HESITANT FUZZY INFORMATION MEASURES DERIVED FROM T-NORMS AND S-NORMS

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Abstract. In this contribution, we first introduce the concept of metrical T-norm-based similarity measure for hesitant fuzzy sets (HFSs) by using the concept of T-norm-based distance measure. Then, the relationship of the proposed metrical T-norm-based similarity measures with the other kind of information measure, called the metrical T-norm-based entropy measure is discussed. The main feature of the proposed metrical T-norm-based similarity measures is a possibility of comparing similarity between HFSs without regarding what value is returned by the similarity measure. To illustrate the application of the proposed metrical T-norm-based similarity measures, we consider two problems of medical diagnosis and pattern recognition to compare the proposed metrical T-norm-based similarity measures with a number of the existing HFS similarity measures.

1. Introduction

Hesitant fuzzy set (HFS) which was first introduced by Torra [17] can be viewed as a generalization of fuzzy sets that may reflect effectively the human's hesitancy in the actual decision making. Among the various kinds of topics in the HFS theory, similarity and similarity-based entropy measures have become a subject that generates much interest in various fields, such as decision making [6]-[10], [15, 20, 28], medical diagnosis [11, 16], pattern recognition [4, 14, 18, 25] and so on.

So far, the following five axioms are the only properties considered for the similarity measures of the HFSs M, N: (1) (Normality) $0 \leq \mathbf{S}(M,N) \leq 1$; (2) (Complementarity) $\mathbf{S}(M,M^c) = 0$ if and only if $M \in \{O^*, I^*\}$; (3) (Reflexivity) $\mathbf{S}(M,N) = 1$ if and only if $M = N$; (4) (Symmetry) $\mathbf{S}(M,N) = \mathbf{S}(N,M)$; (5) (Monotonicity) $\mathbf{S}(M,O) \leq \mathbf{S}(M,N)$ and $\mathbf{S}(M,O) \leq \mathbf{S}(N,O)$, if $M \preceq N \preceq O$.

Although, there exist a large number of scholars in which the properties of similarity measures for fuzzy sets are investigated in a more general theoretical framework [2, 3], such investigations are not yet widely available in the theory of HFSs. More specially, the monotonicity axiom, which plays a key role in defining a similarity measure, has been considered in the weaker form and cannot meet our intuitions. The reason for this occurrence is that we sometimes are interested in comparing similarities without regarding what value the similarity measure returns. By the

Received: April 2017; Revised: September 2017; Accepted: December 2017

Key words and phrases: Hesitant fuzzy set, Metrical T-norm-based information measure, Medical diagnosis, Pattern recognition.

use of the property (5) above, we are only able to compare $\mathbf{S}(M,O)$ with $\mathbf{S}(M,N)$, or $\mathbf{S}(M,O)$ with $\mathbf{S}(N,O)$ whenever $M \preceq N \preceq O$.

The above-mentioned reasons motivate us to investigate the metrical T-norm-based similarity measures between HFSs in the case where the pairwise comparison of distinct HFSs is required.

The rest of the paper is organized as follows: In Section 2, we recall some notions and definitions that are used throughout this paper. Section 3 is devoted to introducing the concept of metrical T-norm-based similarity measures for HFEs on the basis of the concept of T-norm-based distance measure, and then the HFE metrical T-norm-based entropy measures are constructed by the use of the proposed similarity measures. Finally, Section 4 presents two applications of the HFS metrical T-norm-based similarity measures to medical diagnosis and pattern recognition problems. This paper is concluded in Section 5.

2. Basic Notions and Definitions

Let us begin with the definitions of T-norm, S-norm (or T-conorm) and negation together with some relevant properties (see [12]).

Definition 2.1. The function $T : [0,1] \times [0,1] \to [0,1]$ (respectively, $S : [0,1] \times [0,1] \to [0,1]$) is defined as a T-norm (respectively, S-norm) if it is non-decreasing in each variable, commutative, associative and has neutral element 1 (respectively, 0). We say that $\aleph : [0,1] \to [0,1]$ is a negation if it is non-increasing in its variable, with boundary conditions $\aleph(0) = 1$ and $\aleph(1) = 0$.

Example 2.2. The most common and basic T- norms and S-norms are:

- *Algebraic T- norm and Algebraic S-norm*: $T_A(a,b) = ab$, and $S_A(a,b) = a + b - ab$;
- *Drastic T- norm and Drastic S-norm* : $T_D(a,b) = \begin{cases} a, & \text{b=1;} \\ b, & \text{a=1;} \\ 0, & \text{otherwise,} \end{cases}$ and $S_D(a,b) = \begin{cases} a, & \text{b=0;} \\ b, & \text{a=0;} \\ 1, & \text{otherwise;} \end{cases}$
- *Minimum T-norm and Maximum S-norm*: $T_M(a,b) = \min\{a,b\}$, and $S_M(a,b) = \max\{a,b\}$;
- *Lukasiewicz T- norm and Lukasiewicz S-norm*: $T_L(a,b) = \max\{0, a+b-1\}$, and $S_L(a,b) = \min\{1, a+b\}$.

The most common negations are:

- *Standard negation*: $\aleph(a) = 1 - a$;
- *Sugeno class of negations*: $\aleph(a) = \frac{1-a}{1+\lambda a}, \quad \lambda > -1$;
- *Yager class of negations*: $\aleph(a) = (1 - a^{\nu})^{\frac{1}{\nu}}, \quad \nu > 0$.

Definition 2.3. [12] We say that a T-norm T and a S-norm S are *dual* with respect to a negation $\aleph$, denoted by $\langle T, S, \aleph \rangle$ if and only if

$$T(a,b) = \aleph(S(\aleph(a), \aleph(b))), \quad \text{and} \quad S(a,b) = \aleph(T(\aleph(a), \aleph(b))). \tag{1}$$

It is easy to show that all the above-mentioned T-norms and their corresponding S-norms are dual with respect to the standard negation $\aleph(a) = 1 - a$. Hereafter, we only concentrate on the dual pair of T-norm and S-norm with respect to the standard negation because of their interesting relationships.

Proposition 2.4. [12] *For any dual pair of T-norm T and S-norm S, it holds that*

$$T_D \leq T \leq T_M \leq S_M \leq S \leq S_D. \tag{2}$$

In the following, we propose a distance measure for fuzzy sets by the help of the dual pair of T-norm and S-norm defined on them.

Definition 2.5. [1] Given a dual pair of T-norm and S-norm, we define the function $\mathfrak{D} : [0,1] \times [0,1] \to [0,1]$ by

$$\mathfrak{D}(a,b) = \begin{cases} \text{S(a,b)-T(a,b)}, & a \neq b; \\ 0, & \text{a=b}. \end{cases} \tag{3}$$

The function $\mathfrak{D}$, which is hereafter referred to as *the T-norm-based distance measure of fuzzy sets*, satisfies the following axioms which are necessary for a distance measure.

Theorem 2.6. [1] *Given a dual pair of T-norm and S-norm, the function $\mathfrak{D} : [0,1] \times [0,1] \to [0,1]$ given by (3) fulfills the following properties:*

- *($\mathfrak{D}1$)* $0 \leq \mathfrak{D}(a,b) \leq 1$*; (Normality axiom)*
- *($\mathfrak{D}2$)* $\mathfrak{D}(a,b) = 0$ *if and only if* $a = b$*; (Reflexivity axiom)*
- *($\mathfrak{D}3$)* $\mathfrak{D}(a,b) = \mathfrak{D}(b,a)$*; (Symmetry axiom)*
- *($\mathfrak{D}4$)* $\mathfrak{D}(a,c) \leq \mathfrak{D}(a,b) + \mathfrak{D}(b,c)$ *if* $\min\{a,c\} \leq b \leq \max\{a,c\}$*. (Quasi-triangle inequality axiom)*

Definition 2.7. In the case that a T-norm-based distance measure for fuzzy sets satisfies the axioms ($\mathfrak{D}1$)-($\mathfrak{D}4$) given in Theorem 2.6 together with

- ($\mathfrak{D}'4$) $\mathfrak{D}(a,c) \geq \max\{\mathfrak{D}(a,b), \mathfrak{D}(b,c)\}$ if $a \leq b \leq c$, (Comparability axiom)

then, we call it a metrical T-norm-based distance measure for fuzzy sets.

