzy reasoning by extending the natural distance between fuzzy sets to intuitionistic fuzzy sets based on two basic inference models: Intuitionistic fuzzy modus ponens and Intuitionistic fuzzy modus tollens, and some results about continuity, approximation and robustness are proved based on Łukasiewicz implication.

The rest of this paper is organized as follows. In section 2, we review some necessary definitions, propositions and theorems, and propose the definition of continuity and approximation properties in the end. In section 3, we use some lemmas related to this paper to discuss the continuity, approximation properties and robustness of triple implication method for intuitionistic fuzzy reasoning with respect to Intuitionistic Fuzzy Modus Ponens and Intuitionistic Fuzzy Modus Tollens models. The final section includes our conclusions.

2. **Preliminaries**

In this section, we firstly review some basic concepts and results about intuitionistic fuzzy sets.

Definition 2.1. [1, 2, 4, 5] An intuitionistic fuzzy set on the nonempty universe of discourse X is given by $A = \{(x, A_t(x), A_f(x))|x \in X\}$ where

$$A_t : X \to [0, 1],\ A_f : X \to [0, 1]$$

with the condition

$$0 \leqslant A_t(x) + A_f(x) \leqslant 1, \forall x \in X.$$

$A_t(x)$ and $A_f(x)$ denote the membership function and the non-membership function of x to A, respectively.

It is clear that the intuitionistic fuzzy set A in X can be written as:

$$A(x) = (u, v),\ 0 \leqslant u + v \leqslant 1,\ u, v \in [0, 1],\ \forall x \in X.$$

As a generalization of fuzzy sets, intuitionistic fuzzy sets extend the character value from $[0,1]$ to the triangle domain $L^* = \{(u,v) \in [0,1]^2 | u+v \leqslant 1\}$(for a detailed geometrical interpretation of L^*, please see [1, 3, 5]).

If $A_t(x) = 1 - A_f(x)$, $\forall x \in X$, then intuitionistic fuzzy sets degenerate into fuzzy sets. We denote IF(X) as the set of all intuitionistic fuzzy sets in X and $F(X)$ as the set of all fuzzy sets in X.

Let $\alpha, \beta \in L^*, \alpha = (a_1, a_2), \beta = (b_1, b_2)$, we can define a partial order on L^* as follows:

$\alpha \leqslant \beta$ iff $a_1 \leqslant b_1, a_2 \geqslant b_2$.

Obviously, $\alpha \bigwedge \beta = (a_1 \bigwedge b_1, a_2 \bigvee b_2)$,$\alpha \bigvee \beta = (a_1 \bigvee b_1, a_2 \bigwedge b_2)$, $0^* = (0,1)$ and $1^* = (1,0)$ are the smallest element and the greatest element of L^*, respectively. It's easy to verify the fact that $(L^*, \leqslant)$ is a complete lattice.

Definition 2.2. [8, 13, 14] (i) A function $\otimes_{L^*} : {L^*}^2 \to L^*$ is an intuitionistic fuzzy triangular norm (t-norm in short), if it is a commutative, associative and increasing function with neutral element 1^*.
(ii) A function $\oplus_{L^*} : {L^*}^2 \to L^*$ is an intuitionistic fuzzy triangular conorm (t-conorm in short), if it is a commutative, associative and increasing function with neutral element 0^*.
(iii) A binary function $\to_{L^*}: {L^*}^2 \to L^*$ satisfying the conditions:

$$0^* \to_{L^*} 0^* = 0^* \to_{L^*} 1^* = 1^* \to_{L^*} 1^* = 1^*$$

and

$$1^* \to_{L^*} 0^* = 0^*$$

is called an intuitionistic fuzzy implication.
(iv) An intuitionistic fuzzy negation $' : L^* \to L^*$ satisfies, for all $\alpha, \beta \in L^*$, the following properties:

(1) ${0^*}^{'} = 1^*$ and ${1^*}^{'} = 0^*$.
(2) If $\alpha \geqslant \beta$, then $\alpha^{'} \leqslant \beta^{'}$.

Now we give two binary operations $\otimes_{L^*}$,$\oplus_{L^*}$ on L^* as follows:

$$\alpha \otimes_{L^*} \beta = (a_1 \otimes b_1, a_2 \oplus b_2),\ \alpha \oplus_{L^*} \beta = (a_1 \oplus b_1, a_2 \otimes b_2),$$

where $\oplus$ is the dual t-conorm of the t-norm $\otimes$.

First, we should clarify that two binary operations are effective, i.e., $\alpha \otimes_{L^*} \beta, \alpha \oplus_{L^*} \beta \in L^*$. Since $\alpha, \beta \in L^*$, i.e., $(a_1, a_2) \in [0,1]^2, (b_1, b_2) \in [0,1]^2, a_1 + a_2 \leqslant 1, b_1 + b_2 \leqslant 1$, it follows from the monotonicity of $\oplus$ and the duality of $\otimes$, $\oplus$ that $a_1 \otimes b_1 + a_2 \oplus b_2 \leqslant a_1 \otimes b_1 + (1-a_1) \oplus (1-b_1) = a_1 \otimes b_1 + 1 - a_1 \otimes b_1 = 1$, i.e., $\alpha \otimes_{L^*} \beta \in L^*$.

Similarly, $\alpha \oplus_{L^*} \beta \in L^*$.

It is easy to verify that $\otimes_{L^*}$ and $\oplus_{L^*}$ as defined above are intuitionistic t-norm and intuitionistic t-conorm respectively.

Definition 2.3. [31] Let X be non-empty universe of discourse, $A_1, A_2 \in F(X)$, $F(X)$ denotes the set of all fuzzy sets in X, then the natural distance between A_1

and A_2 is given by:

$$d_n(A_1, A_2) = \bigvee_{x\in X} |A_1(x) - A_2(x)|.$$

Definition 2.4. Let X be non-empty universe of discourse, $A_1, A_2 \in IF(X)$, then the natural distance between A_1 and A_2 can be given by:

(i) $d_{N_1}(A_1, A_2) = d_n\big(A_{1t}, A_{2t}\big) \bigvee d_n\big(A_{1f}, A_{2f}\big)$.

(ii) $d_{N_2}(A_1, A_2) = \frac{1}{2}\Big(d_n\big(A_{1t}, A_{2t}\big) + d_n\big(A_{1f}, A_{2f}\big)\Big)$.

Proof. We only prove that d_{N_1} is a distance on $IF(X)$, the proof of d_{N_2} can be given in a similar way.

Firstly, for any $A_1 = (A_{1t}, A_{1f}), A_2 = (A_{2t}, A_{2f}) \in IF(X)$, it is clear that $d_{N_1}(A_1, A_2) \geqslant 0$. Suppose that $A_1 = A_2$, then we have $A_{1t} = A_{2t}$ and $A_{1f} = A_{2f}$. Notice that d_n is the natural distance on $F(X)$, so it follows from $d_n(A_{1t}, A_{2t}) = d_n(A_{1f}, A_{2f}) = 0$ that $d_{N_1}(A_1, A_2) = 0$. Conversely, if $d_{N_1}(A_1, A_2) = 0$, i.e., $d_n(A_{1t}, A_{2t}) \bigvee d_n(A_{1f}, A_{2f}) = 0$, then $d_n(A_{1t}, A_{2t}) = d_n(A_{1f}, A_{2f}) = 0$, we have $A_{1t} = A_{2t}$ and $A_{1f} = A_{2f}$. Therefore, $A_1 = A_2$.

Secondly, it is obvious that $d_{N_1}(A_1, A_2) = d_{N_1}(A_2, A_1)$ holds for any $A_1, A_2 \in IF(X)$.

