ON THE FUZZY SET THEORY AND AGGREGATION FUNCTIONS: HISTORY AND SOME RECENT ADVANCES

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Abstract. Several fuzzy connectives, including those proposed by Lotfi Zadeh, can be seen as linear extensions of the Boolean connectives from the scale $\{0, 1\}$ into the scale $[0, 1]$. We discuss these extensions, in particular, we focus on the dualities arising from the Boolean dualities. These dualities allow to transfer the results from some particular class of extended Boolean functions, e.g., from conjunctive functions, into some other distinguished classes, e.g., into fuzzy implications. We also stress the role of aggregation functions in fuzzy set theory. Then we continue with several recent advances and new directions in aggregation theory. In particular, we discuss some generalizations of monotonicity, additivity and maxitivity issues. Finally, some applications of aggregation functions are sketched.

1. Introduction

Aggregation functions belong to the background of many mathematical branches. They have a long history, probably the first trace concerning aggregation in a written form can be found in the Moscow papyrus, going back to around 1850 B.C. [3], where the Problem 14 deals with the computation of the volume of a pyramidal frustum. Aggregation is a genuine part of the backgrounds of classical logic (Boolean aggregation), many-valued logics (aggregation on the related truth scales), decision sciences, multiobjective optimization, etc. Some other application fields of aggregation functions will be mentioned later. Though some particular aggregation functions and related theories were deeply developed in the framework of specific areas — as, for example, triangular norms in the framework of probabilistic metric spaces [44, 45], or copulas in the framework of multivariate analysis [47, 41], an independent aggregation theory has been built by Klir and Folger in [27] (for n-ary aggregation functions), see also [28] (for extended aggregation functions defined for any arity). Development of each new branch dealing with aggregation functions depends on the state of the art of aggregation theory, and on the contrary, it has a great impact on the development of some other branches of aggregation theory. This was also the case of fuzzy set theory and its further generalizations.

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The aim of this paper is to discuss the links between fuzzy set theory and aggregation theory, including latest developments. The paper is organized as follows. In the next section, we discuss the original Zadeh's approach to fuzzy connectives which can be seen as linear extensions of the related Boolean connectives. An important fact which should be emphasized is that these linear extensions can be characterized by simple sets of axioms, and that abandoning linearity has allowed to open a space for the investigation of new types of fuzzy connectives. This was, for example, the case of triangular norms which have served for modeling intersections of fuzzy sets. For more details we recommend [26]. Dualities play an important role in both fuzzy set theory and aggregation theory. To stress this fact, in Section 3, we provide a deeper discussion of several dualities induced by dualities of Boolean functions. Recent advances of aggregation theory have opened a new perspective for development of fuzzy set theory. Therefore, in Section 4, we recall some important new advances achieved in aggregation theory. In Section 5, we briefly mention several application fields in which aggregation theory has been substantially applied. Finally, Section 6 contains several concluding remarks.

2. **Fuzzy Connectives**

When Lotfi Zadeh formed the basis of fuzzy set theory in his distinguished paper *Fuzzy sets* published in 1965, see [56], the mathematical background of the introduced set operations as well as algebraic operations on fuzzy sets was built on a point-wise approach and the related operations on the unit interval $[0, 1]$. Without going deeper into details, we recall the introduced algebraic and set operations defined by means of them:

- negation $n_Z(x) = 1 - x$ (defines the complement of fuzzy sets),
- conjunction $x \wedge y = \min\{x, y\}$ (defines the intersection of fuzzy sets),
- disjunction $x \vee y = \max\{x, y\}$ (defines the union of fuzzy sets),
- product $x \cdot y$ (defines the algebraic product of fuzzy sets),
- algebraic sum $x + y$ defined for $x + y \leq 1$ (defines the algebraic sum of fuzzy sets).

Note that in [56], there is also mentioned the probabilistic sum given by $x+y-xy$, $(x, y) \in [0, 1]^2$. All these functions can be seen as (piece-wise) linear extensions of basic Boolean functions. Note that apart from the constant Boolean functions, we have two unary and fourteen binary Boolean functions. The Zadeh negation n_Z is a linear extension of the Boolean negation $n\colon \{0, 1\} \to \{0, 1\}$, $n(0) = 1$, $n(1) = 0$. For binary Boolean functions $\{0, 1\}^2 \to \{0, 1\}$ there are three possible approaches to their linear extensions (interpolations) acting on $[0, 1]^2$. The first approach is based on the simplexes A, B, see Figure 1 (left), the second one on the simplexes C, D, visualized in Figure 1 (right), and the third one is a coordinate-wise linear extension.