Here are some examples of the T-norm-based distance measures for fuzzy sets that are defined by the use of different dual pairs of T-norms and S-norms presented in Example 2.2:

- The T-norm-based distance measure associated with the dual pair of minimum T-norm T_M and maximum S-norm S_M:
$$\mathfrak{D}_{T_M}(a,b) = |a - b|; \tag{4}$$
- The T-norm-based distance measure associated with the dual pair of algebraic T-norm T_A and algebraic S-norm S_A:
$$\mathfrak{D}_{T_A}(a,b) = a + b - 2ab, \quad (for\ a \neq b); \tag{5}$$
- The T-norm-based distance measure associated with the dual pair of Lukasiewicz T-norm T_L and Lukasiewicz S-norm S_L:
$$\mathfrak{D}_{T_L}(a,b) = \begin{cases} a + b, & a + b \leq 1; \\ 2 - (a + b), & a + b \geq 1, \end{cases}, \quad (for\ a \neq b). \tag{6}$$

By a simple investigation, we can show that among the above T-norm-based distance measures $\mathfrak{D}_{T_M}$, $\mathfrak{D}_{T_A}$ and $\mathfrak{D}_{T_L}$, only $\mathfrak{D}_{T_M}$ is a metrical T-norm-based distance measure.

Although the concept of fuzzy set has been widely and successfully applied in many different areas to model some type of uncertainty, the limitation of this concept is still more serious in case of dealing with imprecise and vague information when different sources of vagueness appear simultaneously. Due to this fact and to overcome such limitations, Torra [17] introduced a new extension of fuzzy sets, and called it hesitant fuzzy set (HFS). This concept can be used to model a situation where an expert might consider different degrees of membership of an element x in a set A. Later, Xia and Xu [20] put forward the seminal definition of HFS with an easier mathematical representation:

Definition 2.8. [20] Let X be a fixed reference set. A HFS on X is defined in terms of a function that when it is applied to X returns a subset of $[0,1]$, and moreover it can be expressed as:

$$M = \{\langle x, h_M(x)\rangle : x \in X\},$$

where $h_M(x)$ denotes the possible membership degrees of the element $x \in X$ to the set M, and it is called hereafter the hesitant fuzzy element (HFE) of M.

Remark 2.9. Notice that the number of values in different HFEs may be different. Suppose that $l(h_M(x))$ stands for the number of values in $h_M(x)$. Hereafter, the following assumptions are made: (see [5]-[10]) (A1) All the elements in each $h_M(x)$ are arranged in increasing order, and then $h_M^{\sigma(j)}(x)$ is referred to as the jth largest value in $h_M(x)$. (A2) If, for some $x \in X$, $l(h_M(x)) \neq l(h_N(x))$, then $l_x = \max\{l(h_M(x)), l(h_N(x))\}$. In order to compare correctly in this case, the two HFEs $h_M(x)$ and $h_N(x)$ should have the same length l_x. If there are fewer elements in $h_M(x)$ than those in $h_N(x)$, an extension of $h_M(x)$ should be considered optimistically by repeating its maximum element until it has the same length with $h_N(x)$.

Definition 2.10. [9] A partial order $\preceq$ on the HFEs $h_M = \{h_M^{\sigma(j)}\}_{j=1}^{l_x}$ and $h_N = \{h_N^{\sigma(j)}\}_{j=1}^{l_x}$ is defined as:

$$h_M \preceq h_N \qquad \text{if and only if} \qquad h_M^{\sigma(j)} \leq h_N^{\sigma(j)}, \qquad j = 1, 2, ..., l_x. \tag{7}$$

It can be easily seen that the partial order $\preceq$ on HFEs is reflexive, antisymmetric, and transitive.

There are numerous contributions, for instance [7]-[9], [17] and [20] to help readers for being familiar with fundamentals of the theory of HFEs.

3. Metrical T-norm-based Similarity Measures for HFEs

Definition 3.1. Let $h_M = \{h_M^{\sigma(j)}\}_{j=1}^{l_x}$ and $h_N = \{h_N^{\sigma(j)}\}_{j=1}^{l_x}$ be two HFEs on X. We define

$$h_{[M-N]} = \{\mathfrak{D}(h_M^{\sigma(j)}, h_N^{\sigma(j)})\}_{j=1}^{l_x}, \tag{8}$$

as the difference-valued HFE if the function $\mathfrak{D}$ is a T-norm-based distance measure introduced in Definition 2.5, that is,

$$\mathfrak{D}(h_M^{\sigma(j)}, h_N^{\sigma(j)}) = \begin{cases} S(h_M^{\sigma(j)}, h_N^{\sigma(j)}) - T(h_M^{\sigma(j)}, h_N^{\sigma(j)}), & h_M^{\sigma(j)} \neq h_N^{\sigma(j)}; \\ 0, & h_M^{\sigma(j)} = h_N^{\sigma(j)}, \end{cases} \quad j = 1, ..., l_x, \tag{9}$$

where $T(.,.)$ and $S(.,.)$ denote a dual pair of T-norm and S-norm with respect to a negation $\aleph$.
For the samples of dual pair of T-norm and S-norm, one can refer to Example 2.2 where all the mentioned T-norms and their corresponding S-norms are dual with respect to the standard negation $\aleph(a) = 1 - a$.

In the case where $\mathfrak{D}$ is a metrical T-norm-based distance measure introduced in Definition 2.7, then $h_{[M-N]}$ is called *the metrical difference-valued HFE.*

Farhadinia [6] presented a series of score functions $\mathfrak{S} : [0,1]^{l_x} \to [0,1]$ for ranking the HFEs $h_M = \{h_M^{\sigma(j)}\}_{j=1}^{l_x}$ and $h_N = \{h_N^{\sigma(j)}\}_{j=1}^{l_x}$ where each score functions $\mathfrak{S}$ satisfies the *monotone non-decreasing property*, that is, if $h_M^{\sigma(1)} \leq h_N^{\sigma(1)}$,...,$h_M^{\sigma(l_x)} \leq h_N^{\sigma(l_x)}$, then $\mathfrak{S}(h_M^{\sigma(1)}, ..., h_M^{\sigma(l_x)}) \leq \mathfrak{S}(h_N^{\sigma(1)}, ..., h_N^{\sigma(l_x)})$, and also the score functions $\mathfrak{S}$ satisfies the *boundary conditions* $\mathfrak{S}(0, ..., 0) = 0$ and $\mathfrak{S}(1, ..., 1) = 1$.

Let $h_M = \{h_M^{\sigma(j)}\}_{j=1}^{l_x}$ be a HFE. Followings are some HFE score functions $\mathfrak{S}$ that are investigated deeply in [6]:

$$\mathfrak{S}_{AM}(h_M) = \frac{1}{l_x} \sum_{j=1}^{l_x} h_M^{\sigma(j)}, \quad \text{(Arithmetic-mean score function)} \tag{10}$$

$$\mathfrak{S}_{GM}(h_M) = (\prod_{j=1}^{l_x} h_M^{\sigma(j)})^{\frac{1}{l_x}}, \quad \text{(Geometric-mean score function)} \tag{11}$$

$$\mathfrak{S}_{Min}(h_M) = \min\{h_M^{\sigma(1)}, h_M^{\sigma(2)}, ..., h_M^{\sigma(l_x)}\}. \quad \text{(Minimum score function)} \tag{12}$$

Definition 3.2. [22] Let $h_M = \{h_M^{\sigma(j)}\}_{j=1}^{l_x}$, $h_N = \{h_N^{\sigma(j)}\}_{j=1}^{l_x}$ and $h_O = \{h_O^{\sigma(j)}\}_{j=1}^{l_x}$ be three HFEs on X. Then $\mathbf{S}$ is called a similarity measure for HFEs if it possesses the following properties:

(S0): $0 \leq \mathbf{S}(h_M, h_N) \leq 1$;
(S1): $\mathbf{S}(h_M, h_{M^c}) = 0$ if and only if $h_M \in \{O^*, I^*\}$;
(S2): $\mathbf{S}(h_M, h_N) = 1$ if and only if $h_M = h_N$;
(S3): $\mathbf{S}(h_M, h_N) = \mathbf{S}(h_N, h_M)$;
(S4): $\mathbf{S}(h_M, h_O) \leq \mathbf{S}(h_M, h_N)$ and $\mathbf{S}(h_M, h_O) \leq \mathbf{S}(h_N, h_O)$, if $h_M \preceq h_N \preceq h_O$.

Sometimes, we are interested in comparing similarities without regarding what value the similarity measure returns. By the aforementioned property of comparisons, we are only able to compare $\mathbf{S}(h_M, h_O)$ with $\mathbf{S}(h_M, h_N)$, or $\mathbf{S}(h_M, h_O)$ with $\mathbf{S}(h_N, h_O)$ whenever we have $h_M \preceq h_N \preceq h_O$. This issue motivates us to investigate the T-norm-based similarity measures between HFSs where the pairwise comparison of distinct HFEs can be achieved.

