A PRIMER ON FUZZY OPTIMIZATION MODELS AND METHODS

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Abstract. Fuzzy Linear Programming models and methods has been one of the most and well studied topics inside the broad area of Soft Computing. Its applications as well as practical realizations can be found in all the real world areas. In this paper a basic introduction to the main models and methods in fuzzy mathematical programming, with special emphasis on those developed by the authors, is presented. As a whole, Linear Programming problems with fuzzy costs, fuzzy constraints and fuzzy coefficients in the technological matrix are analyzed. Finally, future research and development lines are also pointed out by focusing on fuzzy sets based heuristic algorithms.

1. Introducction

It was in 1965, [32], that Professor Lofti A. Zadeh, an American of Iranian extract, first put forward the idea of the fuzzy set. This enabled a member to belong to a set in a gradual way, as opposed to absolutely, as stated by classical set theory. In other words, membership could be could be ascribed a value within the $[0, 1]$ interval instead of the $\{0, 1\}$ set. The applications and developments that have arisen from this simple concept have been such that it is nigh on impossible to calculate the volume of business they generate in today's world. The functioning of a whole range of products depends directly on the concept, from everyday appliances like the washing machine, the microwave oven, the camera ... to highly sophisticated systems like braking systems in trains, control of furnaces, etc.

The need for an optimal solution, or the best solution among those available, in a properly proposed problem is the rationale behind studying the theories and proposing methodologies appropriate to the scientific field in which the problem arises. More specifically, although still a very broad area, is an important type of problems, known as optimization problems, which are generally associated to finding the maximum or minimum value that a specific function can attain within a previously defined set. Everything that is relative to these problems can be classified within the doctrinal field of Mathematical Programming, which covers a huge range of situations, be these linear cases, non linear cases, randomness, single decision maker, several decision makers etc.

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Of all the models included in Mathematical Programming, the most and best studied is the single objective linear case (dealt with under Linear Programming), which has also turned out to have the most practical benefits. The methods and models of Linear Programming have useful applications in the areas of Engineering, Economics, Mathematics, Operative Research and Artificial Intelligence as well as in other disciplines related to optimization to a greater or lesser degree. They constitute a more than suitable theoretical basis on which to tackle highly complex situations in an elegant and efficient manner.

Although, as mentioned above, Linear Programming models and techniques have received the most attention, it is for this very reason - together with their elegance and efficiency, which make them so adaptable to new technological contexts - that they are key elements in the latest scientific developments, e.g. their incorporation and implementation in systems generating models of Decision Support Systems. Linear Programming is, therefore, firmly embedded in one of the most promising lines of development in Artificial Intelligence and even after more than half a century of use it remains at the leading edge of scientific progress.

Within the context of Decision Support Systems, and within the framework of Artificial Intelligence, the main aim is to obtain automatic systems which, starting from implementations which allow near human intelligence actions, are capable of acting as would a person on each occasion. This means that whatever the Linear Programming models we use under these conditions, will not in general be well known, established ones, since they will need to be redefined to meet the new context.

It is very well known that a real problem is usually approached that, while perfectly comprehensible, are difficult to represent effectively: "transport costs will be about 750 euros", "profit will be 30%", etc. When dealing with such figures, which clearly do not have to be of a probabilistic nature, we generally force the figures to take on values which we understand as being most representative of the real figures, e.g. 750 and 30%. Hence, we deal with what can be described as deformed problems, which can give solutions which may be optimal for the problem considered but which differ greatly from the true solution to the original problem, whose real values might have been 742 and 28.5%.

A correct representation of the information is, therefore, essential Decision Support Systems and Artificial Intelligence, as well as in other disciplines, since it is a guarantee for obtaining correct solutions and also because, depending on the fuzzy version we use, we may be dealing with different optimum concepts, and hence different optimizations.

Hereinafter, when we refer to imprecision we understand what is generally known as fuzziness, i.e. that linguistic vagueness that makes perfect sense to human beings despite a lack of any exact information ("I don't know how old he is, but he's young.") We choose this version of fuzziness since we understand that it is, in general, the closest and most suitable for the developments in Artificial Intelligence which concern us. What we normally do when defining something is take objects from the real world as linguistically labelled concepts in the referential domain under consideration. The imprecision of any linguistic label reflects the distance between the objects labelled and a referential point, which in each case will depend on the context. Hence, we can model human reasoning and communication in quite a suitable way.

In such a context, a Linear Programming (LP) problem can often be formulated much more specifically as:

$$Max\ \{z = cx\ /\ Ax \leq b, x \geq 0\}$$

where A is a matrix of real numbers with a dimension $m \times n$, b is a vector in $\mathbb{R}^m$ and c is a costs vector in $\mathbb{R}^n$.

From the earlier considerations, we can now suppose that the decider expresses, knows or formulates the data of the problem in a fuzzy way which is nevertheless perfectly clear to the decision taker: "performance this year will be better than last year"; "there will be a high number of man hours"; "gross salary stands at around 20,000 euros" etc. It is in such an environment of optimization that Fuzzy Linear Programming (FLP) is born.

While FLP has its theoretical precedents in 1970 in the great work on Decision Theory by R. Bellman and L.A. Zadeh [1], FLP problems were formally born in 1974, the year in which two separate papers [25] and [35] propose the same model to deal with LP problems, i.e. that the set of constraints be given by a fuzzy set. Despite the coincidence, the works approached the solution from different points of view, and, thus, employed different methods, which led to a solution constituted by a single point, which can, therefore, be considered as outside the fuzzy context of the calculation. Later on, it was demonstrated that these methods were particular cases of a more general method which allowed a context-dependent fuzzy solution, and which encompassed the solutions put forward in the studies cited [27].

Specifically, the central problem in FLP lies in solving an LP problem within the set of fuzzy constraints.

$$\begin{array}{ll} Max & z = cx \\ s.t.: & \\ & Ax \leq_f b \\ & x \geq 0 \end{array}$$

where it is supposed that the decision maker can accept moderate violations in the fulfilment of the constraints, with these violations being measured through certain membership functions

$$\mu_i : \mathbb{R} \to [0,1],\ \ i = 1, \ldots, m$$

which the decision maker himself establishes.