Thirdly, let $A_1 = (A_{1t}, A_{1f}), A_2 = (A_{2t}, A_{2f}), A_3 = (A_{3t}, A_{3f}) \in IF(X)$, it follows from the transitivity of d_n that $d_n(A_{1t}, A_{3t}) \leqslant d_n(A_{1t}, A_{2t}) + d_n(A_{2t}, A_{3t})$ and $d_n(A_{1f}, A_{3f}) \leqslant d_n(A_{1f}, A_{2f}) + d_n(A_{2f}, A_{3f})$. Since

$$\begin{aligned} d_{N_1}(A_1, A_3) &= d_n(A_{1t}, A_{3t}) \vee d_n(A_{1f}, A_{3f}) \\ &\leqslant \big[d_n(A_{1t}, A_{2t}) + d_n(A_{2t}, A_{3t})\big] \vee \big[d_n(A_{1f}, A_{2f}) + d_n(A_{2f}, A_{3f})\big] \\ &\leqslant \big[d_n(A_{1t}, A_{2t}) \vee d_n(A_{1f}, A_{2f})\big] + \big[d_n(A_{2t}, A_{3t}) \vee d_n(A_{2f}, A_{3f})\big] \\ &\leqslant d_{N_1}(A_1, A_2) + d_{N_1}(A_2, A_3). \end{aligned}$$

In summary, d_{N_1} is a distance on $IF(X)$. This completes the proof. □

Remark 2.5. It can be easily proved that both d_{N_1} and d_{N_2} given by definition 2.3 are distances on IF(X).

Proposition 2.6. [56, 57] *$(L^*, \otimes_{L^*}, 1^*)$ is a commutative monoid and $\otimes_{L^*}$ is isotone, $(L^*, \oplus_{L^*}, 0^*)$ is a commutative monoid and $\oplus_{L^*}$ is isotone.*

Proposition 2.7. [56, 57] *Let $\otimes$ be a left-continuous t-norm and I be the index set, then*

(i) $\otimes_{L^*}$ is a left-continuous intuitionistic t-norm on L^*, i.e.,

$$(\bigvee_{i\in I} \alpha_i) \otimes_{L^*} \gamma = \bigvee_{i\in I} (\alpha_i \otimes_{L^*} \gamma), \forall \alpha_i, \gamma \in L^*.$$

(ii) $\oplus_{L^*}$ is a right-continuous intuitionistic t-conorm on L^*, i.e.,

$$(\bigwedge_{i\in I} \alpha_i) \oplus_{L^*} \gamma = \bigwedge_{i\in I} (\alpha_i \oplus_{L^*} \gamma), \forall \alpha_i, \gamma \in L^*.$$

Theorem 2.8. [56, 57] *Let $\otimes_{L^*}$ be an intuitionistic t-norm induced by a left-continuous t-norm $\otimes$, then there exists a binary operation $\rightarrow_{L^*}$ on L^* such that*

$$\gamma \otimes_{L^*} \alpha \leqslant \beta \ iff \ \gamma \leqslant \alpha \rightarrow_{L^*} \beta, \tag{1}$$

and $\rightarrow_{L^*}$ is given by

$$\alpha \rightarrow_{L^*} \beta = \bigvee \{\eta \in L^* | \eta \otimes_{L^*} \alpha \leqslant \beta\}. \tag{2}$$

Definition 2.9. [56] $(\otimes_{L^*}, \rightarrow_{L^*})$ is called an intuitionistic adjoint pair and $\rightarrow_{L^*}$ is called a residual intuitionistic implication if $(\otimes_{L^*}, \rightarrow_{L^*})$ satisfies the residual principle (1).

Proposition 2.10. [56] *Let $\otimes_{L^*}$ be an intuitionistic t-norm on L^* and $(\otimes_{L^*}, \rightarrow_{L^*})$ be an intuitionistic adjoint pair on L^*, then*

(i) $\alpha \rightarrow_{L^*} \beta = 1^*$ iff $\alpha \leqslant \beta$.
(ii) $\gamma \leqslant \alpha \rightarrow_{L^*} \beta$ iff $\alpha \leqslant \gamma \rightarrow_{L^*} \beta$.
(iii) $\gamma \rightarrow_{L^*} (\alpha \rightarrow_{L^*} \beta) = \alpha \rightarrow_{L^*} (\gamma \rightarrow_{L^*} \beta)$.
(iv) $1^* \rightarrow_{L^*} \alpha = \alpha$.
(v) $\beta \rightarrow_{L^*} \bigwedge_{i \in I} \alpha_i = \bigwedge_{i \in I} (\beta \rightarrow_{L^*} \alpha_i)$.
(vi) $\bigvee_{i \in I} \beta_i \rightarrow_{L^*} \alpha = \bigwedge_{i \in I} (\beta_i \rightarrow_{L^*} \alpha)$.
(vii) $\rightarrow_{L^*}$ is antitone in the first variable and isotone in the second variable.

Now we extend the triple implication method of Fuzzy Modus Ponens and Fuzzy Modus Tollens on intuitionistic fuzzy sets. We consider the problem of Intuitionistic Fuzzy Modus Ponens and Intuitionistic Fuzzy Modus Tollens as follows:

Suppose that	$A(x) \rightarrow_{L^*} B(y)$		$\cdots$ major premise
and given	$A^*(x)$		$\cdots$ minor premise
calculate		$B^*(y)$	$\cdots$ conclusion

Suppose that	$A(x) \rightarrow_{L^*} B(y)$		$\cdots$ major premise
and given		$B^*(y)$	$\cdots$ minor premise
calculate	$A^*(x)$		$\cdots$ conclusion

where $A(x), A^*(x)$ are the intuitionistic fuzzy sets on X, respectively, $B(y), B^*(y)$ are the intuitionistic fuzzy sets on Y, respectively, and $\rightarrow_{L^*}$ is the residual intuitionistic fuzzy implication on L^*. We denote $A(x) = \big(A_t(x), A_f(x)\big), B(y) = \big(B_t(y), B_f(y)\big), A^*(x) = \big(A_t^*(x), A_f^*(x)\big), B^*(y) = \big(B_t^*(y), B_f^*(y)\big), A_{-f}(x) = 1 - A_f(x), B_{-f}(y) = 1 - B_f(y), A_{-f}^*(x) = 1 - A_f^*(x), B_{-f}^*(y) = 1 - B_f^*(y)$. Clearly, $A_t, A_f, A_t^*, A_f^*, A_{-f}, A_{-f}^*$ are the fuzzy sets on X, respectively, and $B_t, B_f, B_t^*, B_f^*, B_{-f}, B_{-f}^*$ are the fuzzy sets on Y, respectively.

Because $\rightarrow_{L^*}$ is the residual intuitionistic fuzzy implication on L^*, we could smoothly reach the extension of the Triple Implication Principle for Intuitionistic Fuzzy Modus Ponens as follows: $B^*(y)$ should be the smallest intuitionistic fuzzy set on Y satisfying

$$\big(A(x) \rightarrow_{L^*} B(y)\big) \rightarrow_{L^*} \big(A^*(x) \rightarrow_{L^*} B^*(y)\big) = 1^*, \tag{3}$$

under the order of L^*.

Similarly, the Triple Implication Principle for Intuitionistic Fuzzy Modus Tollecs can be extended as follows: $A^*(x)$ should be the biggest intuitionistic fuzzy set on X satisfying (3) under the order of L^*.

Theorem 2.11. [56, 57] *Suppose that the implication $\to_{L^*}$ in IFMP is the residual intuitionistic implication induced by a left-continuous t-norm $\otimes$, then the triple implication solution B^* of IFMP is given by the following formula*

$$B^*(y) = \bigvee_{x\in X} \left\{A^*(x) \otimes_{L^*} \left(A(x) \to_{L^*} B(y)\right)\right\},\ y \in Y. \tag{4}$$

and the decomposition expression of $B^*(y)$ is given by

$$B^*(y) = \Big(\bigvee_{x\in X} \Big\{A_t^*(x) \otimes \Big(\big(A_t(x) \to B_t(y)\big) \bigwedge \big(A_{-f}(x) \to B_{-f}(y)\big)\Big)\Big\}, \bigwedge_{x\in X} \left\{A_f^*(x) \oplus \left(1 - A_{-f}(x) \to B_{-f}(y)\right)\right\} \Big). \tag{5}$$

Theorem 2.12. [56] *Let the implication $\to_{L^*}$ in IFMT be the residual intuitionistic implication induced by a left-continuous t-norm $\otimes$, then the triple implication solution A^* of IFMT is given by the following formula*

$$A^*(x) = \bigvee_{y\in Y} \Big\{\big(A(x) \to_{L^*} B(y)\big) \to_{L^*} B^*(y)\Big\},\ x \in X. \tag{6}$$

and the decomposition expression of $A^*(x)$ is given by

$$A^*(x) = \Big(\bigwedge_{y\in Y} \Big\{ \Big(\Big(\big(A_t(x) \to B_t(y)\big) \bigwedge \big(A_{-f}(x) \to B_{-f}(y)\big)\Big) \to B_t^*(y)\Big) \bigwedge \Big(\big(A_{-f}(x) \to B_{-f}(y)\big) \to B_{-f}^*(y)\Big)\Big\}, \bigvee_{y\in Y} \left\{1 - \left(A_{-f}(x) \to B_{-f}(y)\right) \to B_{-f}^*(y)\right\} \Big). \tag{7}$$

Definition 2.13. Let X, Y be non-empty universe of discourse, $A \to B$ be the known rule. An inference method for solving IFMP is an $IF(X) \to IF(Y)$ mapping g, i.e., each input $A^* \in IF(X)$ corresponds to an output $B^* = g(A^*)$. Let d_N be the natural distance between intuitionistic fuzzy sets.