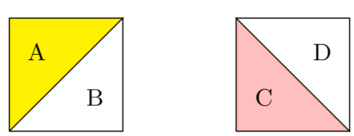

Figure 1. Simplexes for Linear Extensions of Binary Boolean Functions

For example, considering the Boolean disjunction, by means of the described three types of extensions we obtain Zadeh's disjunction, Łukasiewicz's disjunction (extending the algebraic sum) and the probabilistic sum, respectively, see Figure 2.

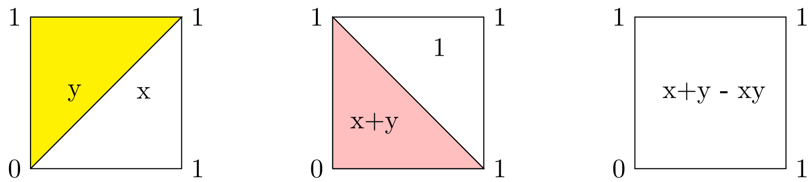

FIGURE 2. Linear Extensions of the Boolean Disjunction: Zadeh's Disjunction (left); Łukasiewicz's Disjunction (Middle); Probabilistic Sum (Right)

Similarly, one can linearly extend the Boolean conjunction, see Figure 3.

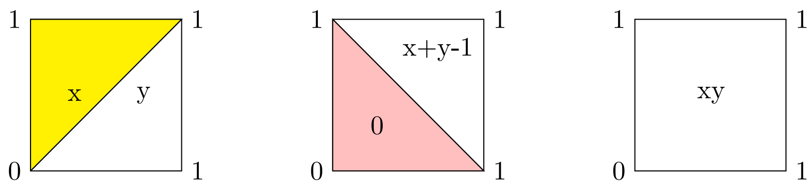

FIGURE 3. Linear Extensions of the Boolean Conjunction: Zadeh's Conjunction (Left); Łukasiewicz's Conjunction (Middle); Product (Right)

Note that the above mentioned approaches can also be applied to other connectives known from Boolean logic. For illustration, we add the extension of the Boolean implication, see Figure 4. Another example, the Sheffer stroke (NAND operator) [46, 42], can be extended as is shown in Figure 5.

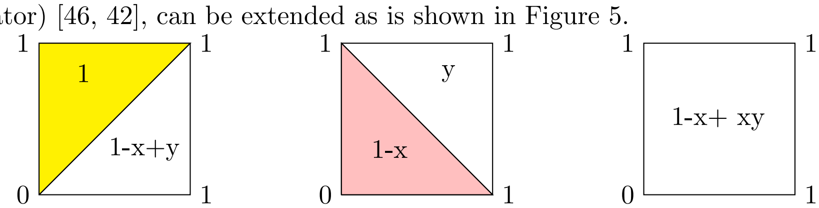

FIGURE 4. Linear Extensions of the Boolean Implication: Łukasiewicz's Implication (left); Kleene-Dienes's Implication (Middle); Reichenbach's Implication (Right)

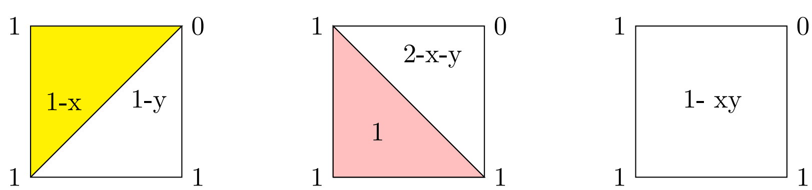

FIGURE 5. Linear Extensions of the Sheffer Stroke to $[0, 1]^2$

Observe that in [56], Zadeh also mentioned the convex sums of fuzzy sets, i.e., the point-wise extensions of weighted arithmetic means. Later, several other extensions preserving the monotonicity of the considered Boolean functions were considered. Note that all Boolean functions can be derived from the Boolean conjunction (or disjunction or the stroke), considering the Boolean constants 0 and 1, and the lattice operations $\vee$ and $\wedge$, too, and thus the major interest in extending Boolean functions has been focused on conjunctions. Let us recall at least such extensions as triangular norms, copulas, quasi-copulas, or semi-copulas. All of them are aggregation functions with annihilator 0. For more information on aggregation functions we refer, e.g., to [6, 7, 13, 22]. Some of these aggregation classes have led to the independent fields of study. For example, triangular norms are deeply studied in monograph [26], and from some recent non-standard applications we recall, e.g., [40]. For copula theory, including several applications, we recommend the monographs [25, 41].