The lines of research which have been followed from this initial approach are numerous but they can be grouped under the following sections:

a) Extensions of the previous model for more complex problems. This has been the case particularly in multi-objective problems, although also worthy of mention are the studies in other areas like Stochastic or Fractional Programming.
b) Methods for solving different problems.
c) Applications within specific domains (transport, games, hydraulic policies, agriculture, reasoning from propositional knowledge, etc.).

Initially, we will tackle the basic theoretical elements necessary to this article. While not descending to the trivial level, we present the most elementary ideas on fuzzy sets and numbers. Below, we put forward (within the fuzzy context) the most typical problems and methods in FLP. In order to finalize we propose an important and very promising future research line, as it is that of Fuzzy Sets based Heuristic Algorithms to solve optimization problems.

2. **Basic Concepts**

One basic concept is that of the fuzzy number. From the point of view of a fuzzy number's being a fuzzy set in RR, it can be stated that the notion of a fuzzy number appears in 1965 with the appearance of L.A. Zadeh's famous paper [32].

Nevertheless, fuzzy numbers really appear on the scene around 1978, with the papers by S. Nahmias on fuzzy variables, and D. Dubois and H. Prade on handling imprecise quantities. Since then, the study of possible definitions of fuzzy numbers and, in particular, how to manage and compare them, has aroused a lot of interest within the field of fuzzy sets, [31].

This section introduces the elementary notions and operations of fuzzy sets leading to the concept of fuzzy number. Once these have been established, the remaining part of this section is devoted to the comparison of two fuzzy numbers. This is a complex problem since, given the imprecise nature of the quantities considered, e.g. A and B, it cannot be guaranteed a priori that $A \leq B$ or that $B \leq A$. Instead, these properties will be verified simultaneously and with certain degrees of fulfilment. This means that there are many ways of comparing two fuzzy numbers, which in the specialist literature has been developed through the so-called comparison indices.

2.1. **Introduction to the Fuzzy Set Concept.**

Let X be a set, whose elements are we will denote by x, and A a subset of X. The membership of one element x of X to the subset A is given by the characteristic function:

$$\mu_A(x) = \begin{cases} 1 & \text{iff } \; x \in A \\ 0 & \text{iff } \; x \notin A \end{cases}$$

where $\{0, 1\}$ is the so-called valuation set.

If the valuation set is the real interval $[0, 1]$, A is called a fuzzy set [32] and μ_A measures the degree of membership of element x in A. A is characterised by the the set of pairs $\{(x, \mu_A(x)),\ x \in X\}$.

Two fuzzy sets, A and B are considered equal iff: $\forall x \in X,\ \mu_A(x) = \mu_B(x)$.

Definition 2.1. [32] Given a fuzzy set $A = \{(x, \mu_A(x))\}$, its support is defined as the ordinary set $Sop(A) = \{x \in X \ / \ \mu_A(x) > 0\}$.

Definition 2.2. [32] Given a fuzzy set A, we give the name α-cut of that set to the ordinary set $A_\alpha = \{x \in X \ / \ \mu_A(x) \geq \alpha\}$ con $\alpha \in [0, 1]$.

It is clearly seen that the sets A_α, $\alpha \in [0, 1]$ constitute a decreasing succession. If $\alpha_1 \geq \alpha_2 \Leftrightarrow A_{\alpha_1} \subseteq A_{\alpha_2},\ \alpha_1, \alpha_2 \in [0, 1]$.

Theorem 2.3. *(Representation Theorem) If A is a fuzzy set and A_α its α-cuts, $\alpha \in [0,1]$, it is verified that*

$$A = \bigcup_{\alpha \in [0,1]} \alpha A_\alpha$$

taking this formal notational as the equality between the membership functions of both sets. If $\mu_{A_\alpha}(x)$ denote the characteristic function of A, a particular case of the membership function:

$$\mu_{A_\alpha}(x) = \begin{cases} 1 & \textit{iff} \;\; x \in A_\alpha \\ 0 & \textit{otherwise} \end{cases}$$

membership function of the fuzzy set A can be expressed in terms of the characteristic functions of its α-cuts, according to the formula

$$\mu_A(x) = \sup_{\alpha \in [0,1]} \min(\alpha, \mu_{A_\alpha}(x))$$

Definition 2.4. [32] A fuzzy set is convex iff its α-cuts are convex.

A definition equivalent to convexity is that A is convex iff $\forall x_1, x_2 \in X$, $\forall \lambda \in [0,1]$, $\mu_A(\lambda x_1 + (1-\lambda)x_2) \geq \min(\mu_A(x_1), \mu_A(x_2))$.

Definition 2.5. The height of a fuzzy set $hgt(A) = \sup_{x \in X} \mu_A(x)$.

Definition 2.6. A fuzzy set is said to be normalized iff $\exists x \in X$ in which $\mu_A(x) = 1$.

2.2. **Fuzzy Numbers.**

Definition 2.7. [12] A fuzzy number A is a μ_A set of the real straight, it is convex and normalized such that

a) $\exists x_0 \in \mathbb{R} \;/\; \mu_A(x_0) = 1$, which is generally called mode, and
b) μ_A is in parts continuous.

Any fuzzy number is therefore characterised by a membership function $\mu_A : \mathbb{R} \to [0,1]$ and any function like the above gives a fuzzy number where $\forall x \in \mathbb{R}$, $\mu_A(x)$ is the degree of membership of x to the fuzzy number A.

We will denote by $F(\mathbb{R})$ the set of membership functions on $\mathbb{R}$. Thus, when talking about the fuzzy number we can refer both to the element $A \in F(\mathbb{R})$ and to $\mu_A \in F(\mathbb{R})$.

A fuzzy number A is said to be of the type $\mathcal{L}-\mathcal{R}$, if and only if its membership function μ_A is of the form

$$\mu_A(x) = \begin{cases} \mathcal{L}\left[\frac{(m-x)}{\alpha}\right] & \text{for } \; x \leq m, \;\; \alpha > 0 \\ \\ \mathcal{R}\left[\frac{(x-m)}{\beta}\right] & \text{for } \; x \geq m, \;\; \beta > 0 \end{cases}$$

where m is the mode of A and α (β) is the width on the left (right), $\mathcal{L}$ and $\mathcal{R}$ represent a function on the left or right of m, $\mathcal{L}$ is non decreasing and $\mathcal{R}$ is not increasing. We will abbreviate the fuzzy number A by $A = (m-\alpha, m, m+\beta)_{\mathcal{L}R}$.