(i) g is said to be continuous at A under metric d_N, if for any $\varepsilon > 0$, there exist $\delta > 0$ such that $d_N\big(g(A^*), g(A)\big) < \varepsilon$ whenever $d_N(A^*, A) < \delta$ for any $A^* \in IF(X)$.

(ii) g is said to be of approximation property at $B \in IF(Y)$ under metric d_N, if for any $\varepsilon > 0$, there exists $\delta > 0$ such that $d_N\big(g(A^*), B\big) < \varepsilon$ whenever $d_N(A^*, A) < \delta$ for any $A^* \in IF(X)$.

(iii) g is said to be reductive if $g(A) = B$ holds, i.e., $B^* = B$ holds whenever $A^* = A$.

Definition 2.14. Let X, Y be non-empty universe of discourse, $A \to B$ be the known rule. A method for solving IFMT is an $IF(Y) \to IF(X)$ mapping h, i.e., each input $B^* \in IF(Y)$ corresponds to an output $A^* = h(B^*)$. Let d_N be the natural distance between intuitionistic fuzzy sets.

(i) h is said to be continuous at B under metric d_N, if for any $\varepsilon > 0$, there exists $\delta > 0$ such that $d_N\big(h(B^*), h(B)\big) < \varepsilon$ whenever $d_N(B^*, B) < \delta$ for any $B^* \in IF(Y)$.

(ii) h is said to be of approximation property at $A \in IF(X)$ under metric d_N, if for any $\varepsilon > 0$, there exists $\delta > 0$ such that $d_N\big(h(B^*), A)\big) < \varepsilon$ whenever $d_N(B^*, B) < \delta$ for any $B^* \in IF(Y)$.

(iii) h is said to be reductive if $h(B) = A$ holds, i.e., $A^* = A$ holds whenever $B^* = B$.

Remark 2.15. The choice of distance in Definition 2.13 and Definition 2.14 is not unique. It could be any other one, such as the Hamming distance and the Euclidean distance.

Remark 2.16. (i) It can be easily seen that an inference method g for Intuitionistic Fuzzy Modus Ponens must be of approximation property if g is both continuous and reductive, and the same conclusion also holds with respect to the method h for Intuitionistic Fuzzy Modus Tollens.

(ii) The continuity of g reflects the fact that if the input A^* is very close to A then the output B^* (i.e., $g(A^*)$) can be very close to $g(A)$. The approximation property of g means that if the input A^* is very close to A then the output B^* is very close to B.

A similar analysis can be done with the method for Intuitionistic Fuzzy Modus Tollens.

3. The Continuity, Approximation Property and Robustness of Triple Implication Method

As is well known that fuzzy logic is the mathematical foundation of fuzzy reasoning while the completeness theorem is one of the important criteria to judge whether logic system is good or bad, the fuzzy logic system based on Łukasiewicz implication is one of the most important fuzzy logic system due to the completeness theorem holds in it. In addition, since Łukasiewicz implication is continuous and is of good transitivity, so it is one of the most common used implication operators in fuzzy reasoning.

In this paper, we focus on the analysis of $\otimes_{L^*}$ and its residuum $\to_{L^*}$ induced by Łukasiewicz t-norm. In the subsequent sections, unless otherwise stated, $\otimes$, $\oplus$, $\to$ always represents Łukasiewicz t-norm, Łukasiewicz t-conorm and Łukasiewicz implication operator. And $a \otimes b = (a+b-1) \bigvee 0$, $a \oplus b = (a+b) \bigwedge 1$, $a \to b = (1-a+b) \bigwedge 1$. To facilitate future disussion, it is necessary to introduce the following three lemmas.

Lemma 3.1. [35] *Let f and g be bounded, real valued functions on a set X. Then*

(i) $|\bigvee_{x\in X} f(x) - \bigvee_{x\in X} g(x)| \leqslant \bigvee_{x\in X} |f(x)-g(x)|$,
(ii) $|\bigwedge_{x\in X} f(x) - \bigwedge_{x\in X} g(x)| \leqslant \bigvee_{x\in X} |f(x)-g(x)|$.

Lemma 3.2. *For any $a, b, c, d \in [0,1]$, then*

(i) $|a\otimes b - c\otimes d| \leqslant |a-c| + |b-d|, |a\otimes b - c\otimes b| \leqslant |a-c|$.
(ii) $|a\otimes b - c\otimes d| \leqslant 2(|a-c| \bigvee |b-d|)$.
(iii) $|a\oplus b - c\oplus d| \leqslant |a-c| + |b-d|, |a\oplus b - c\oplus b| \leqslant |a-c|$.
(iv) $|a\oplus b - c\oplus d| \leqslant 2(|a-c| \bigvee |b-d|)$.

Proof. Firstly, it is obvious that $|a-c|+|b-d| \leqslant 2(|a-c|\bigvee|b-d|)$.
We only need to prove that $|a\otimes b - c\otimes d| \leqslant |a-c|+|b-d|$, $|a\oplus b-c\oplus d| \leqslant |a-c|+|b-d|$. By using Lemma 3.1(i), we can easily get that

$$\begin{aligned}|a\otimes b - c\otimes d| &= \Big|(a+b-1)\vee 0 - (c+d-1)\vee 0\Big| \\ &\leqslant |a-c+b-d| \leqslant |a-c|+|b-d|.\end{aligned}$$

It follows from Lemma 3.1(ii) that

$$\begin{aligned}|a\oplus b - c\oplus d| &= \Big|(a+b)\wedge 1 - (c+d)\wedge 1\Big| \\ &\leqslant |a-c+b-d| \leqslant |a-c|+|b-d|.\end{aligned}$$

This completes the proof. □

Lemma 3.3. *For any $a, b, c, d \in [0,1]$, then*

(i) $|a\to b - c\to d| \leqslant |a-c|+|b-d|, |a\to b - c\to b| \leqslant |a-c|$.
(ii) $|a\to b - c\to d| \leqslant 2(|a-c|\bigvee|b-d|)$.

Proof. Similarly, by using Lemma 3.1(ii), we have

$$\begin{aligned}|a\to b - c\to d| &= \Big|(1-a+b)\wedge 1 - (1-c+d)\wedge 1\Big| \\ &\leqslant |c-a+b-d| \leqslant |a-c|+|b-d|.\end{aligned}$$

This completes the proof. □

Remark 3.4. Lemma 3.2 also holds when $\otimes$ is Minimum t-norm or Product t-norm.