3. **Dualities**

Denote by $\mathcal{B}$ the set of all binary Boolean functions. Recall that a duality on $\mathcal{B}$ is any involutive mapping $\varphi\colon \mathcal{B} \to \mathcal{B}$, i.e., a mapping satisfying $\varphi \circ \varphi = Id$, where Id denotes the identity on $\mathcal{B}$. Dualities between Boolean functions can be extended to dualities between extended Boolean functions, i.e., functions $F\colon [0,1]^2 \to [0,1]$ such that $F|_{\{0,1\}^2}$ is a Boolean function. Here we only recall eight basic dualities on the set $\mathcal{EB}$ of all extended binary Boolean functions:

For any extended Boolean function F and all $(x,y) \in [0,1]^2$ we put

$$\begin{aligned}
\varphi_0(F)(x,y) &= F(x,y) \quad \text{(i.e., } \varphi_0 \text{ is the identity on } \mathcal{EB});\\
\varphi_1(F)(x,y) &= F(x,1-y);\\
\varphi_2(F)(x,y) &= F(1-x,y);\\
\varphi_3(F)(x,y) &= F(1-x,1-y).
\end{aligned}$$

$$\begin{aligned}
\varphi_4(F)(x,y) &= 1-F(x,y);\\
\varphi_5(F)(x,y) &= 1-F(x,1-y);\\
\varphi_6(F)(x,y) &= 1-F(1-x,y);\\
\varphi_7(F)(x,y) &= 1-F(1-x,1-y).
\end{aligned}$$

Observe that the group structure of the set of all basic dualities $\{\varphi_0, \ldots, \varphi_7\}$ was described in our recent paper [36]. Note that there are also other dualities, not described above, such as φ_8, $\varphi_8(F)(x,y) = F(y,x)$. Considering the eight dualities $\varphi_1, \ldots, \varphi_8$ and their compositions we can generate a new group of dualities consisting of 16 elements.

If we consider binary aggregation functions, i.e., functions $F\colon [0,1]^2 \to [0,1]$ which are monotone and satisfy the boundary conditions $F(0,0) = 0$ and $F(1,1) = 1$, the duality φ_7 is simply called duality, with notation $\varphi_7(F) = F^d$. Then, for an extended conjunction F, F^d is an extended disjunction and vice-versa. Moreover, for an extended conjunction F, $\varphi_5(F)$ is a fuzzy implication, $\varphi_4(F)$ is an extended NAND operator (the Sheffer stroke), $\varphi_3(F)$ is an extended NOR operator, $\varphi_2(F)$ is an extended coimplication, etc.

Observe that the introduced dualities allow to transfer the results known for some particular subclass of aggregation functions into another class. For example, based on the duality φ_7, all results known for triangular norms (e.g., the structure of continuous triangular norms) can be transformed into the corresponding results for triangular conorms. Similarly, considering the duality φ_5, we can transform the results known for particular conjunctions (e.g., ordinal sum of copulas) into the corresponding results for implications.

A similar discussion concerning dualities can be done for any arity $n > 2$. For example, the duality φ_7 extended to the n-ary case is of the form $\varphi_7(F)(x_1, \ldots, x_n) = 1 - F(1 - x_1, \ldots, 1 - x_n)$. This duality is a standard duality of functions mapping n inputs from the real unit interval [0,1] into a single output from the same scale.

4. Some Recent Advances in Aggregation Theory

Aggregation functions for a fixed arity $n > 1$ were formally introduced by Klir and Folger [27], and for an unfixed arity by Kolesárová and Komorníková [28]. They can be seen as a common framework for conjunctive and disjunctive fuzzy connectives.

Aggregation functions for a fixed arity $n > 1$ are defined as monotone functions $F\colon [0,1]^n \to [0,1]$ satisfying the conditions $F(0, \ldots, 0) = 0$ and $F(1, \ldots, 1) = 1$.