Definition 2.8. [13] A plane fuzzy number is an A fuzzy number such that

$$\exists(m_1, m_2) \in \mathbb{R},\ m_1 < m_2 \text{ and } \mu_A(x) = 1,\ \forall x \in [m_1, m_2]$$

A plane fuzzy number can model a fuzzy interval. An A plane fuzzy number of type $\mathcal{L} - \mathcal{R}$ is defined as

$$(1) \qquad \mu_A(x) = \begin{cases} \mathcal{L}\left[\frac{(m_1 - x)}{\alpha}\right] & \text{for } \leq m_1,\ \alpha > 0 \\ \mathcal{R}\left[\frac{(x - m_2)}{\beta}\right] & \text{for } x \geq m_2,\ \beta > 0 \\ 1 & \text{for } m_1 \leq x \leq m_2 \\ 0 & \text{otherwise} \end{cases}$$

This will be more briefly denoted by $(m_1 - \alpha, m_1, m_2, m_2 + \beta)_{\mathcal{LR}}$.

It is clear that depending on the $\mathcal{L}$ and $\mathcal{R}$ functions, we will obtain different types of fuzzy numbers.

We will consider numbers as fuzzy, plane, linear and normalised, those whose analytical membership function is as follows.

A plane fuzzy number, which we will denote by $u_j^f = (r_j, \underline{u}_j, \overline{u}_j, R_j)$ will have the membership function

$$(2) \qquad \forall v \in \mathbb{R},\ \mu_{u_j^f}(v) = \begin{cases} \frac{(v - r_j)}{(\underline{u}_j - r_j)} & \text{if } r_j \leq v \leq \underline{u}_j \\ \frac{(R_j - v)}{(R_j - \overline{u}_j)} & \text{if } \overline{u}_j \leq v \leq R_j \\ 1 & \text{if } \underline{u}_j \leq v \leq \overline{u}_j \\ 0 & \text{otherwise} \end{cases}$$

From now on we will frequently use fuzzy numbers expressed as linear combinations $y^f = \sum_j u_j^f x_j$ with $x_j \in \mathbb{R},\ j = 1, \ldots, n$.

In [26] we find the membership function of those numbers, which we express below.

Proposition 2.9. *If $y^f = \sum_j u_j^f x_j = u^f x$ is a linear expression in which the u_j^f, $j = 1, \ldots, n$, are fuzzy numbers linear membership functions given by $u_j^f = (r_j, \underline{u}_j, \overline{u}_j, R_j)$ and $x_j \geq 0$, $j = 1, \ldots, n$, then the membership function of y^f is*

$$\mu(z) = \begin{cases} \frac{(z - rx)}{(\underline{u}x - rx)} & \textit{if } x > 0 \textit{ and } rx \leq z \leq \underline{u}x \\ \frac{(Rx - z)}{(Rx - \overline{u}x)} & \textit{if } x > 0 \textit{ and } \overline{u}x \leq z \leq Rx \\ 1 & \textit{if } \underline{u}x \leq z \leq \overline{u}x \\ 0 & \textit{otherwise} \end{cases}$$

where $r = (r_1, \ldots, r_n)$, $\underline{u} = (\underline{u}_1, \ldots, \underline{u}_n)$, $\overline{u} = (\overline{u}_1, \ldots, \overline{u}_n)$ and $R = (R_1, \ldots, R_n)$.

2.3. **Methods for Comparing Fuzzy Numbers.**

A constant problem over the last 15 years has been that of the distribution of imprecise quantities, and hence the comparison of fuzzy numbers. The many and varied approaches to the problem mean that a wide range of methods exist to make the comparison in question. An excellent collection of techniques, methods and approaches can be found in [34] and [31].

We will use the ways of comparing fuzzy numbers exclusively to analyse the repercussion of using various methods of comparison in a Fuzzy Linear Programming problem. Thus, it is not our aim here to review all the possible ways of comparing.

The solution to the problem can be shortened in either of the following ways, depending on whether the method used is based on the definition of an ordering function or on the comparison of alternatives.

2.3.1. **Methods Based on the Definition of an Ordering Function.**

We will consider $A,\ B \in F(\mathbb{R})$. A simple method to compare these lies in the definition of a function $g : F(\mathbb{R}) \rightarrow \mathbb{R}$. If the function $g(\cdot)$ is known, then

$$\begin{gathered} g(A) < g(B) \Leftrightarrow \text{ A is less than B} \\ g(A) > g(B) \Leftrightarrow \text{ A is greater than B} \\ g(A) = g(B) \Leftrightarrow \text{ A is equal to B} \end{gathered}$$

g is usually called a linear ordering function if

1) $\forall A, B \in F(\mathbb{R}),\ g(A + B) = g(A) + g(B)$
2) $\forall r \in \mathbb{R},\ r > 0,\ g(rA) = rg(A),\ \forall A \in F(\mathbb{R})$

In this case, the indices can be classified according to whether the ordering function is linear or not.

2.3.2. **Methods Based on the Comparison of Alternatives.**

These methods consist of obtaining the fuzzy set of the optimal alternatives:

$$O^f = \{i, \mu_{O^f}(i)\},\ \ \mu_{O^f}(i) = \mu_{O^f}(A^i),\ \ A^i \in F(\mathbb{R})$$

where $\mu_{O^f}(i)$ represents the degree to which the ith alternative can be considered the best.

Finally, we underline in spite of the huge wealth of methods for comparing fuzzy numbers, as yet few indices have been studied since it is perfectly justifiable for each human decision taker to use their own method of comparison independently of any method described in the literature. A detailed study in this respect can be found in [22] where an artificial neuronal network is used which learns the ordering criteria of each decision taker considered.

3. **Fuzzy Linear Programming: Methods and Models**

An LP problem is generally set out as

$$Max\ \{z = cx/Ax \leq b, x \geq 0\}$$

where A is matrix $m \times n$ of real numbers, $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$.

Obviously, it is assumed here that the decision taker has exact information on the elements intervening in the problem. Even were this the case, the decision taker usually finds it more convenient to express his knowledge in linguistic terms, i.e. through conventional linguistic labels [33], rather than by using high precision numerical data. Thus, it makes perfect sense to talk about optimization problems from a vague predicate approach as it is understood that this vagueness arises from the way we use to express the decision taker's knowledge and not from any random event. In short, it is supposed that the imprecision of the data defining the problem is fuzzy.