Definition 3.3. Let $h_M = \{h_M^{\sigma(j)}\}_{j=1}^{l_x}$, $h_N = \{h_N^{\sigma(j)}\}_{j=1}^{l_x}$, $h_O = \{h_O^{\sigma(j)}\}_{j=1}^{l_x}$ and $h_P = \{h_P^{\sigma(j)}\}_{j=1}^{l_x}$ be four HFEs on X, and suppose that $\mathfrak{S}$ is a HFE score function calculated by using one of the above formulas (1)-(1). If the function $\mathbb{S}$ possesses the following properties:

- **($\mathbb{S}$0):** $0 \leq \mathbb{S}[h_M, h_N] \leq 1$;
- **($\mathbb{S}$1):** $\mathbb{S}[h_M, h_{M^c}] = 0$ if and only if $h_M \in \{O^*, I^*\}$;
- **($\mathbb{S}$2):** $\mathbb{S}[h_M, h_N] = 1$ if and only if $h_M = h_N$;
- **($\mathbb{S}$3):** $\mathbb{S}[h_M, h_N] = \mathbb{S}[h_N, h_M]$;
- **($\mathbb{S}$4):** $\mathbb{S}[h_M, h_N] \leq \mathbb{S}[h_O, h_P]$, if $\mathfrak{S}(h_{[M-N]}) \geq \mathfrak{S}(h_{[O-P]})$,

then the function $\mathbb{S}$ is called a T-norm-based similarity measure for HFEs if $h_{[M-N]}$ and $h_{[O-P]}$ are difference-valued HFEs introduced in Definition 3.1.

Once again, we emphasize that $\mathbb{S}$ is called a metrical T-norm-based similarity measure for HFEs if $h_{[M-N]}$ and $h_{[O-P]}$ are metrical difference-valued HFEs.

Theorem 3.4. *A metrical T-norm-based similarity measure for HFEs is a similarity measure for HFEs.*

Proof. Let $h_M = \{h_M^{\sigma(j)}\}_{j=1}^{l_x}$, $h_N = \{h_N^{\sigma(j)}\}_{j=1}^{l_x}$, $h_O = \{h_O^{\sigma(j)}\}_{j=1}^{l_x}$ and $h_P = \{h_P^{\sigma(j)}\}_{j=1}^{l_x}$ be four HFEs on X. Furthermore, suppose that $\mathbb{S}$ is a metrical T-norm-based similarity measure for HFEs. We need to prove that $\mathbb{S}$ satisfies the properties (**S**0)-(**S**4) in Definition 3.2. Obviously, the properties ($\mathbb{S}$0)-($\mathbb{S}$3) in Definition 3.3 are the same as (**S**0)-(**S**3) in Definition 3.2. To show that the metrical T-norm-based similarity measure $\mathbb{S}$ satisfies (**S**4), we take $h_M \preceq h_N \preceq h_O$. Thus, from definition of metrical difference-valued HFEs $h_{[M-N]}$, $h_{[N-O]}$ and $h_{[M-O]}$, we conclude that $h_{[M-N]} \preceq h_{[M-O]}$ and $h_{[N-O]} \preceq h_{[M-O]}$. These results together with the monotone non-decreasing property of the score function $\mathfrak{S}$ give rise to $\mathfrak{S}(h_{[M-N]}) \leq \mathfrak{S}(h_{[M-O]})$ and $\mathfrak{S}(h_{[N-O]}) \leq \mathfrak{S}(h_{[M-O]})$. Taking into account the latter relations and the property ($\mathbb{S}$4), we get $\mathbb{S}[h_M, h_N] \geq \mathbb{S}[h_M, h_O]$ and $\mathbb{S}[h_N, h_O] \geq \mathbb{S}[h_M, h_O]$ which imply that the metrical T-norm-based similarity measure $\mathbb{S}$ satisfies (**S**4). □

Theorem 3.5. *Let $h_M = \{h_M^{\sigma(j)}\}_{j=1}^{l_x}$ and $h_N = \{h_N^{\sigma(j)}\}_{j=1}^{l_x}$ be two HFEs on X, and $\aleph$ be a negation operator with the properties considered in Definition 2.1. Then, $\mathbb{S}[h_M, h_N] = \aleph(\mathfrak{S}(h_{[M-N]}))$ defines a metrical T-norm-based similarity measure for HFEs.*

Proof. It is necessary to show that $\mathbb{S}[h_M, h_N] = \aleph(\mathfrak{S}(h_{[M-N]}))$ satisfies the properties ($\mathbb{S}$0)-($\mathbb{S}$4) in Definition 3.3. Suppose that $h_M = \{h_M^{\sigma(j)}\}_{j=1}^{l_x}$, $h_N = \{h_N^{\sigma(j)}\}_{j=1}^{l_x}$, $h_O = \{h_O^{\sigma(j)}\}_{j=1}^{l_x}$ and $h_P = \{h_P^{\sigma(j)}\}_{j=1}^{l_x}$ are four HFEs on X.

Property ($\mathbb{S}$0) follows immediately from definition of negation $\aleph$ given in Definition 2.1.

Property ($\mathbb{S}$1) is resulted from

$$\mathbb{S}[h_M, h_{M^c}] = \aleph(\mathfrak{S}(h_{[M-M^c]})) = \aleph(\mathfrak{S}(1, ..., 1)) = \aleph(1) = 0.$$

Property ($\mathbb{S}2$) is proven by

$$\mathbb{S}[h_M, h_N] = 1 \quad \text{if and only if} \quad \aleph(\mathfrak{S}(h_{[M-N]})) = 1 \quad \text{if and only if} \quad \mathfrak{S}(h_{[M-N]}) = 0 \quad \text{if and only if} \quad h_M = h_N.$$

Property ($\mathbb{S}3$) follows immediately from the symmetry property of $h_{[M-N]}$ where $h_{[M-N]} = h_{[N-M]}$.
Property ($\mathbb{S}4$) is true because if $\mathfrak{S}(h_{[M-N]}) \geq \mathfrak{S}(h_{[O-P]})$, then $\aleph(\mathfrak{S}(h_{[M-N]})) \leq \aleph(\mathfrak{S}(h_{[O-P]}))$ which results in $\mathbb{S}[h_M, h_N] \leq \mathbb{S}[h_O, h_P]$. □

Here are some examples of the metrical T-norm-based similarity measures for HFEs using different negation functions [12]:

- The Sugeno class of negations $\aleph(a) = \frac{1-a}{1+\lambda a}$ with $\lambda > -1$ defines the similarity
$$\mathbb{S}[h_M, h_N] = \frac{1 - \mathfrak{S}(h_{[M-N]})}{1 + \lambda\mathfrak{S}(h_{[M-N]})}, \quad \lambda > -1; \tag{13}$$
- The Yager class of negations $\aleph(a) = (1 - a^\nu)^{\frac{1}{\nu}}$ with $\nu > 0$ defines the similarity
$$\mathbb{S}[h_M, h_N] = (1 - \mathfrak{S}(h_{[M-N]})^\nu)^{\frac{1}{\nu}}, \quad \nu > 0. \tag{14}$$

Taking the formulas (13) and (14) into account, we list in Table 1 some metrical T-norm-based similarity measures for HFEs by using several HFE score functions given by (10)-(12).

Negation $\aleph(a)$	Score function $\mathfrak{S}(h_M)$	Metrical T-norm-based similarity measure $\mathbb{S}[h_M, h_N]$
	$\frac{1}{l_x}\sum_{j=1}^{l_x} h_M^{\sigma(j)}$	$\frac{1-(\frac{1}{l_x}\sum_{j=1}^{l_x}\lvert h_M^{\sigma(j)}-h_N^{\sigma(j)}\rvert)}{1+\lambda(\frac{1}{l_x}\sum_{j=1}^{l_x}\lvert h_M^{\sigma(j)}-h_N^{\sigma(j)}\rvert)}$, $\lambda > -1$
$\frac{1-a}{1+\lambda a}, \quad \lambda > -1$	$(\prod_{j=1}^{l_x} h_M^{\sigma(j)})^{\frac{1}{l_x}}$	$\frac{1-(\prod_{j=1}^{l_x}\lvert h_M^{\sigma(j)}-h_N^{\sigma(j)}\rvert)^{\frac{1}{l_x}}}{1+\lambda(\prod_{j=1}^{l_x}\lvert h_M^{\sigma(j)}-h_N^{\sigma(j)}\rvert)^{\frac{1}{l_x}}}$, $\lambda > -1$
	$\min\{h_M^{\sigma(1)}, h_M^{\sigma(2)}, ..., h_M^{\sigma(l_x)}\}$	$\frac{1-(\min\{\lvert h_M^{\sigma(1)}-h_N^{\sigma(1)}\rvert,\ldots,\lvert h_M^{\sigma(l_x)}-h_N^{\sigma(l_x)}\rvert\})}{1+\lambda(\min\{\lvert h_M^{\sigma(1)}-h_N^{\sigma(1)}\rvert,\ldots,\lvert h_M^{\sigma(l_x)}-h_N^{\sigma(l_x)}\rvert\})}$, $\lambda > -1$
	$\frac{1}{l_x}\sum_{j=1}^{l_x} h_M^{\sigma(j)}$	$(1 - (\frac{1}{l_x}\sum_{j=1}^{l_x}\lvert h_M^{\sigma(j)} - h_N^{\sigma(j)}\rvert)^\nu)^{\frac{1}{\nu}}$, $\nu > 0$
$(1-a^\nu)^{\frac{1}{\nu}}, \quad \nu > 0$	$(\prod_{j=1}^{l_x} h_M^{\sigma(j)})^{\frac{1}{l_x}}$	$(1 - ((\prod_{j=1}^{l_x}\lvert h_M^{\sigma(j)} - h_N^{\sigma(j)}\rvert)^{\frac{1}{l_x}})^\nu)^{\frac{1}{\nu}}$, $\nu > 0$
	$\min\{h_M^{\sigma(1)}, h_M^{\sigma(2)}, ..., h_M^{\sigma(l_x)}\}$	$(1 - (\min\{\lvert h_M^{\sigma(1)} - h_N^{\sigma(1)}\rvert, ..., \lvert h_M^{\sigma(l_x)} - h_N^{\sigma(l_x)}\rvert\})^\nu)^{\frac{1}{\nu}}$, $\nu > 0$