Theorem 3.5. *The triple implication method g for IFMP based on the Łukasiewicz implication is continuous at A under the metric d_{N_1}.*

Proof. It follows from Definition 2.4 that

$$d_{N_1}\big(g(A^*), g(A)\big) = d_n\big(g(A^*)_t, g(A)_t\big) \bigvee d_n\big(g(A^*)_f, g(A)_f\big). \tag{8}$$

From (8) and (5), we can first obtain that

$$\begin{aligned}d_n\big(g(A^*)_t, g(A)_t\big) &= \bigvee_{y\in Y} |g(A^*)_t(y) - g(A)_t(y)| \\ &= \bigvee_{y\in Y}\Big|\bigvee_{x\in X}\Big\{A_t^*(x)\otimes\Big(\big(A_t(x)\to B_t(y)\big)\bigwedge\big(A_{-f}(x)\to B_{-f}(y)\big)\Big)\Big\} \\ &\quad - \bigvee_{x\in X}\Big\{A_t(x)\otimes\Big(\big(A_t(x)\to B_t(y)\big)\bigwedge\big(A_{-f}(x)\to B_{-f}(y)\big)\Big)\Big\}\Big|.\end{aligned} \tag{9}$$

Then by Lemma 3.1(i) and Lemma 3.2(i), we have

$$\begin{aligned} d_n\big(g(A^*)_t, g(A)_t\big) \leqslant & \bigvee_{y\in Y}\bigvee_{x\in X}\Big|A_t^*(x)\otimes\Big(\big(A_t(x)\to B_t(y)\big)\bigwedge\big(A_{-f}(x)\to B_{-f}(y)\big)\Big) \\ & - A_t(x)\otimes\Big(\big(A_t(x)\to B_t(y)\big)\bigwedge\big(A_{-f}(x)\to B_{-f}(y)\big)\Big)\Big| \\ \leqslant & \bigvee_{x\in X}|A_t^*(x)-A_t(x)|. \end{aligned} \tag{10}$$

Similarly, by (8) and (5), we can easily get that

$$\begin{aligned} d_n\big(g(A^*)_f, g(A)_f\big) &= \bigvee_{y\in Y}|g(A^*)_f(y)-g(A)_f(y)| \\ &= \bigvee_{y\in Y}\Big|\bigwedge_{x\in X}\big\{A_f^*(x)\oplus\big(1-A_{-f}(x)\to B_{-f}(y)\big)\big\} \\ &\quad - \bigwedge_{x\in X}\big\{A_f(x)\oplus\big(1-A_{-f}(x)\to B_{-f}(y)\big)\big\}\Big|. \end{aligned} \tag{11}$$

According to equation(11), Lemma 3.1(ii) and Lemma 3.2(iii), it is obvious that

$$\begin{aligned} d_n\big(g(A^*)_f, g(A)_f\big) \leqslant & \bigvee_{y\in Y}\bigvee_{x\in X}\Big|A_f^*(x)\oplus\big(1-A_{-f}(x)\to B_{-f}(y)\big) \\ & - A_f(x)\oplus\big(1-A_{-f}(x)\to B_{-f}(y)\big)\Big| \\ \leqslant & \bigvee_{y\in Y}\bigvee_{x\in X}|A_f^*(x)-A_f(x)| = \bigvee_{x\in X}|A_f^*(x)-A_f(x)|. \end{aligned} \tag{12}$$

Therefore, for any $\varepsilon > 0$, by taking $\delta = \varepsilon$, when $d_{N_1}(A^*, A) < \delta$, then by equations(8), (10) and (12) we have

$$\begin{aligned} d_{N_1}\big(g(A^*), g(A)\big) &\leqslant \Big(\bigvee_{x\in X}|A_t^*(x)-A_t(x)|\Big)\bigvee\Big(\bigvee_{x\in X}|A_f^*(x)-A_f(x)|\Big) \\ &= d_{N_1}(A^*, A) < \delta = \varepsilon. \end{aligned}$$

This completes the proof of Theorem. □

Theorem 3.6. *Based on the Łukasiewicz implication, the triple implication method g for IFMP is continuous at A under the metric d_{N_2}.*

Proof. It follows from Definition 2.4 that

$$d_{N_2}\big(g(A^*), g(A)\big) = \frac{1}{2}\Big(d_n\big(g(A^*)_t, g(A)_t\big) + d_n\big(g(A^*)_f, g(A)_f\big)\Big). \tag{13}$$

By using the same method as in the proof of Theorem 3.5, we can obtain that

$$d_n\big(g(A^*)_t, g(A)_t\big) \leqslant \bigvee_{x\in X}|A_t^*(x)-A_t(x)|. \tag{14}$$

$$d_n\big(g(A^*)_f, g(A)_f\big) \leqslant \bigvee_{x\in X}|A_f^*(x)-A_f(x)|. \tag{15}$$

Therefore, for any $\varepsilon > 0$, by taking $\delta = \varepsilon$, when $d_{N_2}(A^*, A) \leqslant \delta$, then by equations (13), (14) and (15) we have

$$\begin{aligned} d_{N_2}\big(g(A^*), g(A)\big) &\leqslant \frac{1}{2}\Big(d_n\big(g(A^*)_t, g(A)_t\big) + d_n\big(g(A^*)_f, g(A)_f\big)\Big) \\ &= d_{N_2}(A^*, A) < \delta = \varepsilon. \end{aligned}$$

This completes the proof of Theorem. □

Lemma 3.7. [56] *If $\to_{L^*}$ is a residual intuitionistic implication induced by a left-continuous t-norm, then the triple implication method of IFMP given by Theorem 2.11 is reductive whenever A is a normal intuitionistic fuzzy set on X, i.e., there exists x_0 in X such that $A(x_0) = 1^*$.*

The following Corollary can be easily get by Lemma 3.7 and Remark 2.16.

Corollary 3.8. *Let $A \to B$ be the known rule, $A \in IF(X), B \in IF(Y)$. If A^* is a normal intuitionistic fuzzy set, then the triple implication method g for IFMP is of approximation property.*

Theorem 3.9. *The triple implication method h for IFMT based on the Łukasiewicz implication is continuous at B under the metric d_{N_1}.*

Proof. Firstly, it is easy for us to know that $|B^*_{-f}(y) - B_{-f}(y)| = |B^*_f(y) - B_f(y)|$. From equation (7) and equation (8), we could obtain that

$$\begin{aligned} d_n\big(h(B^*)_t, h(B)_t\big) &= \bigvee_{x\in X} \big|h(B^*)_t(x) - h(B)_t(x)\big| \\ &= \bigvee_{x\in X}\Bigg| \bigwedge_{y\in Y} \Big\{ \Big(\Big(\big(A_t(x) \to B_t(y)\big) \bigwedge \big(A_{-f}(x) \to B_{-f}(y)\big)\Big) \to B^*_t(y)\Big) \\ &\quad \bigwedge \Big(\big(A_{-f}(x) \to B_{-f}(y)\big) \to B^*_{-f}(y)\Big)\Big\} \\ &\quad - \bigwedge_{y\in Y} \Big\{ \Big(\Big(\big(A_t(x) \to B_t(y)\big) \bigwedge \big(A_{-f}(x) \to B_{-f}(y)\big)\Big) \to B_t(y)\Big) \\ &\quad \bigwedge\big((A_{-f}(x) \to B_{-f}(y)) \to B_{-f}(y)\big)\Big\}\Bigg|. \end{aligned} \tag{16}$$

Then by using Lemma 3.1(ii) twice, we have

$$\begin{aligned} &d_n\big(h(B^*)_t, h(B)_t\big) \\ &\leqslant \bigvee_{x\in X}\bigvee_{y\in Y} \Big(\Big| \Big(\big(A_t(x) \to B_t(y)\big) \bigwedge \big(A_{-f}(x) \to B_{-f}(y)\big)\Big) \to B^*_t(y) \\ &\quad - \Big(\big(A_t(x) \to B_t(y)\big) \bigwedge \big(A_{-f}(x) \to B_{-f}(y)\big)\Big) \to B_t(y)\Big| \\ &\quad \bigvee \Big|\big(A_{-f}(x) \to B_{-f}(y)\big) \to B^*_{-f}(y) - \big(A_{-f}(x) \to B_{-f}(y)\big) \to B_{-f}(y)\Big|\Big). \end{aligned} \tag{17}$$

It follows from Lemma 3.3(i) that

$$\begin{aligned} d_n\big(h(B^*)_t, h(B)_t\big) &\leqslant \bigvee_{x\in X}\bigvee_{y\in Y}\Big(|B_t^*(y)-B_t(y)|\bigvee|B_{-f}^*(y)-B_{-f}(y)|\Big)\\ &=\bigvee_{y\in Y}\Big(|B_t^*(y)-B_t(y)|\bigvee|B_f^*(y)-B_f(y)|\Big)\\ &=\Big(\bigvee_{y\in Y}|B_t^*(y)-B_t(y)|\Big)\bigvee\Big(\bigvee_{y\in Y}|B_f^*(y)-B_f(y)|\Big). \end{aligned} \tag{18}$$