The first case of optimization problems with fuzzy approach appeared in the literature more than three decades ago [1], in an article which put forward the now classical, key concepts of constraint, objective and fuzzy optimal decision.

As with LP in conventional optimization, so have FLP methods been the subject of most study in the fuzzy context. While not exhaustive, there are three main types of FLP problem, depending on the imprecision established in the constraints, on the coefficients of the technological matrix or on the costs which define the objective function. The rest of this section is devoted to each of these.

Models and methods to solve these problems abound in the literature, especially for the case in which f and g_i, $i \in M$, are linear functions [9, 10, 16, 37]. In some cases precise solutions are obtained, while in others these are fuzzy and more in line with the approach to the problem. The latter offer a set of good alternatives and encompass the more precise solutions obtained using other methods. Finally though, it is the decision taker who must choose.

While we find many FLP models in the literature [14, 6, 8, 17, 9, 20, 19] ..., the majority only suppose vagueness for some of the elements described in the model. At the end of this section, a general FLP model is presented in which all the elements are fuzzy. To this end, the most important problems in fuzzy linear programming will be presented along with the general FLP method, [5]. From the said model it is easy to derive each particular case of the FLP problem, and these are in agreement with its characteristics.

3.1. **Linear Programming with Fuzzy Constraints.**

We consider the case in which the decision taker assumes that there is a certain tolerance in the fulfilment of constraints i.e. a certain degree of violation is allowed and this is established by the decision maker himself, [5]. This supposition can be represented for each constraint as follows:

$$a_i x \leq_f b_i, \; i \in M = \{1, 2, \ldots, m\}$$

and this can be modelled using a membership function

$$\mu_i : \mathbb{R} \to [0,1] \; / \; \mu_i(a_i x) = \begin{cases} 1 & \text{if } \; a_i x \leq b_i \\ f_i(a_i x) & \text{if } \; b_i \leq a_i x \leq b_i + t_i \\ 0 & \text{if } \; a_i x \geq b_i + t_i \end{cases}$$

These functions express that the decision taker is tolerating violations in each constraint up to a value of $b_i + t_i,\ i \in M$. Functions f_i are assumed to ne non decreasing and continuous for these constraints.

Function μ_i is defined for each $x \in X$ and it gives the degree of fulfilment for each i-th constraint for $x \in X$.

The associated problem is represented as follows

$$\begin{array}{ll} Max & z = cx \\ s.t.: & \\ & Ax \leq_f b \\ & x \geq 0 \end{array} \tag{3}$$

wheree $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, A is a matrix $m \times n$ of real numbers.

Although the origin of (3) is found in [1], this problem was developed in [25] and [36], where additional hypotheses on the nature of the objective function were considered, although these do not concern us here.

Three approaches - [25, 36, 27] - may be considered to solve (3). In particular, and using the Representation Theorem for fuzzy sets, [27] shows how to find a fuzzy solution for (3) through the auxiliary parametric PL problem

$$\begin{array}{ll} Max & z = cx \\ s.t.: & \\ & Ax \leq g(\alpha) \\ & x \geq 0,\ \alpha \in [0,1] \end{array}$$

where $g(\alpha) \in \mathbb{R}^m$ is a column vector defined by the inverse functions of the f_i, $i \in M$. The linearity and dimension of (3) are clearly maintained in the model.

The solutions proposed in [25] and [36] arise from the fuzzy solution proposed in [27] for particular values of the parameter $\alpha \in [0,1]$.

Supposing the linearity of the f_i, we obtain that the auxiliary model which solves (3) is

$$\begin{array}{ll} Max & z = cx \\ s.t.: & \\ & Ax \leq b + t(1-\alpha) \\ & x \geq 0,\ \alpha \in [0,1] \end{array} \tag{4}$$

with $t = (t_1, \ldots, t_m) \in \mathbb{R}^m$.

If we denote $S(\alpha) = \{x \in \mathbb{R}^n \ / \ z(x) = \max\ cx,\ x \in X(\alpha)\}$ with $X(\alpha) = \{x \in \mathbb{R}^n \ / \ Ax \leq b + t(1-\alpha),\ \alpha \in [0,1]\}$, we define a fuzzy solution (3) as,

Definition 3.1. The fuzzy solution to (3) is a fuzzy set with the membership function

$$\lambda(x) = \begin{cases} \sup\limits_{x \in S(\alpha)} \alpha & \text{if } x \in \bigcup\limits_{\alpha} S(\alpha) \\ 0 & \text{otherwise} \end{cases}$$

We consider (3) and the imprecision of the constraints represented by non lineal membership functions

$$\mu_i^{'}(x) = \begin{cases} 1 & \text{if } a_i x \le b_i \\ f_i'(a_i x) & \text{if } b_i \le a_i x \le b_i + t_i \\ 0 & \text{if } a_i x \ge b_i + t_i \end{cases}$$

where the function $f_i'(\cdot)$ is supposed strictly decreasing and continuous, $f_i'(b_i) = 1$ and $f_i'(b_i + t_i) = 0$.

If we use the approach proposed in [27] and we apply a similar discussion to the one above for the linear case, then the optimal fuzzy solution for (3) can be obtained from the optimal parametric solution of the problem

$$\begin{array}{ll} Max & z = cx \\ s.t.: & \\ & Ax \le g'(\alpha) \\ & x \ge 0,\ \alpha \in [0,1] \end{array}$$

where $g'(\alpha) = f'^{-1}(\alpha)$, $\forall \alpha \in [0, 1]$.

A relation between the solutions obtained in the linear and non linear case is shown in [11]. The subsequent results are shown in [11].

Proposition 3.2. *Let $[a, b]$ be a real interval and $f : [a, b] \to [0, 1]$ is continuous, linear and strictly decreasing with $f(a) = 1$ and $f(b) = 0$. For any other strictly decreasing continuous function $f' : [a, b] \to [0, 1]$, such that $f'(a) = 1$ and $f(b) = 0$, there exists a function $r : [0, 1] \to [0, 1]$ such that $r(\cdot) \circ f(\cdot) = f'(\cdot)$.*

Proposition 3.3. *We consider the FLP problem (3). We denote $x(\cdot)$ and $x'(\cdot)$ to the optimal fuzzy solutions for this problem using linear and non linear membership functions for the imprecision of the constraints. Thus $x'(\alpha) = x(r^{-1}(\alpha))$, where $r(\cdot)$ is the solution obtained for the previous proposition.*

With these results, the value of the objective function will be $z'(\alpha) = cx'(\alpha) = cx(r^{-1}(\alpha))$.