TABLE 1. Some Metrical T-norm-based Similarity Measures for HFEs

3.1. **Metrical T-norm-based Entropy Measures for HFEs.** Xu and Xia [23] put forward some axioms to describe the fuzziness degree of a HFE. They gave the axiomatic definition of entropy for HFEs as follows:

Definition 3.6. Let $h_M = \{h_M^{\sigma(j)}\}_{j=1}^{l_x}$ and $h_N = \{h_N^{\sigma(j)}\}_{j=1}^{l_x}$ be two HFEs on X. Then E is called an entropy for HFEs if it possesses the following properties:

(E0): $0 \leq E(h_M) \leq 1$;
(E1): $E(h_M) = 0$ if and only if $h_M \in \{O^*, I^*\}$;
(E2): $E(h_M) = 1$ if and only if $h_M^{\sigma(j)} + h_M^{\sigma(l_x-j+1)} = 1$ for $j = 1, ..., l_x$;
(E3): $E(h_M) = E(h_{M^c})$;
(E4): $E(h_M) \leq E(h_N)$, if $h_M^{\sigma(j)} \leq h_N^{\sigma(j)}$ for $h_N^{\sigma(j)} + h_N^{\sigma(l_x-j+1)} \leq 1$ or $h_M^{\sigma(j)} \geq h_N^{\sigma(j)}$ for $h_N^{\sigma(j)} + h_N^{\sigma(l_x-j+1)} \geq 1$ where $j = 1, ..., l_x$.

Indeed, the above definition is developed based on the axiomatic definition of a fuzzy set. As Yager [24] and Zeng and Guo [26] related the fuzziness to the lack of distinction between a set and its complement, we will extend this idea to propose some entropy measures for HFEs which return the similarity degree between a HFE and its complement.

On the basis of the aforementioned metrical T-norm-based similarity measure, the entropy measure for HFEs is constructed as follows:

Theorem 3.7. *Let $h_M = \{h_M^{\sigma(j)}\}_{j=1}^{l_x}$ be a HFE on X, and $\mathbb{S}$ be a metrical T-norm-based similarity measure for HFEs. Then,*

$$E(h_M) = \mathbb{S}[\phi(h_M), \phi(h_{M^c})] \tag{15}$$

is an entropy measure of the HFE h_M whenever the function ϕ on HFEs satisfies the following properties:

(ϕ0): $\phi(O^*) = O^*$, $\phi(I^*) = I^*$;
(ϕ1): $\phi(h_{M^c}) = (\phi(h_M))^c$;
(ϕ2): *ϕ is strictly monotonically increasing.*

Proof. It is sufficient to prove that $E(h_M) = \mathbb{S}[\phi(h_M), \phi(h_{M^c})]$ satisfies the properties ($E0$)-($E4$) in Definition 3.6. Suppose that $h_M = \{h_M^{\sigma(j)}\}_{j=1}^{l_x}$ is a HFE on X.

Property ($E0$) follows immediately from the property ($\mathbb{S}0$) of the metrical T-norm-based similarity measure $\mathbb{S}$ for HFEs.

Property ($E1$) is resulted from

$$\begin{aligned} &E(h_M) = 0 \quad \text{if and only if} \quad \mathbb{S}[\phi(h_M), \phi(h_{M^c})] = 0 \quad \text{if and only if} \\ &\phi(h_M) \in \{O^*, I^*\} \quad \text{if and only if} \quad \phi(h_M) = O^*, \ \textit{or}\ \phi(h_M) = I^* \\ &\text{if and only if} \quad h_M \in \{O^*, I^*\}. \end{aligned}$$

Property ($E2$) is proven by

$$\begin{aligned} &E(h_M) = 1 \quad \text{if and only if} \quad \mathbb{S}[\phi(h_M), \phi(h_{M^c})] = 1 \quad \text{if and only if} \\ &\phi(h_M) = \phi(h_{M^c}) \quad \text{if and only if} \\ &h_M = h_{M^c} \quad \text{if and only if} \quad h_M^{\sigma(j)} = h_{M^c}^{\sigma(j)} := 1 - h_M^{\sigma(l_x-j+1)}, \\ &(j = 1, ..., l_x) \quad \text{if and only if} \\ &h_M^{\sigma(j)} + h_M^{\sigma(l_x-j+1)} = 1, \quad (j = 1, ..., l_x). \end{aligned}$$

Property ($E3$) is straightforward.

To show the property ($E4$) is true, we consider the following two cases:

Case 1. If $h_M^{\sigma(j)} \leq h_N^{\sigma(j)}$ for $h_N^{\sigma(j)} + h_N^{\sigma(l_x-j+1)} \leq 1$, then, we have $h_M^{\sigma(j)} \leq h_N^{\sigma(j)}$ for $h_N^{\sigma(j)} \leq 1 - h_N^{\sigma(l_x-j+1)} = h_{N^c}^{\sigma(j)}$. Therefore, $h_M^{\sigma(j)} \leq h_N^{\sigma(j)} \leq h_{N^c}^{\sigma(j)}$. On the other hand, we conclude from $h_M^{\sigma(j)} \leq h_N^{\sigma(j)}$ that $1 - h_M^{\sigma(j)} \geq 1 - h_N^{\sigma(j)}$, that is, $h_{M^c}^{\sigma(j)} \geq h_{N^c}^{\sigma(j)}$. Taking the latter inequalities into account, we deduce that $h_M^{\sigma(j)} \leq h_N^{\sigma(j)} \leq h_{N^c}^{\sigma(j)} \leq h_{M^c}^{\sigma(j)}$. Keeping the definition of a metrical difference-valued HFE given in Definition 3.1 in mind, we get that $h_{[M-M^c]} \geq h_{[N-N^c]}$.

Case 2. If $h_M^{\sigma(j)} \geq h_N^{\sigma(j)}$ for $h_N^{\sigma(j)} + h_N^{\sigma(l_x-j+1)} \geq 1$, then, we have $h_M^{\sigma(j)} \geq h_N^{\sigma(j)}$ for $h_N^{\sigma(j)} \geq 1 - h_N^{\sigma(l_x-j+1)} = h_{N^c}^{\sigma(j)}$. Therefore, $h_M^{\sigma(j)} \geq h_N^{\sigma(j)} \geq h_{N^c}^{\sigma(j)}$. On the other hand, we conclude from $h_M^{\sigma(j)} \geq h_N^{\sigma(j)}$ that $1 - h_M^{\sigma(j)} \leq 1 - h_N^{\sigma(j)}$, that is, $h_{M^c}^{\sigma(j)} \leq h_{N^c}^{\sigma(j)}$. Taking the latter inequalities into account, we deduce that $h_M^{\sigma(j)} \geq h_N^{\sigma(j)} \geq h_{N^c}^{\sigma(j)} \geq h_{M^c}^{\sigma(j)}$. Keeping the definition of a metrical difference-valued HFE given in Definition 3.1 in mind, we get once again that $h_{[M-M^c]} \geq h_{[N-N^c]}$.