Similarly, by using equation (7) and equation (8), we can easily get that

$$\begin{aligned} d_n\big(h(B^*)_f, h(B)_f\big) &= \bigvee_{x\in X}|h(B^*)_f(x)-h(B)_f(x)|\\ &=\bigvee_{x\in X}\Big|\bigvee_{y\in Y}\big\{1-\big(A_{-f}(x)\to B_{-f}(y)\big)\to B_{-f}^*(y)\big\}\\ &\quad-\bigvee_{y\in Y}\big\{1-\big(A_{-f}(x)\to B_{-f}(y)\big)\to B_{-f}(y)\big\}\Big|. \end{aligned} \tag{19}$$

From (19), by using Lemma 3.1(i) and Lemma 3.3(i), it is obvious that

$$\begin{aligned} d_n\big(h(B^*)_f, h(B)_f\big) &\leqslant \bigvee_{x\in X}\bigvee_{y\in Y}\big|\big(A_{-f}(x)\to B_{-f}(y)\big)\to B_{-f}^*(y)\\ &\quad-\big(A_{-f}(x)\to B_{-f}(y)\big)\to B_{-f}(y)\big|\\ &\leqslant \bigvee_{x\in X}\bigvee_{y\in Y}|B_{-f}^*(y)-B_{-f}(y)|=\bigvee_{y\in Y}|B_f^*(y)-B_f(y)|. \end{aligned} \tag{20}$$

Therefore, for any $\varepsilon>0$, by taking $\delta=\varepsilon$, when $d_{N_1}(B^*,B)\leqslant\delta$, then by equations (8), (18) and (20), we can easily obtain that

$$\begin{aligned} d_{N_1}\big(h(B^*), h(B)\big) &\leqslant \Big(\bigvee_{y\in Y}|B_t^*(y)-B_t(y)|\Big)\bigvee\Big(\bigvee_{y\in Y}|B_f^*(y)-B_f(y)|\Big)\\ &\bigvee\Big(\bigvee_{y\in Y}|B_f^*(y)-B_f(y)|\Big)\\ &=\Big(\bigvee_{y\in Y}|B_t^*(y)-B_t(y)|\Big)\bigvee\Big(\bigvee_{y\in Y}|B_f^*(y)-B_f(y)|\Big)\\ &=d_{N_1}(B^*,B)<\delta=\varepsilon. \end{aligned}$$

This completes the proof of Theorem. □

Theorem 3.10. *Based on the Łukasiewicz implication, the triple implication method h for IFMT is continuous at B under the metric d_{N_2}.*

Proof. Using the same method as in the Theorem 3.9, we can obtain that

$$\begin{aligned} d_n\big(h(B^*)_t, h(B)_t\big) &\leqslant \Big(\bigvee_{y\in Y}|B_t^*(y)-B_t(y)|\Big)\bigvee\Big(\bigvee_{y\in Y}|B_f^*(y)-B_f(y)|\Big)\\ &\leqslant \bigvee_{y\in Y}|B_t^*(y)-B_t(y)|+\bigvee_{y\in Y}|B_f^*(y)-B_f(y)|. \end{aligned} \tag{21}$$

$$d_n\big(h(B^*)_f, h(B)_f\big) \leqslant \bigvee_{y\in Y} |B_f^*(y) - B_f(y)|. \tag{22}$$

Therefore, for any $\varepsilon > 0$, by taking $\delta = \dfrac{1}{2}\varepsilon$, when $d_{N_2}(B^*, B) < \delta$, then by equations (13), (21) and (22), it is obvious that

$$\begin{aligned} d_{N_2}\big(h(B^*), h(B)\big) &\leqslant \frac{1}{2}\Big(\bigvee_{y\in Y}|B_t^*(y) - B_t(y)| + 2\bigvee_{y\in Y}|B_f^*(y) - B_f(y)|\Big) \\ &\leqslant \bigvee_{y\in Y}|B_t^*(y) - B_t(y)| + \bigvee_{y\in Y}|B_f^*(y) - B_f(y)| \\ &= 2d_{N_2}(B^*, B) \\ &< 2\delta = \varepsilon. \end{aligned}$$

Hence, there exists $\delta > 0$ such that $d_{N_2}\big(h(B^*), h(B)\big) < \varepsilon$ when $d_{N_2}(B^*, B) < \delta$. This completes the proof of the Theorem. □

Theorem 3.11. *Let B_1^* and B_2^* be the triple implication solutions of IFMP (A_1, B_1, A_1^*) and IFMP (A_2, B_2, A_2^*) based on the Łukasiewicz implication, respectively. Suppose that $d_{N_1}(A_1, A_2) \leqslant \delta_1, d_{N_1}(B_1, B_2) \leqslant \delta_2, d_{N_1}(A_1^*, A_2^*) \leqslant \delta_3$, then*

$$d_{N_1}(B_1^*, B_2^*) \leqslant 4\delta_1 \bigvee 4\delta_2 \bigvee 2\delta_3.$$

Proof. We firstly take the natural distance between the membership functions of B_1^* and B_2^* into consideration, by definition 2.3 and equation (5) we have

$$\begin{aligned} &d_n(B_{1t}^*, B_{2t}^*) \\ &= \bigvee_{y\in Y}\Big|\bigvee_{x\in X}\Big\{A_{1t}^*(x) \otimes \Big(\big(A_{1t}(x) \to B_{1t}(y)\big) \bigwedge \big(A_{1-f}(x) \to B_{1-f}(y)\big)\Big)\Big\} \\ &\quad - \bigvee_{x\in X}\Big\{A_{2t}^*(x) \otimes \Big(\big(A_{2t}(x) \to B_{2t}(y)\big) \bigwedge \big(A_{2-f}(x) \to B_{2-f}(y)\big)\Big)\Big\}\Big|. \end{aligned} \tag{23}$$

From (23), Lemma 3.1(i) and Lemma 3.2(ii), we can easily obtain that

$$\begin{aligned} &d_n(B_{1t}^*, B_{2t}^*) \\ &\leqslant \bigvee_{y\in Y}\bigvee_{x\in X}\Big|A_{1t}^*(x) \otimes \Big(\big(A_{1t}(x) \to B_{1t}(y)\big) \bigwedge \big(A_{1-f}(x) \to B_{1-f}(y)\big)\Big) \\ &\quad - A_{2t}^*(x) \otimes \Big(\big(A_{2t}(x) \to B_{2t}(y)\big) \bigwedge \big(A_{2-f}(x) \to B_{2-f}(y)\big)\Big)\Big| \\ &\leqslant \bigvee_{y\in Y}\bigvee_{x\in X} 2\Big(\Big|A_{1t}^*(x) - A_{2t}^*(x)\Big| \bigvee \Big|\big(A_{1t}(x) \to B_{1t}(y)\big) \bigwedge \big(A_{1-f}(x) \to B_{1-f}(y)\big) \\ &\quad - \big(A_{2t}(x) \to B_{2t}(y)\big) \bigwedge \big(A_{2-f}(x) \to B_{2-f}(y)\big)\Big|\Big). \end{aligned} \tag{24}$$

Then it follows from (24) and Lemma 3.1(ii) that

$$\begin{aligned} &d_n(B_{1t}^*, B_{2t}^*) \\ &\leqslant \bigvee_{y\in Y}\bigvee_{x\in X} 2\Big(|A_{1t}^*(x) - A_{2t}^*(x)| \bigvee |A_{1t}(x) \to B_{1t}(y) - A_{2t}(x) \to B_{2t}(y)| \\ &\quad \bigvee |A_{1-f}(x) \to B_{1-f}(y) - A_{2-f}(x) \to B_{2-f}(y)|\Big). \end{aligned} \tag{25}$$