This demonstrates that by solving an LP problem with fuzzy constraints modelled by linear membership functions, we can obtain the fuzzy solution to the same problem modelled by non linear membership functions. For the case in which the LP problem has fuzzy constraints modelled piecewise defined membership functions, we obtain a similar result, [11]. Thus the generality is not lost if we always suppose the fuzzy constraint problem as having linear membership functions.

3.2. **Linear Programming with Fuzzy Costs.**

In this case, the decision taker does not know the exact values of the coefficients c. The situation is represented by the following FLP problem, [5].

$$\begin{array}{ll} Max & z = c^f x \\ s.t.: & \\ & Ax \le b \\ & x \ge 0 \end{array} \tag{5}$$

with $c^f \in (F(\mathbb{R}))^n$ and supposing membership functions of the following form
(6)

$$\mu_j : \mathbb{R} \to [0,1],\ j \in N = \{1,\dots,n\}\ \ /\ \ \mu_j(v) = \begin{cases} 0 & \text{if } v \le r_j \text{ or } v \ge R_j \\ h_j(v) & \text{if } r_j \le v \le \underline{c}_j \\ g_j(v) & \text{if } \overline{c}_j \le v \le R_j \\ 1 & \text{if } \underline{c}_j \le v \le \overline{c}_j \end{cases}$$

where $h_j(\cdot)$ and $g_j(\cdot)$ are strictly continuous increasing and decreasing functions, respectively, such that $h_j(\underline{c}_j) = g_j(\overline{c}_j) = 1$, $\forall j \in N$.

Although there exists a wide range of h_j and g_j functions (linear, exponential, logarithmic, parabolic, concave and convex, etc.), fuzzy costs are usually considered as plane fuzzy numbers with linear functions $h_j(\cdot)$ and $g_j(\cdot)$. Hence, for the number $(r_j, \underline{c}_j, \overline{c}_j, R_j)$, these functions will be given by:

$$h_j(v) = \begin{cases} \frac{v-r_j}{\underline{c}_j - r_j} & r_j \le v \le \underline{c}_j \\ 0 & \text{otherwise} \end{cases},\ g_j(u) = \begin{cases} \frac{R_j-u}{R_j-\overline{c}_j} & \overline{c}_j \le u \le R_j \\ 0 & \text{otherwise} \end{cases}$$

There are various approaches [8, 23, 26] to solve (5). It is demonstrated in [10] that the method proposed in [8] gives a formal context to find the solution of (5) and encompasses the methods proposed in [23, 26, 20].

The fuzzy solution to this problem proposed in [8] can be obtained from the solution to the following multiobjective parametric problem

$$\begin{array}{ll} Max & z = [c^1x, c^2x, \dots, c^{2^n}x] \\ s.t.: & \\ & Ax \le b,\ x \le 0 \\ & c_j^k \in \{h_j^{-1}(1-\alpha), g_j^{-1}(1-\alpha)\} \\ & \alpha \in [0,1],\ k = 1,\dots,2^n,\ j \in N \end{array}$$

3.3. **Linear Programming with Fuzzy Numbers in the Technological Matrix.**

Now we consider that the coefficients in the technological matrix and the coefficients of the right hand size are represented by fuzzy numbers, with the costs that define the objective function being real, [5].

This type of FLP problem is set out in the following terms

$$\begin{array}{ll} Max & z = cx \\ s.t.: & \\ & a_i^f x \le_f b_i^f,\ i \in M \\ & x \ge 0 \end{array} \tag{7}$$

where for each $i \in M$, $a_i^f = (a_{i1}^f, \dots, a_{in}^f)$, $a_{ij}^f \in F(\mathbb{R})$, $j \in N$, $b_i^f \in F(\mathbb{R})$, $x \in X = \{x \in \mathbb{R}^n\ /\ a_i^f x \le_f b_i^f,\ i \in M,\ x \ge 0\}$ and $c \in \mathbb{R}^n$.

An almost identical version of (7) was the starting point for this type of models (a similar problem was put forward in [26] but supposing imprecision in the objective as well). In order to solve (7) with a fuzzy solution rather than a precise one, as in (3), we can suppose that violations in its constraint coupling are admitted up to a maximum amplitude of a value t_i^f, $i \in M$, [9]. Note that unlike (3), t_i^f has to be a fuzzy number on account of the nature of the coefficients taking part in each constraint.

From this viewpoint a method is proposed in [9] to solve the general model (7). The approach is based on the substitution of the set of constraints of (7) by the following convex, fuzzy set:

$$a_i^f x \leq_g b_i^f + t_i^f(1-\alpha),\ i \in M,\ \alpha \in [0,1]$$

where $a_i^f = (a_{i1}^f, \ldots, a_{in}^f)$, t_i^f is a number fixed by the decision taker which sets the violation tolerance in the constraint, and $\leq_g$ is a relation between these numbers.

Thus, problem (7) takes the following form

$$\begin{array}{ll} Max & z = cx \\ s.t.: & \\ & a_i^f x \leq_g b_i^f + t_i^f(1-\alpha),\ i \in M \\ & x \geq 0,\ \alpha \in [0,1] \end{array}$$

The solution to the original problem is obtained in [7] by particularization (in the auxiliary problem) for each different method of comparison of fuzzy numbers.