As observed from the above two cases, both of them return the same result, that is, $h_{[M-M^c]} \geq h_{[N-N^c]}$. Since the function ϕ is strictly monotonically increasing with respect to its argument, thus it holds $\phi(h_{[M-M^c]}) \geq \phi(h_{[N-N^c]})$. By the increasing property of the score function $\mathfrak{S}$, we achieve $\mathfrak{S}(\phi(h_{[M-M^c]})) \geq \mathfrak{S}(\phi(h_{[N-N^c]}))$, and consequently, $\mathbb{S}[\phi(h_M), \phi(h_{M^c})] \leq \mathbb{S}[\phi(h_N), \phi(h_{N^c})]$ which implies that $E(h_M) \leq E(h_N)$. □

We intend in this study to refer those entropy measures that are derived from the metrical T-norm-based similarity measures as the metrical T-norm-based entropy measures for HFEs.

In view of Theorem 3.7, we can construct below some metrical T-norm-based entropy measures for HFEs using different strictly monotonically increasing functions ϕ:

- If we take $\phi(x) = x$ together with the similarity $\mathbb{S}[h_M, h_N]$ given by (13) or (14), then

$$E(h_M) = \mathbb{S}[h_M, h_{M^c}] = \frac{1 - \mathfrak{S}(h_{[M-M^c]})}{1 + \lambda\mathfrak{S}(h_{[M-M^c]})}, \quad \lambda > -1, \tag{16}$$

$$E(h_M) = \mathbb{S}[h_M, h_{M^c}] = (1 - \mathfrak{S}(h_{[M-M^c]})^{\nu})^{\frac{1}{\nu}}, \quad \nu > 0; \tag{17}$$

- If we take $\phi(x) = \sin(\frac{\Pi}{2}x)$ together with the similarity $\mathbb{S}[h_M, h_N]$ given by (13) or (14), then

$$E(h_M) = \mathbb{S}[\sin(\frac{\Pi}{2}h_M), \sin(\frac{\Pi}{2}h_{M^c})] = \frac{1 - \mathfrak{S}(h_{[\sin(\frac{\Pi}{2}(h_M)) - \sin(\frac{\Pi}{2}(h_{M^c}))]})}{1 + \lambda\mathfrak{S}(h_{[\sin(\frac{\Pi}{2}(h_M)) - \sin(\frac{\Pi}{2}(h_{M^c}))]})},$$
$$\lambda > -1, \tag{18}$$

$$E(h_M) = \mathbb{S}[\sin(\frac{\Pi}{2}h_M), \sin(\frac{\Pi}{2}h_{M^c})] = (1 - \mathfrak{S}(h_{[\sin(\frac{\Pi}{2}(h_M)) - \sin(\frac{\Pi}{2}(h_{M^c}))]})^{\nu})^{\frac{1}{\nu}},$$
$$\nu > 0. \tag{19}$$

Taking the formulas (16)-(19) into account, we list in Table 2 some metrical T-norm-based entropy measures for HFEs by using several HFE score functions given by (10)-(12).

Score function $\mathfrak{S}(h_M)$	$\mathbb{S}[h_M, h_{M^c}]$	Metrical T-norm-based entropy measure $E(h_M)$
$\frac{1}{l_x}\sum_{j=1}^{l_x} h_M^{\sigma(j)}$	$\frac{1-\mathfrak{S}(h_{[M-M^c]})}{1+\lambda\mathfrak{S}(h_{[M-M^c]})}$	$\frac{1-(\frac{1}{l_x}\sum_{j=1}^{l_x} \lvert h_M^{\sigma(j)}+h_M^{l_x-\sigma(j)+1}-1\rvert)}{1+\lambda(\frac{1}{l_x}\sum_{j=1}^{l_x} \lvert h_M^{\sigma(j)}+h_M^{l_x-\sigma(j)+1}-1\rvert)}$
$(\prod_{j=1}^{l_x} h_M^{\sigma(j)})^{\frac{1}{l_x}}$		$\frac{1-(\prod_{j=1}^{l_x} \lvert h_M^{\sigma(j)}+h_M^{l_x-\sigma(j)+1}-1\rvert)^{\frac{1}{l_x}}}{1+\lambda(\prod_{j=1}^{l_x} \lvert h_M^{\sigma(j)}+h_M^{l_x-\sigma(j)+1}-1\rvert)^{\frac{1}{l_x}}}$
$\min\{h_M^{\sigma(1)}, h_M^{\sigma(2)}, \ldots, h_M^{\sigma(l_x)}\}$		$(1-(\min\{\lvert h_M^{\sigma(1)}+h_M^{l_x-\sigma(1)+1}-1\rvert, \ldots, \lvert h_M^{\sigma(n)}+h_M^{l_x-\sigma(l_x)+1}-1\rvert\}))/(1+\lambda(\min\{\lvert h_M^{\sigma(1)}+h_M^{l_x-\sigma(1)+1}-1\rvert, \ldots, \lvert h_M^{\sigma(l_x)}+h_M^{l_x-\sigma(l_x)+1}-1\rvert\}))$
$\frac{1}{l_x}\sum_{j=1}^{l_x} h_M^{\sigma(j)}$	$(1-\mathfrak{S}(h_{[M-M^c]})^{\nu})^{\frac{1}{\nu}}$	$(1-(\frac{1}{l_x}\sum_{j=1}^{l_x} \lvert h_M^{\sigma(j)}+h_M^{l_x-\sigma(j)+1}-1\rvert)^{\nu})^{\frac{1}{\nu}}$
$(\prod_{j=1}^{l_x} h_M^{\sigma(j)})^{\frac{1}{l_x}}$		$(1-((\prod_{j=1}^{l_x} \lvert h_M^{\sigma(j)}+h_M^{l_x-\sigma(j)+1}-1\rvert)^{\frac{1}{l_x}})^{\nu})^{\frac{1}{\nu}}$
$\min\{h_M^{\sigma(1)}, h_M^{\sigma(2)}, \ldots, h_M^{\sigma(l_x)}\}$		$(1-(\min\{\lvert h_M^{\sigma(1)}+h_M^{l_x-\sigma(1)+1}-1\rvert, \ldots, \lvert h_M^{\sigma(n)}+h_M^{l_x-\sigma(l_x)+1}-1\rvert\})^{\nu})^{\frac{1}{\nu}}$

TABLE 2. Some Metrical T-norm-based Entropy Measures for HFEs Using $\phi(x) = x$. The Parameters in the Above Formulas are Defined as $\lambda > -1$ and $\nu > 0$

Many other formulas for a metrical T-norm-based entropy measure for HFEs can be derived by using other strictly monotonically increasing functions ϕ such as $\phi(x) = \frac{\lambda^x-1}{\lambda-1}$ with $\lambda > 1$, and $\phi(x) = \log_\lambda((\lambda-1)x+1)$ with $\lambda > 1$.

Theorem 3.8. *Let $h_M = \{h_M^{\sigma(j)}\}_{j=1}^{l_x}$ and $h_N = \{h_N^{\sigma(j)}\}_{j=1}^{l_x}$ be two HFEs on X, and E be a metrical T-norm-based entropy measure for HFEs. Then,*

$$\mathbb{S}[h_M, h_N] = E(\frac{1}{2} - \frac{1}{2}h_{[M-N]}) \tag{20}$$

is a metrical T-norm-based similarity measure for HFEs.

Proof. It is necessary to show that $\mathbb{S}[h_M, h_N] = E(\frac{1}{2} - \frac{1}{2}h_{[M-N]})$ satisfies the properties ($\mathbb{S}$0)-($\mathbb{S}$4) in Definition 3.3. Suppose that $h_M = \{h_M^{\sigma(j)}\}_{j=1}^{l_x}$, $h_N = \{h_N^{\sigma(j)}\}_{j=1}^{l_x}$, $h_O = \{h_O^{\sigma(j)}\}_{j=1}^{l_x}$ and $h_P = \{h_P^{\sigma(j)}\}_{j=1}^{l_x}$ are four HFEs on X.

Property ($\mathbb{S}$0) follows immediately from the property (E0) in Definition 3.6.

Property ($\mathbb{S}$1) is resulted from

$$\begin{aligned} &\mathbb{S}[h_M, h_{M^c}] = 0 \quad \text{if and only if} \quad E(\frac{1}{2} - \frac{1}{2}h_{[M-M^c]}) = 0 \\ &\text{if and only if} \quad \frac{1}{2} - \frac{1}{2}h_{[M-M^c]} \in \{O^*, I^*\} \quad \text{if and only if} \\ &h_M \in \{O^*, I^*\}. \end{aligned}$$

Property ($\mathbb{S}2$) is proven by

$$\mathbb{S}[h_M, h_N] = 1 \quad \text{if and only if} \quad E(\frac{1}{2} - \frac{1}{2}h_{[M-N]}) = 1$$
$$\text{if and only if} \quad \frac{1}{2} - \frac{1}{2}h^{\sigma(j)}_{[M-N]} + \frac{1}{2} - \frac{1}{2}h^{\sigma(l_x-j+1)}_{[M-N]} = 1 \ (j = 1, ..., l_x)$$
$$\text{if and only if} \quad h^{\sigma(j)}_{[M-N]} + h^{\sigma(l_x-j+1)}_{[M-N]} = 0 \ (j = 1, ..., l_x)$$
$$\text{if and only if} \quad h^{\sigma(j)}_{[M-N]} = 0 \ (j = 1, ..., l_x) \quad \text{if and only if}$$
$$h_M = h_N.$$

Property ($\mathbb{S}3$) follows immediately from the symmetry property of $h_{[M-N]}$ where $h_{[M-N]} = h_{[N-M]}$.