As already mentioned, an overview of aggregation functions can be found in monographs [6, 7, 22], but also in many papers published in distinguished journals. A discussion concerning some new trends in aggregation can be found in our recent paper "Quo vadis aggregation?", see [39].

Apart from already mentioned applications of binary aggregation functions in fuzzy set theory as fuzzy connectives, we have many other applications. For example, n-ary aggregation functions are considered in fuzzy rule-based systems [53], and there are also numerous applications in decision support based on fuzzy modeling [54, 52], in fuzzy logic [23, 26], fuzzy games [11], etc.

Recall that according to the classification of aggregation functions by Dubois and Prade in [15], aggregation functions we can be divided into four classes. Following their proposal, we distinguish:

- **Conjunctive aggregation functions:** aggregation functions whose output never exceed any input, in other words,

  $$\forall\, \mathbf{x} = (x_1, \ldots, x_n) \in [0,1]^n, \quad F(\mathbf{x}) \le \min_{i=1}^{n} x_i = Min(x_1, \ldots, x_n),$$

  i.e., $F \le Min$;
- **Disjunctive aggregation functions:** those ones whose output is always greater than or equal to any input, in other words,

  $$\forall\, \mathbf{x} = (x_1, \ldots, x_n) \in [0,1]^n, \quad F(\mathbf{x}) \ge \max_{i=1}^{n} x_i = Max(x_1, \ldots, x_n),$$

  i.e., $F \ge Max$;
- **Averaging aggregation functions:** their output is always bounded from both below and above by some input values, i.e.,

  $$Min \le F \le Max;$$

- **Hybrid aggregation function:** those ones which do not satisfy any of the previous properties.

Note that this classification can also be used on non-linear scales, such as lattices or posets. Observe that considering the standard duality φ_7 discussed in the previous section, we get that disjunctive aggregation functions are dual to conjunctive aggregation functions, and vice-versa. Next, averaging as well as hybrid aggregation functions (as classes), are self-dual. This means that any aggregation function F which is self-dual, i.e., satisfies the property $\varphi_7(F) = F$, is necessarily either averaging or hybrid. For example, the arithmetic mean is an averaging aggregation function, while 0.5-median, $Med_{0.5}\colon [0,1]^n \to [0,1]$, given by

$$Med_{0.5}(x_1, \ldots, x_n) = med(x_1, 0.5, x_2, 0.5, \ldots, 0.5, x_n)$$

is a hybrid aggregation function. Observe that there are also refined classifications [34] and [30].

We now recall some typical representatives of the above introduced four classes of aggregation functions:

Conjunctive aggregation functions: triangular norms [44, 26, 1]; copulas [47, 41, 25, 17]; quasi-copulas [2, 19, 22]; semicopulas [5, 16]. Note that in fuzzy set theory, semicopulas are also called conjunctors.

Recall that all mentioned classes are bounded from above by the greatest conjunctive aggregation function Min. The bottom element of triangular norms and semicopulas as well, is the drastic product T_D whose binary form is

$$T_D(x, y) = \begin{cases} 0 & \text{if } (x, y) \in [0, 1[^2, \\ \min\{x, y\} & \text{otherwise.} \end{cases}$$

On the other hand, the bottom element of binary copulas and quasi-copulas is the so-called Łukasiewicz t-norm T_L given by $T_L(x, y) = \max\{0, x + y - 1\}$. Note that in the framework of probability theory and statistics this function is called the Fréchet-Hoeffding lower bound and denoted by W. Let us emphasize that T_L is an associative function and for $n \geq 3$ its n-ary form is the bottom element of the class of all n-ary quasi-copulas (which form a bounded complete lattice), and the best possible lower bound for n-ary copulas (but not a copula). Recall that the class of all copulas with a fixed arity is closed neither under sup nor inf. If their sup and inf closure is considered, the class of all quasi-copulas is obtained. Observe that except for triangular norm, all mentioned classes of conjunctive aggregation functions are convex. For for more information on these aggregation functions we refer to the cited references.

Disjunctive aggregation functions: due to their duality with conjunctive aggregation functions, we mostly consider duals to already introduced classes of conjunctive aggregation functions, including triangular conorms, dual copulas, dual quasi-copulas or dual semi-copulas (often called disjunctors in fuzzy set theory).