3.4. **A General Model for Fuzzy Linear Programming.**

A general FLP model, [5], which encompasses all the above cases, is a problem of the type:

$$\begin{array}{ll} Max & z = \sum_{j=1}^{n} c_j^f x_j \\ s.t.: & \\ & \sum_{j=1}^{n} a_{ij}^f x_j \leq_f b_i^f \\ & x_j \geq 0,\ i \in M,\ j \in N \end{array} \tag{8}$$

where the fuzzy elements considered are given by:

a) For each cost $\exists \mu_j \in F(\mathbb{R})$ such that $\mu_j : \mathbb{R} \to [0,1],\ j \in N$, which define the fuzzy costs vector.
b) For each row $\exists \mu_i \in F(\mathbb{R})$ such that $\mu_i : \mathbb{R} \to [0,1],\ i \in M$, which define the fuzzy number on the right hand side.
c) For each $i \in M$ and $j \in N$ $\exists \mu_{ij} \in F(\mathbb{R})$ such that $\mu_{ij} : \mathbb{R} \to [0,1]$, which define the fuzzy numbers in the technological matrix.
d) For each row $\exists \mu^i \in F[F(\mathbb{R})]$ such that $\mu^i : F(\mathbb{R}) \to [0,1]$, $i \in M$ the degree of fulfilment for the fuzzy number $a_{i1}^f x_1 + a_{i2}^f x_2 + \ldots a_{in}^f x_n,\ i \in M$ with respect to the i-th constraint, i.e. the similarity of these numbers and the corresponding b_i^f with respect to the i-th constraint.

One method, [4], for solving the general model (8) consists of substituting the constraints set of (8) by a convex fuzzy set.

Let g be an ordering function of fuzzy numbers, and function $\psi : F(\mathbb{R}) \times F(\mathbb{R}) \to F(\mathbb{R})$ such that

$$\psi(a_i^f x, b_i^f) = \begin{cases} t_i^f & \text{if } a_i^f x \leq_g b_i^f \\ t_i^f \ominus a_i^f x \oplus b_i^f & \text{if } b_i^f \leq_g a_i^f x \leq_g b_i^f \oplus t_i^f \\ 0 & \text{if } a_i^f x \leq_g b_i^f \oplus t_i^f \end{cases}$$

with $t_i^f \in F(\mathbb{R})$ such that its support is included in $\mathbb{R}^+$ and $\leq_g$ is a relation which measures the fact that $A \leq_g B$, $\forall A, B \in F(\mathbb{R})$ and $\ominus$ and $\oplus$ are the most usual operations between fuzzy numbers.

Definition 3.4. The membership function associated to the fuzzy constraint $a_i^f x \leq_f b_i^f$, with t_i^f being a fuzzy number which gives maximum violation in the verification of the i-th constraint, is the following

$$\mu^i : F(\mathbb{R}) \to [0,1] \ / \ \mu^i(a_i^f x, b_i^f) = \frac{g(\psi(a_i^f x, b_i^f))}{g(t_i^f)} \tag{9}$$

where g is a fuzzy number ordering function.

If we consider problem (8), $\leq_f$ with membership functions (9), then using the fuzzy number representation theorem, we obtain that

$$\mu^i(a_i^f x, b_i^f) \geq \alpha \Leftrightarrow \frac{g(f(a_i^f x, b_i^f))}{g(t_i^f)} \geq \alpha \Leftrightarrow \frac{g(t_i^f \ominus a_i^f x \oplus b_i^f))}{g(t_i^f)} \geq \alpha \Leftrightarrow$$

$$g(t_i^f) - g(a_i^f x) + g(b_i^f) \geq g(t_i^f)\alpha \Leftrightarrow g(a_i^f x) \leq g(b_i^f \oplus t_i^f(1-\alpha)) \Leftrightarrow$$

$$a_i^f x \leq_g b_i^f + t_i^f(1-\alpha)$$

where $\leq_g$ is the relation corresponding to g.

Hence, an auxiliary problem to solve (8) is the following:

$$\begin{array}{ll} Max & \sum_{j=1}^{n} c_j^f x_j \\ s.t.: & \\ & \sum_{j=1}^{n} a_{ij}^f x_j \leq_g b_i^f + t_i^f(1-\alpha), \ i \in M \\ & x \geq 0, \ \alpha \in [0,1], \ j \in N \end{array} \tag{10}$$

If in problem (8) there were no fuzzy numbers in its formulation, but only fuzzy constraints, this approach would coincide with the corresponding model

$$\begin{array}{ll} Max & z = cx \\ s.t.: & \\ & Ax \leq b + t(1-\alpha) \\ & x \geq 0, \ \alpha \in [0,1] \end{array}$$

In other words, in the case of a fuzzy constraint $a_i x \leq_f b_i$, the membership function associated to this constraint will be of the form

$$\mu^i : F(\mathbb{R}) \to [0,1] \;/\; \mu^i(a_i x, b_i) = \frac{g(f(a_i x, b_i))}{g(t_i)}$$

where g is the classical order in $\mathbb{R}$ and t_i is the maximum violation in the fulfilment of the i-th constraint ($t_i \in \mathbb{R}$).

$$\mu^i(a_i x, b_i) = \begin{cases} 1 & a_i x \leq b_i \\ (1 - \frac{a_i x - b_i}{t_i}) & b_i \leq a_i x \leq b_i + t_i \\ 0 & a_i x \geq b_i + t_i \end{cases}$$

To solve this problem we can use the different comparison relations of fuzzy numbers, both in the constraints and in the objective, or we can use comparison relations in the constraints and a-cuts in the objective. This will lead to our obtaining various conventional models, which will allow for a properly fuzzy solution.

4. **Fuzzy Sets Based Heuristic**

Optimization methods based on fuzzy logic do not end with FLP. Indeed, the easy solving of real problems of ever greater dimensions, thanks to the greater power and lower cost of computers, the impossibility of obtaining exact solutions in all cases and the need to provide answers for a host of practical cases (sequencing problems, design of routes, location, etc.) have all led to the growing use of heuristic type algorithms as valuable tools which can provide answers which exact algorithms are unable to provide. Thus, in recent years a huge and varied range of techniques has appeared which has sprung from the idea that satisfaction is better than optimization or, in other words, that rather than being unable to find the optimal solution to a problem it is preferable to provide a solution that satisfies a user's previously described needs. And these techniques have proved to be extraordinarily effective. Examples of these techniques are the algorithms Tabu Search, Simulated Annealing, GRASP ("Greedy Randomized Adaptive Search Procedure"), Genetics, or the more recent, Memetics, VNS (Variable Neighbourhood Search), Ant Colonies, Scatter Search, Constraints Programming. In short, there is a wealth of interest in this field along with a lack of a minimal theoretical framework within which to set, relate and compare these algorithms.

It may be stated that in the majority of cases these heuristics are inspired by some real model in nature, society, physics..., and have been used to produce theoretical models which meet the circumstances under consideration. Thus, solutions have been found for cases that until very recently could not be dealt with using traditional techniques. However, the solutions have not been optimal in the vast majority of cases. They have been "near-optimal" solutions, which have frequently been obtained applying criteria that differ from the classical "achieve the best value of the objective function" since they take into account subjective characteristics established by the decision taker.