Property ($\mathbb{S}4$) is true because of the following reasoning. We know that

$$E(\frac{1}{2} - \frac{1}{2}h_{[M-N]}) \leq E(\frac{1}{2} - \frac{1}{2}h_{[O-P]}) \tag{21}$$

whenever

Case 1: $\frac{1}{2} - \frac{1}{2}h^{\sigma(j)}_{[M-N]} \leq \frac{1}{2} - \frac{1}{2}h^{\sigma(j)}_{[O-P]}$ for $j \in J_1 = \{j \in J \mid \frac{1}{2} - \frac{1}{2}h^{\sigma(j)}_{[O-P]} + \frac{1}{2} - \frac{1}{2}h^{\sigma(l_x-j+1)}_{[O-P]} \leq 1\}$ and

Case 2: $\frac{1}{2} - \frac{1}{2}h^{\sigma(j)}_{[M-N]} \geq \frac{1}{2} - \frac{1}{2}h^{\sigma(j)}_{[O-P]}$ for $j \in J_2 = \{j \in J \mid \frac{1}{2} - \frac{1}{2}h^{\sigma(j)}_{[O-P]} + \frac{1}{2} - \frac{1}{2}h^{\sigma(l_x-j+1)}_{[O-P]} \geq 1\}$,

where $J_1 \cup J_2 = J = \{1, ..., l_x\}$.

By a simple calculation, we obtain that

Case 1: $h^{\sigma(j)}_{[M-N]} \geq h^{\sigma(j)}_{[O-P]}$ for $j \in J_1 = \{j \in J \mid \frac{1}{2}h^{\sigma(j)}_{[O-P]} + \frac{1}{2}h^{\sigma(l_x-j+1)}_{[O-P]} \geq 0\} = J$ and

Case 2: $h^{\sigma(j)}_{[M-N]} \leq h^{\sigma(j)}_{[O_P]}$ for $j \in J_2 = \{j \in J \mid h^{\sigma(j)}_{[O-P]} + h^{\sigma(l_x-j+1)}_{[O-P]} \leq 0\} = \emptyset$,

which result in $h^{\sigma(j)}_{[M-N]} \geq h^{\sigma(j)}_{[O-P]}$ for all $j \in J$, that is, $h_{[M-N]} \geq h_{[O-P]}$. Known by the monotonically increasing property of score function, the latter inequality gives rise to $\mathfrak{S}(h_{[M-N]}) \geq \mathfrak{S}(h_{[O-P]})$. Consequently, the inequality (21) is valid if $\mathfrak{S}(h_{[M-N]}) \geq \mathfrak{S}(h_{[O-P]})$. This result is nothing else but the conclusion that $\mathbb{S}[h_M, h_N] \leq \mathbb{S}[h_O, h_P]$ if $\mathfrak{S}(h_{[M-N]}) \geq \mathfrak{S}(h_{[O-P]})$. □

Corollary 3.9. *Let $h_M = \{h^{\sigma(j)}_M\}^{l_x}_{j=1}$ and $h_N = \{h^{\sigma(j)}_N\}^{l_x}_{j=1}$ be two HFEs on X, and E be a metrical T-norm-based entropy measure for HFEs. Then,*

$$\mathbb{S}[h_M, h_N] = E(\frac{1}{2} - \frac{1}{2}h_{[M^c-N^c]}) \tag{22}$$

is a metrical T-norm-based similarity measure for HFEs.

4. Application of HFE Metrical T-norm-based Similarity Measure

The main difference between the present study and the other ones being based on HFS and it extensions, such as interval-valued HFS (IVHFS), dual HFS (DHFS) and so on, is that the former deals directly with the HFS, meanwhile the latter deal with the element of HFS, that is, with the HFE. To see the mentioned difference clearly, it is enough to refer to Example 6 in [13], Example 5.1 in [9], Example 11 in [19], Example 4 in [20], Example 1 in [22], and Section 4 in [21].

However, to demonstrate the application of the proposed metrical T-norm-based similarity measure for HFSs, a medical diagnosis problem and a pattern recognition problem are respectively discussed as follows:

Example 4.1. (Medical diagnosis)

Consider the set of diagnoses $D = \{$Viral fever, Malaria, Typhoid, Stomach problem, Chest problem$\}$. The aim here is to assign a patient with the given values of the symptoms $S = \{$Temperature, Headache, Cough, Stomach pain, Chest pain$\}$ to one of the aforementioned diagnoses.

Three medical experts E_l $(l = 1, 2, 3)$ are invited to provide their possible assessment of diagnoses with respect to symptoms. For each diagnosis with respect to each symptom, all of the medical experts provide anonymously their evaluated values. As an example, for the diagnosis "Viral fever" with respect to the symptom "Temperature", the evaluation value provided by the medical experts E_1 and E_3 is 0.5; and E_2's evaluation value is 0.7. In this regard, and supposing that the importance weights of three medical experts are the same, the evaluation of "Viral fever" with respect to "Temperature" can be represented by a HFE as:

$$h(Viral fever, Temperature) := h_{11} = \{0.5, 0.7\}.$$

Note that the characteristics of the diagnosis "Viral fever" with respect to the symptoms "Headache", "Cough", "Stomach pain", and "Chest pain", which are denoted respectively by HFEs h_{1j} $(j = 2, 3, 4, 5)$, form the HFS H_1 which is indicated in the first row of Table 3. The results evaluated for other diagnoses with respect to symptoms are contained in a hesitant fuzzy decision matrix, shown in Table 3.

Furthermore, suppose that the set of patients is $P = \{$Al, Bob, Joe, Ted$\}$, and the symptoms characteristic for the considered patients are evaluated and given by the three medical experts in the form of a hesitant fuzzy matrix demonstrated in Table 4. Here, the main task is to seek a diagnosis for each patient.

	Temperature	Headache	Cough	Stomach pain	Chest pain
Viral fever	$\{0.5, 0.7\}$	$\{0.3\}$	$\{0.4, 0.6\}$	$\{0.3, 0.4\}$	$\{0.6, 0.7\}$
Malaria	$\{0.4, 0.7\}$	$\{0.5, 0.8\}$	$\{0.5\}$	$\{0.2, 0.3\}$	$\{0.7, 0.8\}$
Typhoid	$\{0.3, 0.4\}$	$\{0.5, 0.6\}$	$\{0.1, 0.4\}$	$\{0.6\}$	$\{0.5, 0.7\}$
Stomach problem	$\{0.5, 0.7\}$	$\{0.5, 0.7\}$	$\{0.7\}$	$\{0.4, 0.5\}$	$\{0.5, 0.8\}$
Chest problem	$\{0.4, 0.5\}$	$\{0.4, 0.6\}$	$\{0.4, 0.5\}$	$\{0.6, 0.7\}$	$\{0.4\}$

Table 3. Symptoms Characteristic for the Considered Diagnoses

As can be seen from Tables 3 and 4, all HFEs are not in the same size. To circumvent this issue, we implement Remark 2.9 where the HFEs with fewer elements are extended optimistically by repeating the maximum element until it has the same length with others. For example, the HFE $h(Viral fever, Headache) := h_{12} = \{0.3\}$ is extended to $\{0.3, 0.3\}$.

	Temperature	Headache	Cough	Stomach pain	Chest pain
Al	$\{0.4\}$	$\{0.5, 0.7\}$	$\{0.6, 0.7\}$	$\{0.2, 0.4\}$	$\{0.1, 0.2\}$
Bob	$\{0.6, 0.7\}$	$\{0.5, 0.8\}$	$\{0.5, 0.6\}$	$\{0.3\}$	$\{0.4, 0.5\}$
Joe	$\{0.2, 0.3\}$	$\{0.5\}$	$\{0.2, 0.4\}$	$\{0.6, 0.7\}$	$\{0.5, 0.7\}$
Ted	$\{0.4\}$	$\{0.4, 0.7\}$	$\{0.3, 0.4\}$	$\{0.7, 0.8\}$	$\{0.5, 0.6\}$

Table 4. Symptoms Characteristic for the Considered Patients

During the process of deriving a diagnosis for each patient, the degree of dependence between the rows of Tables 3 and 4 should be analyzed. For instance, the first rows of Tables 3 and 4 which are regarded as the following two HFSs:

$$\begin{aligned} Al = \{&Temperature, \{0.4, 0.4\}, Headache, \{0.5, 0.7\}, Cough, \{0.6, 0.7\},\\ &Stomachpain, \{0.2, 0.4\}, Chestpain, \{0.1, 0.2\}\};\\ Viralfever = \{&Temperature, \{0.5, 0.7\}, Headache, \{0.3, 0.3\},\\ &Cough, \{0.4, 0.6\}, Stomachpain, \{0.3, 0.4\},\\ &Chestpain, \{0.6, 0.7\}\}, \end{aligned}$$

are taken into account to determine the similarity degree of Al and Viral fever.