According to Lemma 3.3 (ii), we further have

$$d_n(B_{1t}^*, B_{2t}^*) \leqslant \bigvee_{y\in Y}\bigvee_{x\in X} 2\Big(|A_{1t}^*(x) - A_{2t}^*(x)| \bigvee 2\big(|A_{1t}(x) - A_{2t}(x)| \bigvee |B_{1t}(y) - B_{2t}(y)|\big) \bigvee 2\big(|A_{1-f}(x) - A_{2-f}(x)| \bigvee |B_{1-f}(y) - B_{2-f}(y)|\big)\Big)$$
$$= \Big(2\bigvee_{x\in X} |A_{1t}^*(x) - A_{2t}^*(x)|\Big) \bigvee \Big(4\bigvee_{x\in X} |A_{1t}(x) - A_{2t}(x)|\Big) \bigvee \Big(4\bigvee_{y\in Y} |B_{1t}(y) - B_{2t}(y)|\Big) \bigvee \Big(4\bigvee_{x\in X} |A_{1f}(x) - A_{2f}(x)|\Big) \bigvee \Big(4\bigvee_{y\in Y} |B_{1f}(y) - B_{2f}(y)|\Big). \tag{26}$$

Secondly, we consider the natural distance between the non-membership functions of B_1^* and B_2^*, by Definition 2.3, equation (5) and Lemma 3.1(ii) we have

$$d_n(B_{1f}^*, B_{2f}^*) = \bigvee_{y\in Y}\Big|\bigwedge_{x\in X}\big\{A_{1f}^*(x) \oplus \big(1 - A_{1-f}(x) \to B_{1-f}(y)\big)\big\} - \bigwedge_{x\in X}\big\{A_{2f}^*(x) \oplus \big(1 - A_{2-f}(x) \to B_{2-f}(y)\big)\big\}\Big|$$
$$\leqslant \bigvee_{y\in Y}\bigvee_{x\in X}\Big|A_{1f}^*(x) \oplus \big(1 - A_{1-f}(x) \to B_{1-f}(y)\big) - A_{2f}^*(x) \oplus \big(1 - A_{2-f}(x) \to B_{2-f}(y)\big)\Big|. \tag{27}$$

Then it can be easily obtained from (27), Lemma 3.2(iv) and Lemma 3.3(ii) that

$$d_n(B_{1f}^*, B_{2f}^*) \leqslant \bigvee_{y\in Y}\bigvee_{x\in X} 2\Big(\big|A_{1-f}(x) \to B_{1-f}(y) - A_{2-f}(x) \to B_{2-f}(y)\big| \bigvee |A_{1f}^*(x) - A_{2f}^*(x)|\Big)$$
$$\leqslant \Big(2\bigvee_{x\in X} |A_{1t}^*(x) - A_{2t}^*(x)|\Big) \bigvee \Big(4\bigvee_{x\in X} |A_{1f}(x) - A_{2f}(x)|\Big) \bigvee \Big(4\bigvee_{y\in Y} |B_{1f}(y) - B_{2f}(y)|\Big). \tag{28}$$

Thus it follows from equations (8), (26) and (28) that

$$d_{N_1}(B_1^*, B_2^*) \leqslant \Big(2\bigvee_{x\in X} |A_{1t}^*(x) - A_{2t}^*(x)|\Big) \bigvee \Big(2\bigvee_{x\in X} |A_{1f}^*(x) - A_{2f}^*(x)|\Big) \bigvee \Big(4\bigvee_{x\in X} |A_{1t}(x) - A_{2t}(x)|\Big) \bigvee \Big(4\bigvee_{x\in X} |A_{1f}(x) - A_{2f}(x)|\Big) \bigvee \Big(4\bigvee_{y\in Y} |B_{1t}(y) - B_{2t}(y)|\Big) \bigvee \Big(4\bigvee_{y\in Y} |B_{1f}(y) - B_{2f}(y)|\Big)$$
$$= 4d_{N_1}(A_1, A_2) \bigvee 4d_{N_1}(B_1, B_2) \bigvee 2d_{N_1}(A_1^*, A_2^*)$$
$$\leqslant 4\delta_1 \bigvee 4\delta_2 \bigvee 2\delta_3.$$

This completes the proof. □

Theorem 3.12. *Let B_1^* and B_2^* be the triple implication solutions of IFMP (A_1, B_1, A_1^*) and IFMP (A_2, B_2, A_2^*) based on the Łukasiewicz implication, respectively. Suppose that $d_{N_2}(A_1, A_2) \leqslant \delta_1, d_{N_2}(B_1, B_2) \leqslant \delta_2, d_{N_2}(A_1^*, A_2^*) \leqslant \delta_3$, then*

$$d_{N_2}(B_1^*, B_2^*) \leqslant 2\delta_1 + 2\delta_2 + \delta_3.$$

Proof. Firstly we take the natural distance between the membership functions of B_1^* and B_2^* into consideration, by definition 2.3 and equation (5) we have

$$\begin{aligned} &d_n(B_{1t}^*, B_{2t}^*) \\ &\quad= \bigvee_{y\in Y} \Big| \bigvee_{x\in X} \Big\{ A_{1t}^*(x) \otimes \Big(\big(A_{1t}(x) \to B_{1t}(y)\big) \bigwedge \big(A_{1-f}(x) \to B_{1-f}(y)\big) \Big) \Big\} \\ &\quad- \bigvee_{x\in X} \Big\{ A_{2t}^*(x) \otimes \Big(\big(A_{2t}(x) \to B_{2t}(y)\big) \bigwedge \big(A_{2-f}(x) \to B_{2-f}(y)\big) \Big) \Big\} \Big|. \end{aligned} \tag{29}$$

From (29), Lemma 3.1(i) and Lemma 3.2(i), it can be easily obtained that

$$\begin{aligned} d_n(B_{1t}^*, B_{2t}^*) &\leqslant \bigvee_{y\in Y} \bigvee_{x\in X} \Big| A_{1t}^*(x) \otimes \Big(\big(A_{1t}(x) \to B_{1t}(y)\big) \bigwedge \big(A_{1-f}(x) \to B_{1-f}(y)\big) \Big) \\ &\quad - A_{2t}^*(x) \otimes \Big(\big(A_{2t}(x) \to B_{2t}(y)\big) \bigwedge \big(A_{2-f}(x) \to B_{2-f}(y)\big) \Big) \Big| \\ &\leqslant \bigvee_{y\in Y} \bigvee_{x\in X} \Big(\big|A_{1t}^*(x) - A_{2t}^*(x)\big| + \Big| \big(A_{1t}(x) \to B_{1t}(y)\big) \bigwedge \big(A_{1-f}(x) \to B_{1-f}(y)\big) \\ &\quad - \big(A_{2t}(x) \to B_{2t}(y)\big) \bigwedge \big(A_{2-f}(x) \to B_{2-f}(y)\big) \Big| \Big). \end{aligned} \tag{30}$$

Then it follows from (30) and Lemma 3.1(ii) that

$$\begin{aligned} d_n(B_{1t}^*, B_{2t}^*) &\leqslant \bigvee_{y\in Y} \bigvee_{x\in X} \Big(\big|A_{1t}^*(x) - A_{2t}^*(x)\big| + \big|A_{1t}(x) \to B_{1t}(y) - A_{2t}(x) \to B_{2t}(y)\big| \\ &\quad \bigvee \big|A_{1-f}(x) \to B_{1-f}(y) - A_{2-f}(x) \to B_{2-f}(y)\big| \Big) \\ &\leqslant \bigvee_{y\in Y} \bigvee_{x\in X} \Big(\big|A_{1t}^*(x) - A_{2t}^*(x)\big| + \big|A_{1t}(x) \to B_{1t}(y) - A_{2t}(x) \to B_{2t}(y)\big| \\ &\quad + \big|A_{1-f}(x) \to B_{1-f}(y) - A_{2-f}(x) \to B_{2-f}(y)\big| \Big). \end{aligned} \tag{31}$$