As has been made clear throughout this paper, when we speak about human associated subjectivity, or even about nearness to an optimal value, the best way of modelling these situations is through fuzzy sets (Soft Computing).

It is assumed generally that in the first level, the principals constituent of Soft Computing are Approximate Reasoning and the Functional Approximation/Randomised Search. Then in a second level we can find Probabilistic Models, Fuzzy Sets and Systems, Heuristics and Meta-heuristics and Neural Networks. On the one hand, it is evident that since the famous "Fuzzy Boom" of the 90s, Fuzzy Sets and Systems have settled permanently in all the areas of R+D+I. Their applications can be found in several the fields of our daily life, and they are a subject of study in different educational levels. On the other hand, there is no doubt that thanks to the technological potential that we have nowadays, we are witness to discoveries that were unpredictable just only a decade ago.

Computers in particular, do efficiently tasks that seemed to be very laborious, when not impossible, just a short time ago, allowing us to approach problems of great complexity, both in comprehension as well as in dimension, in a great variety of fields. In spite of the huge success achieved by the Fuzzy Sets and Systems, of the important progress produced by the heuristics in a practical way, and of the close relationship between both methodologies, if we set apart the area of Genetic Algorithms, or more generally of Evolutionary Algorithms, not much work has been conducted in the development of Fuzzy Sets-based Heuristics.

To bridge this gap, in the following we will give an overview on the practical applications that fuzzy sets based heuristics algorithms have, by describing three particular fuzzy sets based heuristics for solving optimisation problems. First, and as introduction, we will focus on an already explored approach which seeks the possibility of using fuzzy rules as termination criteria in the algorithms. Second, we will review the basic ideas of a fuzzy sets-based heuristic algorithm that looks for solutions qualified in terms of fuzzy valuations, and whose behaviour is adapted as a response to the state of the search. Finally, we describe a new methodology proposal based on using fuzzy rules to coordinate a set of concurrent heuristics.

4.1. **Fuzzy Stopping Rules for Terminating Algorithms.**

The key point in this section is that FLP methodologies may help to find solutions for problems in which to find an optimum solution is not easy. As it is well known there are a lot of NP problems which cannot effectively be solved in all cases. In these problems the decision-maker must usually accept approximate solutions instead of optimum ones. The aim here is to show how FLP can help classical Mathematical Programming models by providing fuzzy solutions that may be used by the decision-maker as help to quickly obtain a good enough solution for these problems.

Let's justify this fact. An algorithm for solving a general optimisation problem is an iterative process that produces a sequence of points according to a prescribed set of instructions, together with a termination criterion. Usually we are interested in algorithms that generate a sequence that converges to an overall, optimum solution. However, because of the difficulties in the problem, we may have to be satisfied with less favourable solutions. Then the iterative procedure may stop either 1) if a point belonging to a prefixed set (the solution set) is reached, or 2) if some prefixed condition for satisfaction is verified.

But, the conditions for satisfaction are not to be meant as universal ones. They depend on factors such as the decision-maker, the features of the problem, the nature of the information available, ... In any case, assuming that a solution set is prefixed, the algorithm will stop if a point in that solution set is reached. Frequently, however, the convergence to a point in the solution set is not easy because, for example, of the existence of local optimum points, and hence we must redefine some rules for terminating the iterative procedure.

Roughly speaking, the possible criteria to be taken into account for terminating the algorithms are nothing but control rules. Thus these rules could be associated to the two above points: the solution set, and the criteria for terminating the algorithm. As it is clear, fuzziness can be introduced in both points, not assuming it as inherent in the problem, but as help for obtaining, in a more effective way, some solution for satisfying the decision-maker's wishes. This is meant so that the decision-maker might be more comfortable when obtaining a solution expressed in terms of satisfaction instead of optimisation, as is the case when fuzzy control rules are applied to the control processes. Therefore, and in the particular case of optimisation problems [24], it makes sense to consider fuzziness

a) In the Solution Set, i.e., there is a membership function giving the degree with which a point belongs to that set, and
b) On the conditions for satisfaction, and hence Fuzzy Control rules on the criteria for terminating the algorithm.

Let consider, for the sake of illustration, a conventional LP problem

$$Min \ \{cx \ / \ Ax = b \ , x \geq 0\}$$

the main step of the Simplex Algorithm, with the usual denotation, can be summarised as follows,

Let x be an extreme point with basis B, and let R be the matrix corresponding to the nonbasic variables. Compute $c_B B^{-1} R - c_R$. If this vector is non positive then stop, x is an optimal extreme point.

Else select the most positive component $c_B B^{-1} a_j - c_j$ and compute $y_j = B^{-1} a_j$:

- If $y_j = B^{-1} a_j$ is less than or equal to 0 then stop. Objective unbounded.
- If $y_j = B^{-1} a_j$ is neither less than nor equal to 0 then go to next step

Therefore, the termination criterion can be seen as a control rule, and then

- The non positivity of the vector $c_B B^{-1} R - c_R$ could be meant in a soft sense,
- The positivity of the component $c_B B^{-1} a_j - c_j$ could be measured according to some membership function, and
- The accomplishment of $y_j = B^{-1} a_j \leq 0$, if this is viewed as a constraint, could be fuzzified.

If the first possibility is considered, a new main step can be formulated, Let x be an extreme point with basis B. Compute $c_B B^{-1} R - c_R$. If $\forall j = 1, \ldots, n, c_B y_j - c_j <_f 0, cj \in c_R$, where $<_f$ stands for a fuzzy constraint.

Thus this condition would mean that the decision-maker can accept violations in the accomplishment of the control rules, $c_B y_j - c_j < 0$, to obtain a near, and therefore approximate, optimal solution instead of a full optimal one. Results obtained from the application of this heuristic to a number of NP-problems can be seen in [24, 28, 29].