Before doing any other part of the procedure, we recall here some existing similarity measures for HFEs.
Xu and Xia [22] presented a broad range of distance measures for HFEs, and also aimed that any HFE distance measure can produce a similarity measure for two HFEs h_M and h_N by the rule $S(h_M, h_N) = 1 - d(h_M, h_N)$. In this regard, if we consider HFEs h_M and h_N such that $h_M^{\sigma(j)}$ and $h_N^{\sigma(j)}$ denote respectively the jth largest values in h_M and h_N, then some Xu and Xia's [22] HFE similarity measures are as follows:

- The generalized hesitant normalized similarity measure:

$$S_{ghn}(h_M, h_N) := 1 - d_{ghn}(h_M, h_N) = 1 - (\frac{1}{l_x}\sum_{j=1}^{l_x} |h_M^{\sigma(j)} - h_N^{\sigma(j)}|^{\lambda})^{\frac{1}{\lambda}}, \quad \lambda > 0. \tag{23}$$

  In the case that $\lambda = 1$, the generalized hesitant normalized similarity measure becomes the hesitant normalized Hamming similarity measure; and in the case where $\lambda = 2$, the generalized hesitant normalized similarity measure is converted to the hesitant normalized Euclidean similarity measure.
- The generalized hesitant normalized Hausdorff similarity measure:

$$\begin{aligned} S_{ghnh}(h_M, h_N) := 1 - d_{ghnh}(h_M, h_N) = \\ 1 - \max_{1\le j\le l_x} \{|h_M^{\sigma(j)} - h_N^{\sigma(j)}|^{\lambda}\}^{\frac{1}{\lambda}}, \quad \lambda > 0. \end{aligned} \tag{24}$$

- The generalized hybrid hesitant normalized similarity measure:

$$\begin{aligned} &S_{ghhnh}(h_M, h_N) := 1 - d_{ghhnh}(h_M, h_N) = \\ &1 - (\frac{1}{2}[\frac{1}{l_x}\sum_{j=1}^{l_x} |h_M^{\sigma(j)} - h_N^{\sigma(j)}|^{\lambda} + \max_{1\le j\le l_x} \{|h_M^{\sigma(j)} - h_N^{\sigma(j)}|^{\lambda}\}])^{\frac{1}{\lambda}}, \quad \lambda > 0. \end{aligned} \tag{25}$$

Now, we apply the above-mentioned Xu and Xia's similarity measures [22], and the six similarity measures listed in Table 1 to determine the degree of similarity between diagnoses and patients whose results are shown in Tables 5 and 6. There, the notations of $\mathbb{S}_k$ $(k = 1, 2, ..., 6)$ indicate the proposed metrical T-norm-based similarity measures located in the k-row of Table 1.

		The	proposed	similarity	measures	
	$\mathbb{S}_1$	$\mathbb{S}_2$	$\mathbb{S}_3$	$\mathbb{S}_4$	$\mathbb{S}_5$	$\mathbb{S}_6$
$S(Viral\ fever, Al)$	0.6365	0.6699	0.7273	0.9574	0.9600	0.9672
$S(Viral\ fever, Bob)$	**0.7725**	**0.8372**	**0.8667**	**0.9826**	**0.9857**	**0.9919**
$S(Viral\ fever, Joe)$	0.6516	0.6714	0.6821	0.9698	0.9703	0.9735
$S(Viral\ fever, Ted)$	0.6505	0.6741	0.7403	0.9696	0.9732	0.9793
$S(Malaria, Al)$	0.7076	0.8004	0.8136	0.9550	0.9580	0.9590
$S(Malaria, Bob)$	**0.8332**	**0.9077**	**0.9077**	**0.9893**	**0.9908**	**0.9908**
$S(Malaria, Joe)$	0.6224	0.6889	0.7463	0.9655	0.9701	0.9773
$S(Malaria, Ted)$	0.6593	0.7141	0.7273	0.9636	0.9662	0.9672
$S(Typhoid, Al)$	0.6312	0.6765	0.7267	0.9522	0.9551	0.9701
$S(Typhoid, Bob)$	0.6345	0.6776	0.7124	0.9691	0.9714	0.9765
$S(Typhoid, Joe)$	**0.9065**	**0.9636**	**0.9636**	**0.9982**	**0.9990**	**0.9990**
$S(Typhoid, Ted)$	0.8370	0.9141	0.9273	0.9952	0.9970	0.9980
$S(Stomach\ problem, Al)$	0.7288	0.7599	0.8130	0.9667	0.9693	0.9813
$S(Stomach\ problem, Bob)$	**0.7909**	**0.8418**	**0.8909**	**0.9909**	**0.9930**	**0.9970**
$S(Stomach\ problem, Joe)$	0.6599	0.7187	0.7487	0.9654	0.9680	0.9775
$S(Stomach\ problem, Ted)$	0.6819	0.7457	0.7790	0.9728	0.9754	0.9806
$S(Chest\ problem, Al)$	0.6942	0.7154	0.7380	0.9757	0.9765	0.9817
$S(Chest\ problem, Bob)$	0.7220	0.7445	0.7683	0.9798	0.9806	0.9847
$S(Chest\ problem, Joe)$	0.7781	0.7884	0.8242	0.9887	0.9899	0.9930
$S(Chest\ problem, Ted)$	**0.8370**	**0.8777**	**0.8909**	**0.9952**	**0.9960**	**0.9970**

Table 5. Values of Similarity Measures Between Patients and Diagnoses. The Parameters in the Above Formulas are Considered as $\lambda = 1$ and $\nu = 2$

By comparing the results shown in Tables 5 and 6, we observe that Bob suffers from Viral fever, Malaria and Stomach problem, meanwhile, Joe from Typhoid, and Ted from Chest problem.

Example 4.2. (Pattern recognition)

In this portion, a pattern recognition problem is re-considered from the project of the National Social Science Foundation of China (No. 15CJY057) which is entitled as "Research on the motivation, effect and mechanism of the third-party logistics embedded in the global supply chain" to illustrate the efficiency of the new proposed similarity measures.

There exists a tugboat company in Zhabei district, Shanghai city, P. R. China that is encouraged to buy two harbor operational tugs. According to the same property of fleets of tugs in Shanghai city, this company is going to join one fleet which is most similarly managed as itself. To do so, the company has invited a group of experts to evaluate the management philosophy of the three ports and himself. Based on the three evaluation criteria including P_1: the efficiency in

	Xu and Xia's	similarity	measures [22]
	S_{ghn}	S_{ghnh}	S_{ghhnh}
$S(Viral\ fever, Al\ )$	0.7600	0.8500	0.8050
$S(Viral\ fever, Bob)$	**0.8600**	**0.9000**	**0.8800**
$S(Viral\ fever, Joe)$	0.7800	0.8800	0.8300
$S(Viral\ fever, Ted)$	0.7800	0.8600	0.8200
$S(Malaria, Al\ )$	0.8000	0.8700	0.8350
$S(Malaria, Bob)$	**0.9000**	**0.9300**	**0.9150**
$S(Malaria, Joe)$	0.7600	0.8400	0.8000
$S(Malaria, Ted)$	0.7800	0.8700	0.8250
$S(Typhoid, Al\ )$	0.7500	0.8400	0.7950
$S(Typhoid, Bob)$	0.7700	0.8600	0.8150
$S(Typhoid, Joe)$	**0.9500**	**0.9600**	**0.9550**
$S(Typhoid, Ted)$	0.9100	0.9300	0.9200
$S(Stomach\ problem, Al\ )$	0.8200	0.8800	0.8500
$S(Stomach\ problem, Bob)$	**0.8800**	**0.9100**	**0.8950**
$S(Stomach\ problem, Joe)$	0.7800	0.8600	0.8200
$S(Stomach\ problem, Ted)$	0.8000	0.8700	0.8350
$S(Chest\ problem, Al\ )$	0.8100	0.8900	0.8500
$S(Chest\ problem, Bob)$	0.8300	0.9000	0.8650
$S(Chest\ problem, Joe)$	0.8700	0.9200	0.8950
$S(Chest\ problem, Ted)$	**0.9100**	**0.9400**	**0.9250**