For (31), according to Lemma 3.3(i), we further have

$$\begin{aligned} d_n(B_{1t}^*, B_{2t}^*) &\leqslant \bigvee_{y\in Y} \bigvee_{x\in X} \Big(|A_{1t}^*(x) - A_{2t}^*(x)| + |A_{1t}(x) - A_{2t}(x)| + |B_{1t}(y) - B_{2t}(y)| \\ &\quad + |A_{1-f}(x) - A_{2-f}(x)| + |B_{1-f}(y) - B_{2-f}(y)| \Big) \\ &\leqslant \bigvee_{x\in X} |A_{1t}^*(x) - A_{2t}^*(x)| + \bigvee_{x\in X} |A_{1t}(x) - A_{2t}(x)| + \bigvee_{y\in Y} |B_{1t}(y) - B_{2t}(y)| \\ &\quad + \bigvee_{x\in X} |A_{1f}(x) - A_{2f}(x)| + \bigvee_{y\in Y} |B_{1f}(y) - B_{2f}(y)|. \end{aligned} \tag{32}$$

Secondly, we consider the natural distance between the non-membership functions of B_1^* and B_2^*, by Definition 2.3, equation (5) and Lemma 3.1(ii) we have

$$\begin{aligned}
d_n(B_{1f}^*, B_{2f}^*) &= \bigvee_{y\in Y}\Big|\bigwedge_{x\in X}\big\{A_{1f}^*(x)\oplus\big(1-A_{1-f}(x)\to B_{1-f}(y)\big)\big\} \\
&\quad -\bigwedge_{x\in X}\big\{A_{2f}^*(x)\oplus\big(1-A_{2-f}(x)\to B_{2-f}(y)\big)\big\}\Big| \\
&\leqslant \bigvee_{y\in Y}\bigvee_{x\in X}\big|A_{1f}^*(x)\oplus\big(1-A_{1-f}(x)\to B_{1-f}(y)\big) \\
&\quad - A_{2f}^*(x)\oplus\big(1-A_{2-f}(x)\to B_{2-f}(y)\big)\big|. \qquad (33)
\end{aligned}$$

Then it can be easily gotten from (33), Lemma 3.2(iii) and Lemma 3.3(i) that

$$\begin{aligned}
&d_n(B_{1f}^*, B_{2f}^*) \\
&\leqslant \bigvee_{y\in Y}\bigvee_{x\in X}\Big(|A_{1f}^*(x)-A_{2f}^*(x)|+|A_{1-f}(x)\to B_{1-f}(y)-A_{2-f}(x)\to B_{2-f}(y)|\Big) \\
&\leqslant \bigvee_{y\in Y}\bigvee_{x\in X}\Big(|A_{1f}^*(x)-A_{2f}^*(x)|+|A_{1-f}(x)-A_{2-f}(x)|+|B_{1-f}(y)-B_{2-f}(y)|\Big) \\
&\leqslant \bigvee_{x\in X}|A_{1f}^*(x)-A_{2f}^*(x)|+\bigvee_{x\in X}|A_{1f}(x)-A_{2f}(x)|+\bigvee_{y\in Y}|B_{1f}(y)-B_{2f}(y)|. \qquad (34)
\end{aligned}$$

Thus it follows from (13), (32) and (34) that

$$\begin{aligned}
&d_{N_2}(B_1^*, B_2^*) \\
&\leqslant \frac{1}{2}\Big(\bigvee_{x\in X}|A_{1t}^*(x)-A_{2t}^*(x)|+\bigvee_{x\in X}|A_{1f}^*(x)-A_{2f}^*(x)|+\bigvee_{x\in X}|A_{1t}(x)-A_{2t}(x)| \\
&\quad +2\bigvee_{x\in X}|A_{1f}(x)-A_{2f}(x)|+\bigvee_{y\in Y}|B_{1t}(y)-B_{2t}(y)|+2\bigvee_{y\in Y}|B_{1f}(y)-B_{2f}(y)|\Big) \\
&\leqslant \frac{1}{2}\Big(\bigvee_{x\in X}|A_{1t}^*(x)-A_{2t}^*(x)|+\bigvee_{x\in X}|A_{1f}^*(x)-A_{2f}^*(x)|+2\bigvee_{x\in X}|A_{1t}(x)-A_{2t}(x)| \\
&\quad +2\bigvee_{x\in X}|A_{1f}(x)-A_{2f}(x)|+2\bigvee_{y\in Y}|B_{1t}(y)-B_{2t}(y)|+2\bigvee_{y\in Y}|B_{1f}(y)-B_{2f}(y)|\Big) \\
&= 2d_{N_2}(A_1, A_2)+2d_{N_2}(B_1, B_2)+d_{N_2}(A_1^*, A_2^*) \\
&\leqslant 2\delta_1+2\delta_2+\delta_3.
\end{aligned}$$

This completes the proof. □

Theorem 3.13. *Let A_1^* and A_2^* be the triple implication method of IFMT (A_1, B_1, B_1^*) and IFMT(A_2, B_2, B_2^*) based on the Łukasiewicz implication, respectively. Suppose that $d_{N_1}(A_1, A_2) \leqslant \delta_1, d_{N_1}(B_1, B_2) \leqslant \delta_2, d_{N_1}(B_1^*, B_2^*) \leqslant \delta_3$, then*

$$d_{N_1}(A_1^*, A_2^*) \leqslant 4\delta_1 \bigvee 4\delta_2 \bigvee 2\delta_3.$$

Proof. We firstly take the natural distance between the membership functions of A_1^* and A_2^* into consideration, by definition 2.3 and equation (7) we have

$$\begin{aligned} &d_n(A_{1t}^*, A_{2t}^*) \\ &= \bigvee_{x\in X} \Big| \bigwedge_{y\in Y} \Big\{ \Big(\Big(\big(A_{1t}(x) \to B_{1t}(y)\big) \bigwedge \big(A_{1-f}(x) \to B_{1-f}(y)\big) \Big) \to B_{1t}^*(y) \Big) \\ &\bigwedge \Big(\big(A_{1-f}(x) \to B_{1-f}(y)\big) \to B_{1-f}^*(y) \Big) \Big\} \\ &- \bigwedge_{y\in Y} \Big\{ \Big(\Big(\big(A_{2t}(x) \to B_{2t}(y)\big) \bigwedge \big(A_{2-f}(x) \to B_{2-f}(y)\big) \Big) \to B_{2t}^*(y) \Big) \\ &\bigwedge \Big(\big(A_{2-f}(x) \to B_{2-f}(y)\big) \to B_{2-f}^*(y) \Big) \Big\} \Big|. \end{aligned} \tag{35}$$

Using Lemma 3.1(ii) twice, it can be easily obtained that

$$\begin{aligned} &d_n(A_{1t}^*, A_{2t}^*) \\ &\leqslant \bigvee_{x\in X} \bigvee_{y\in Y} \Big(\Big| \Big(\big(A_{1t}(x) \to B_{1t}(y)\big) \bigwedge \big(A_{1-f}(x) \to B_{1-f}(y)\big) \Big) \to B_{1t}^*(y) \\ &- \Big(\big(A_{2t}(x) \to B_{2t}(y)\big) \bigwedge \big(A_{2-f}(x) \to B_{2-f}(y)\big) \Big) \to B_{2t}^*(y) \Big| \\ &\bigvee \big| \big(A_{1-f}(x) \to B_{1-f}(y)\big) \to B_{1-f}^*(y) \\ &- \big(A_{2-f}(x) \to B_{2-f}(y)\big) \to B_{2-f}^*(y) \big| \Big). \end{aligned} \tag{36}$$

Then it follows from Lemma 3.3(ii) that

$$\begin{aligned} &d_n(A_{1t}^*, A_{2t}^*) \\ &\leqslant \bigvee_{x\in X} \bigvee_{y\in Y} \Big(2 \Big| \big(A_{1t}(x) \to B_{1t}(y)\big) \bigwedge \big(A_{1-f}(x) \to B_{1-f}(y)\big) \\ &- \big(A_{2t}(x) \to B_{2t}(y)\big) \bigwedge \big(A_{2-f}(x) \to B_{2-f}(y)\big) \Big| \bigvee 2 \big| B_{1t}^*(y) - B_{2t}^*(y) \big| \\ &\bigvee 2 \Big| A_{1-f}(x) \to B_{1-f}(y) - A_{2-f}(x) \to B_{2-f}(y) \Big| \\ &\bigvee 2 \big| B_{1-f}^*(y) - B_{2-f}^*(y) \big| \Big). \end{aligned} \tag{37}$$