4.2. **FANS: A Fuzzy Adaptive Neighbourhood Search Algorithm.**
The Fuzzy Adaptive Neighborhood Search Method (FANS), [3], is a local search procedure which differs from other local search methods in two aspects. The first one is how solutions are evaluated; within FANS a fuzzy valuation $\mu()$ representing some (maybe fuzzy) property P is used together with the objective function to obtain a "semantic evaluation" of the solution. In this way, we may talk about solutions satisfying P in certain degree. Under this view, we define the semantic neighborhood of a solution s as:

$$N(s) = \{s/\mu(s) > \lambda\}$$

FANS moves between solutions satisfying P with at least certain degree λ, until it became trapped in a local optimum. In this situation the second novel aspect arise: the operator used to construct solutions is changed, so solutions coming from different neighborhoods are explored. This process is repeated once for each of a set of available operators until some finalization criterion for the local search is met.

The fuzzy valuation also enables the algorithm to achieve the qualitative behavior of other classical local search techniques [3].

The scheme of FANS is shown in Fig. 1. The execution of the algorithm finishes when some external condition holds, for example, when the number of cost function evaluations reached certain limit. Each iteration begins with a call to the so called neighborhood scheduler (NS), which is responsible for the generation and selection of the next solution in the optimization path. The call is done with parameters S_{cur} (the current solution), $\mu()$ (the fuzzy valuation), and O (a parameterized operator which is used to construct solutions). The neighborhood scheduler can return two alternative results; either a good enough (in terms of $\mu()$) solution (S_{new}) was found or not.

In the first case S_{new} is taken as the current solution and $\mu()$ parameters are adapted. In this way, the fuzzy valuation is changed as a function of the context or, in other terms, as a function of the state of the search. If NS failed to return an acceptable solution (no solution was good enough in the neighborhood induced by the operator), the parameters of the operator are changed. The strategy for this adaptation is encapsulated in the so called operator scheduler (OS). The next time NS is executed, it will have a modified operator to search for solutions.

When the whole set of operators available was used and the search was still stagnated (TrappedSituation = True), a classical random restart procedure is applied, and FANS continues the search from the new solution.

```
Procedure FANS;
Begin
   While (not-end) Do
      /* The neighborhood scheduler NS is called */
      S_new=NS(O,μ(),S_cur);
      If (S_new is good enough in terms of μ()) Then
         S_cur:=S_new;
         adaptFuzzyValuation(μ(),S_cur);
      Else
         /* NS failed to return a good solution with O */
         /* The operator scheduler will modify the operator */
         O:=OpSchedul();
      Fi
      If (TrappedSituacion() Then
         doEscape();
      Fi
   Od
End.
```

FIGURE 1. Pseudo Code of FANS

4.2.1. **Examples of Application.**
In order to apply FANS to a particular problem, the user must provide definitions for FANS's components. Let's suppose we are given a problem with n variables where each v_i can take discrete values from a set S with elements s_j. For example, if we consider knapsack problems, then $S = \{1, 0\}$ or $\{\text{included, not included}\}$ and n is the number of items available. Then we can obtain "canonical" implementations of FANS methods as follows.

First, we can define a modification operator which randomly selects k variables and produce a random assignemt as $v_i = random(S)$, where $random(S)$ returns any value from the set S.

The value of k, will be modified by the operator scheduler. A simple strategy may be to start with high values for k, say $k = n$, and then doing $k = k - 1$ each time the search became trapped. In this way we obtain very disruptive moves at the beginning, making them finer as the run progresses.

The fuzzy valuation will measure how Similar, or Different are two given solutions, or it can measure the level of Acceptability between a neighbor solution $\hat{s}$ and the current one s. Using the cost of both solution we can define Acceptability as:

$$\mu(s, \hat{s}) = \begin{cases} 1.0 & \text{if } f(\hat{s}) < f(s) \\ (f(\hat{s} - \beta)/(f(s) - \beta) & \text{if } f(s) \le f(\hat{s}) \le \beta \\ 0.0 & \text{if } f(\hat{s}) > \beta \end{cases}$$

If $\hat{s}$ improves the cost of s, it will get the highest value of Acceptability. If $\hat{s}$ diminishes the cost a little it may be considered as acceptable but with lower degree. If $\hat{s}$ is much worst than s, it will get a degree of acceptability of zero.

The last component is the neighborhood scheduler. For its definition we may use a First strategy, which simply returns the first solution obtained with O, satysfing $\mu()$ in certain level λ.

Using such a simple scheme, different versions of FANS were implemented and adapted to solve knapsack problems with single and multiple restrictions, minimization of functions of type $\mathbb{R}^n \rightarrow \mathbb{R}$, and also to solve a lattice model of the protein structure prediction problem. For all of them, comparisons were made against other general purpose heuristics (like genetic algorihtm or simulated annealing) and the results were usually equal or better for FANS. The interested reader may refer to [21] for an updated review of FANS applications.

4.3. **A Fuzzy Rule-based Methodology for Coordinating Metaheuristics.** The main idea if this methodology is well explained with the following metaphore: imagine a committee in charge to solve some concrete problem. The committee is chaired by a Director, who knows the general aspects of the problem and the particular features of each worker/expert of the team. Besides, the Director knows when the problem is reasonably solved (i.e. the team found a good enough solution). Each worker tries to solve the problem using a particular methodology (or perhaps several ones). In this context the set of experts work concurrently, reporting to the Director the partial results they are obtaining. Based on those partial results, the Director may order changes to the workers in order to improve the "performance" of the team.

Under this schematic view, we propose to implement workers as optimization algorithms and the Director's role by means of a fuzzy rule based system. The Director's knowledge could be represented by three groups of rules:

(1) Global rules: representing the knowledge of the problem (type, size, characteristics, ...), and possibly who are the best experts to solve it.
(2) Worker Specific Rules: represent the specific knowledge of the Director about the performance's worker (if exists). Concretely, it stands for knowledge about its behavior across the time, preferences, tuning, etc.
(3) Stopping Rules: set of rules representing the termination criteria that the Director uses. Director is able to recognize when the work should be stopped, based on objective function value and time.

These set of rules try to address the following "common sense" principles:

- If a worker goes fine, don't touch it, but if it performance becomes bad, change somehting to alter its behaviour.
- Keep closer to the good ones and you will be as good as them, simply, communicate to workers the good solutions already found.
- A good food don't change with a microgram more of salt, which esentially address the problem of when to stop

At the time of writing this article, the methodology is being tested on a bioinformatic problem called protein structure prediction. The main preliminary conclusion is that the use of several coordinated and concurrent implementations of FANS, lead to a significant speed up of the computation times.

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