TABLE 6. Values of Similarity Measures Between Patients and Diagnoses. The Parameters in the Above Formulas are Considered as $\lambda = 1$ and $\nu = 2$

decision-making, P_2: the sense of urgency and P_3: the quality of service with the weights $W = (0.35, 0.35, 0.30)$, the fleets E_1, E_2, E_3, and the tugboat company E_0 are evaluated. By the use of aggregating decision information, the evaluation results are represented in the form of the following matrix:

	P_1	P_2	P_3
E_1	$\{0.80, 0.85, 0.90\}$	$\{0.72, 0.75, 0.78\}$	$\{0.85, 0.88, 0.90\}$
E_2	$\{0.86, 0.88\}$	$\{0.82, 0.87\}$	$\{0.70, 0.72\}$
E_3	$\{0.68, 0.72, 0.74, 0.75\}$	$\{0.78, 0.80, 0.85, 0.88\}$	$\{0.82, 0.85, 0.87, 0.90\}$
E_0	$\{0.80, 0.85\}$	$\{0.82, 0.88\}$	$\{0.85, 0.90\}$

TABLE 7. Hesitant Fuzzy Decision Matrix

Needless to say that the above-mentioned problem is a pattern recognition problem where each pattern is described using a HFS. Here, we implement again Xu and Xia's similarity measures [22], and the six similarity measures listed in Table 1 to determine the degree of similarity between the three fleets E_1, E_2 and E_3 and the company E_0 whose results are shown in Table 8. Note that $\mathbb{S}_k$ $(k = 1, 2, ..., 6)$ are referred to as the proposed metrical T-norm-based similarity measures located in the k-row of Table 1.

Before any more progress could be made, we unify the HFEs in Table 7 according to the rule given in Remark 2.9 where the first entry $\{0.80, 0.85, 0.90\}$ in Table 7 is converted to $\{0.80, 0.85, 0.90, 0.90\}$. Here, we should notice that the unification procedure of HFEs in [27] is based on extending HFEs into multi-sets, and it of course differs from that explained above by Remark 2.9.

Zhang et al. [27] presented a number of distance measures for HFEs which produce similarity measures for two HFEs h_M and h_N by the rule $S(h_M, h_N) = 1 - d(h_M, h_N)$. Thus, if we consider the HFEs h_M and h_N such that $h_M^{\sigma(j)}$ and $h_N^{\sigma(j)}$ denote respectively the jth largest values in h_M and h_N, then Zhang et al.'s [27] HFE similarity measure is stated as:

$$S_{ftwg}(h_M, h_N) := 1 - d_{ftwg}(h_M, h_N) = 1 - (\frac{1}{lcm_x} \sum_{j=1}^{lcm_x} |h_M^{\sigma(j)} - h_N^{\sigma(j)}|^{\lambda})^{\frac{1}{\lambda}}, \quad \lambda > 0, \tag{26}$$

where the notation lcm_x in the above formula denotes the *least common multiple* of $\{l(h_M^{\sigma(j)}(x)), l(h_N^{\sigma(j)}(x)), ..., l(h_P^{\sigma(j)}(x))\}$.

In the case where $M = \{\langle x, h_M(x)\rangle : x \in X\}$ and $N = \{\langle x, h_N(x)\rangle : x \in X\}$ are the HFSs on X, then based on Zhang et al.'s [27] generalized hesitant weighted distance measure, we can define

$$\widetilde{S}_{ftwg}(M, N) := 1 - \sum_{i=1}^{n} \omega_i d_{ftwg}(h_M(x_i), h_N(x_i)) =$$

$$1 - \sum_{i=1}^{n} \omega_i (\frac{1}{lcm_x} \sum_{j=1}^{lcm_x} |h_M^{\sigma(j)}(x_i) - h_N^{\sigma(j)}(x_i)|^{\lambda})^{\frac{1}{\lambda}}, \quad \lambda > 0, \tag{27}$$

where ω_i $(i = 1, 2, ..., n)$ denote the weights of $x_i \in X$ $(i = 1, 2, ..., n)$.

Moreover, by the rule of $\widetilde{S}(M, N) := \sum_{i=1}^{n} \omega_i S(h_M(x_i), h_N(x_i))$, we can extend easily the similarity measures of HFEs to that of HFSs. In this regard, Xu and Xia's [22] similarity measures for HFEs given by (23)-(25) are extended to that for HFSs as $\widetilde{S}_{ghn}(M, N)$, $\widetilde{S}_{ghnh}(M, N)$ and $\widetilde{S}_{ghhnh}(M, N)$, respectively. Further, $\widetilde{\mathbb{S}}_k$ $(k = 1, 2, ..., 6)$ are referred to as the proposed metrical T-norm-based similarity measures for HFSs based on the measures located in the k-row of Table 1.

Now, according to the principle of the maximum degree of similarity between HFSs, the process of assigning E_0 to E_i $(i = 1, 2, 3)$ is described by

$$i^* = Arg \max_{1 \le i \le 3} \{\widetilde{S}(E_i, E_0)\}. \tag{28}$$

	$\widetilde{\mathbb{S}}_1$	$\widetilde{\mathbb{S}}_2$	$\widetilde{\mathbb{S}}_3$	$\widetilde{\mathbb{S}}_4$	$\widetilde{\mathbb{S}}_5$	$\widetilde{\mathbb{S}}_6$
$\widetilde{S}(E_1, E_0)$	**0.3205**	**0.9357**	**0.3288**	**0.3493**	**0.3493**	**0.3494**
$\widetilde{S}(E_2, E_0)$	0.3055	0.8792	0.3128	0.3482	0.3482	0.3486
$\widetilde{S}(E_3, E_0)$	0.2668	0.9315	0.2818	0.2992	0.2993	0.2995
i^*	1	1	1	1	1	1

TABLE 8. Values of Similarity Measures Between the Patterns E_i $(i = 1, 2, 3)$ and the Sample E_0

	$\widetilde{S}_{ghn}$	$\widetilde{S}_{ghnh}$	$\widetilde{S}_{ghhnh}$
$\widetilde{S}(E_1, E_0)$	**0.3340**	**0.3442**	**0.3303**
$\widetilde{S}(E_2, E_0)$	0.3246	0.3427	0.3227
$\widetilde{S}(E_3, E_0)$	0.2820	0.2935	0.2780
i^*	1	1	1

Table 9. Values of Similarity Measures Between the Patterns E_i $(i = 1, 2, 3)$ and the Sample E_0

	$\lambda = 1$	$\lambda = 2$	$\lambda = 3$	$\lambda = 4$	$\lambda = 5$	$\lambda = 6$
$\widetilde{S}_{ftwg}(E_1, E_0)$	**0.9538**	**0.9360**	**0.9260**	**0.9192**	**0.9142**	**0.9102**
$\widetilde{S}_{ftwg}(E_2, E_0)$	0.9330	0.9050	0.8875	0.8761	0.8681	0.8623
$\widetilde{S}_{ftwg}(E_3, E_0)$	0.9518	0.9357	0.9255	0.9187	0.9138	0.9101
i^*	1	1	1	1	1	1

Table 10. Values of Similarity Measures Between the Patterns E_i $(i = 1, 2, 3)$ and the Sample E_0

	$\lambda = 7$	$\lambda = 8$	$\lambda = 9$	$\lambda = 10$
$\widetilde{S}_{ftwg}(E_1, E_0)$	0.9069	0.9041	0.9017	0.8996
$\widetilde{S}_{ftwg}(E_2, E_0)$	0.8578	0.8542	0.8513	0.8489
$\widetilde{S}_{ftwg}(E_3, E_0)$	**0.9073**	**0.9050**	**0.9030**	**0.9014**
i^*	3	3	3	3

Table 11. Values of Similarity Measures Between the Patterns E_i $(i = 1, 2, 3)$ and the Sample E_0

From Tables 8, 9, 10 and 11, we can observe that no matter what kind of the proposed HFS metrical T-norm-based similarity measure is used, the unknown company E_0 has been classified to the fleet E_1, meanwhile, Zhang et al. [27]'s HFS similarity measure $\widetilde{S}_{ftwg}$ varies depending on which parameter λ is considered. In addition to the later disadvantage, a HFS should be extended into a multi-set before implementing Zhang et al. [27]'s similarity measure $\widetilde{S}_{ftwg}$, and it is obvious that such an extra unification of HFSs is costly.

5. **Conclusions**

This contribution gave the concept of metrical T-norm-based similarity measure for HFSs together with the concept of HFS metrical T-norm-based entropy measure. The proposed similarity measure equips the theory of HFSs with a possibility of comparing similarity without regarding what value the similarity measure returns. The comparative results of the proposed similarity measures versus a number of the existing measures clearly demonstrated the superior performance of the proposed measures.

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