According to Lemma 3.1(ii), we further have

$$\begin{aligned} &d_n(A_{1t}^*, A_{2t}^*) \\ &\leqslant \bigvee_{x\in X} \bigvee_{y\in Y} \Big(2|B_{1t}^*(y) - B_{2t}^*(y)| \bigvee 2|A_{1t}(x) \to B_{1t}(y) - A_{2t}(x) \to B_{2t}(y)| \\ &\bigvee 2|A_{1-f}(x) \to B_{1-f}(y) - A_{2-f}(x) \to B_{2-f}(y)| \bigvee 2|B_{1-f}^*(y) - B_{2-f}^*(y)| \\ &\bigvee 2|A_{1-f}(x) \to B_{1-f}(y) - A_{2-f}(x) \to B_{2-f}(y)| \Big). \end{aligned} \tag{38}$$

For (38), using Lemma 3.3(ii) again, we can obtain

$$\begin{aligned}
&d_n(A^*_{1t}, A^*_{2t})\\
&\leqslant \bigvee_{x\in X}\bigvee_{y\in Y}\Big(2|B^*_{1t}(y) - B^*_{2t}(y)| \bigvee 4|A_{1t}(x) - A_{2t}(x)| \bigvee 4|B_{1t}(y) - B_{2t}(y)|\\
&\quad \bigvee 4|A_{1-f}(x) - A_{2-f}(x)| \bigvee 4|B_{1-f}(y) - B_{2-f}(y)| \bigvee 2|B^*_{1-f}(y) - B^*_{2-f}(y)|\Big)\\
&= \bigvee_{x\in X}\bigvee_{y\in Y}\Big(4|A_{1t}(x) - A_{2t}(x)| \bigvee 4|A_{1f}(x) - A_{2f}(x)| \bigvee 4|B_{1t}(y) - B_{2t}(y)|\\
&\quad \bigvee 4|B_{1f}(y) - B_{2f}(y)| \bigvee 2|B^*_{1t}(y) - B^*_{2t}(y)| \bigvee 2|B^*_{1f}(y) - B^*_{2f}(y)|\Big). \qquad (39)
\end{aligned}$$

Secondly, we consider the natural distance between the non-membership functions of A^*_1 and A^*_2, by Definition 2.3 and equation (7) we have

$$\begin{aligned}
d_n(A^*_{1f}, A^*_{2f}) = \bigvee_{x\in X}\Big|&\bigvee_{y\in Y}\{1 - (A_{1-f}(x) \to B_{1-f}(y)) \to B^*_{1-f}(y)\}\\
&- \bigvee_{y\in Y}\{1 - (A_{2-f}(x) \to B_{2-f}(y)) \to B^*_{2-f}(y)\}\Big|. \qquad (40)
\end{aligned}$$

Then using the similar method above, it can be easily obtained from (40) that

$$\begin{aligned}
d_n(A^*_{1f}, A^*_{2f}) \leqslant \bigvee_{x\in X}\bigvee_{y\in Y}&(4|A_{1f}(x) - A_{2f}(x)| \bigvee 4|B_{1f}(y) - B_{2f}(y)|\\
&\bigvee 2|B^*_{1f}(y) - B^*_{2f}(y)|). \qquad (41)
\end{aligned}$$

Thus it follows from (8), (39) and (41) that

$$\begin{aligned}
&d_{N_1}(A^*_1,A^*_2) = d_n(A^*_{1t}, A^*_{2t}) \bigvee d_n(A^*_{1f}, A^*_{2f})\\
&\quad\leqslant \bigvee_{x\in X}\bigvee_{y\in Y}\Big(4|A_{1t}(x) - A_{2t}(x)| \bigvee 4|A_{1f}(x) - A_{2f}(x)| \bigvee 4|B_{1t}(y) - B_{2t}(y)|\\
&\qquad \bigvee 4|B_{1f}(y) - B_{2f}(y)| \bigvee 2|B^*_{1t}(y) - B^*_{2t}(y)| \bigvee 2|B^*_{1f}(y) - B^*_{2f}(y)|\Big).\\
&\quad= 4d_{N_1}(A_1, A_2) \bigvee 4d_{N_1}(B_1, B_2) \bigvee 2d_{N_1}(B^*_1, B^*_2)\\
&\quad\leqslant 4\delta_1 \bigvee 4\delta_2 \bigvee 2\delta_3.
\end{aligned}$$

This completes the proof. □

Theorem 3.14. *Let A^*_1 and A^*_2 be the triple implication solutions of IFMT (A_1, B_1, B^*_1) and IFMT (A_2, B_2, B^*_2) based on the Lukasiewicz implication, respectively. Suppose that $d_{N_2}(A_1, A_2) \leqslant \delta_1, d_{N_2}(B_1, B_2) \leqslant \delta_2, d_{N_2}(B^*_1, B^*_2) \leqslant \delta_3$, then*

$$d_{N_2}(A^*_1, A^*_2) \leqslant 2\delta_1 + 2\delta_2 + \delta_3.$$

Proof. The proof is similar to that of above Theorems. □

Remark 3.15. Suppose that δ_i is close to 0 (i=1,2,3), then it follows from Theorem 3.11 and Theorem 3.12 that both $d_{N_1}(B_1^*, B_2^*)$ and $d_{N_2}(B_1^*, B_2^*)$ are close to 0. Based on this fact, we can see that small disturbances on the inputs A, B and A^* imply small changes of the output B^* of IFMP(A, B, A^*), hence the triple implication method of Intuitionistic Fuzzy Modus Ponens has a good behavior of robustness under two kinds of natural distances given by the present paper. The same conclusion can be drawn with respect to the triple implication method of Intuitionistic Fuzzy Modus Tollens.

Remark 3.16. It is obvious that if g is reductive, then B is the solution of IFMP (A, B, A), therefore the continuity (i.e., approximation property) of g is just right its robustness in the case that $A_1 = A_2 = A, B_1 = B_2 = B$ and $A_1^* = A_2^* = A$. In this case, both the continuity (i.e., approximation property) and the robustness reflect that small disturbances on the input do not lead to large deviations of the output.

A similar analysis can be done about the method h for Intuitionistic Fuzzy Modus Tollens.

Remark 3.17. It is worth noting that both the present paper and references [24, 38] above have discussed the robustness of fuzzy reasoning, but the robustness of fuzzy connectives and fuzzy reasoning in the reference [24] is discussed through the sensitivity of interval-valued fuzzy connectives, the robustness of intuitionistic fuzzy connectives in fuzzy reasoning is discussed in the reference [38] through the evaluation of sensitivity in n-order function on the class of intuitionistic fuzzy sets, while the present is through some distances defined on the set of all intuitionistic fuzzy sets to discuss the continuity, approximation property and robustness of the triple implication method of intuitionistic fuzzy reasoning.

4. **Conclusions**

In this paper, we discussed the continuity, approximation property and robustness of triple implication method for Intuitionistic Fuzzy Modus Ponens model by using two kinds of natural distances based on Łukasiewicz implication, and for intuitionistic fuzzy modus tollens model, we use similar method to take continuity and robustness into consideration. Our results showed that both the triple implication methods for Intuitionistic Fuzzy Modus Ponens and Intuitionistic Fuzzy Modus Tollens have continuity and good behavior of robustness. Of course, we should note that the robustness results presented in this paper are conservative in certain sense. A fuzzy knowledge base normally contains many fuzzy rules, as in the case of fuzzy control. We can treat inference with multiple rules as a generalized form of generalized modus ponens. However, since multiple rules are involved, we may obtain different robustness results between combination based inference and individual-rule based inference. This is a problem left for further investigation. Whether triple implication method for intuitionistic fuzzy modus tollens model is of approximation property is another problem left. Finally, the present paper is mainly based on intuitionistic fuzzy implication operator induced by Łukasiewicz implication operator, the results about other intuitionistic fuzzy implication operators are possible topics for future study.

Acknowledgements. Authors would like to express their sincere thanks to the valuable comments and suggestions from editors and reviewers. This work was supported by the National Natural Science Foundation of China (No. 11261032).

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