Averaging aggregation functions: weighted (quasi-arithmetic) means [8], OWA operators and related aggregation functions (GOWA, IOWA, etc.) [55], several kinds of integrals with respect to capacities (i.e., to normed fuzzy measures), including the Choquet, Sugeno, Shilkret and other integrals [50, 22], etc.

Hybrid aggregation functions: uninorms [18], nullnorms [12], compensative operators [33, 57], etc.

Now we turn our attention to some new trends in aggregation theory.

An important stream in aggregation theory is aggregation on lattices. Note that early traces of this approach go back to L-fuzzy set theory introduced by Goguen [21]. From distinguished lattices on which aggregation has been intensively studied, we recall the interval lattice $L([0,1]) = \{[a,b] \mid 0 \leq a \leq b \leq 1\}$ and the lattice $L^*([0,1])$ isomorphic to $L([0,1])$, $L^*([0,1]) = \{(a,b) \mid a, b \in [0,1], a+b \leq 1\}$, which is the background of the intuitionistic fuzzy sets introduced by Atanassov [4]. Further lattices are related, e.g., to hesitant fuzzy sets, neutrosophic fuzzy sets, etc. Interestingly, some new results on general (bounded distributive) lattices have also brought a new light into the standard aggregation theory related to the classical fuzzy set theory, as for example, the result that the only aggregation functions preserving the congruences are the Sugeno integrals [24]. Several other studies are devoted to triangular norms, uninorms, or nullnorms on lattices. Observe, for example, that any uninorm $U\colon [0,1]^2 \to [0,1]$ (i.e., a commutative and associative aggregation function with neutral element) is either conjunctive (extension of the Boolean conjunction) or disjunctive (extension of the Boolean disjunction). This claim is also valid on any bounded chain, but no more on a bounded distributive lattice with incomparable elements (if $U(0_L, 1_L) = a$ is comparable with neutral element e, then $a \in \{0_L, 1_L\}$; else a is incomparable with e).

Another major new stream in aggregation deals with the standard scale $[0,1]$ (or any other real interval), but with modified monotonicity requirements. In particular, directional monotonicity introduced in [9] requires monotonicity in a fixed direction $\vec{r} \in \mathbb{R}^n \setminus \{\vec{0}\}$:

A function $F\colon [0,1]^n \to [0,1]$ is said to be $\vec{r}$-increasing whenever

$$F(\mathbf{x} + c\vec{r}) \geq F(\mathbf{x})$$

for all $\mathbf{x} = (x_1, \ldots, x_n) \in [0,1]^n$ and $c > 0$ such that $\mathbf{x} + c\vec{r} \in [0,1]^n$.

For example, the weighted Lehmer mean $LM_{(w_1,w_2)}\colon [0,1]^2 \to [0,1]$ given by

$$LM_{(w_1,w_2)}(x_1, x_2) = \frac{w_1 x_1^2 + w_2 x_2^2}{w_1 x_1 + w_2 x_2},$$

(with convention $\frac{0}{0} = 0$) where $w_1, w_2 \geq 0$ and $w_1 + w_2 = 1$, is $\vec{r}$-increasing if and only if $\vec{r} = k(w_2, w_1)$ for some positive constant k.

The notion of directional monotonicity allows to extend the class of all aggregation functions into the class of all pre-aggregation functions [32].

Recall that for a fixed $n > 1$, a function $F\colon [0,1]^n \to [0,1]$ is a pre-aggregation function whenever

- F satisfies the boundary conditions $F(0, \ldots, 0) = 0$, $F(1, \ldots, 1) = 1$, and

- F is $\vec{r}$-increasing for some $\vec{r} \in [0, \infty[^n \setminus \{\vec{0}\}$.

Note that aggregation functions are $\vec{r}$-increasing for all directions $\vec{r} \in [0, \infty[^n \setminus \{\vec{0}\}$. As a particular case of directional monotonicity we mention weak monotonicity introduced by Wilkin and Beliakov [51], i.e., directional monotonicity with respect to $\vec{r} = (1, \ldots, 1)$. A typical weakly increasing mean is the above mentioned Lehmer mean with weights $w_1 = w_2$, but also, e.g., the mode. Finally, let us recall some other recently introduced modifications of monotonicity, such as, e.g., ordered directional monotonicity [10] or strengthened ordered directional monotonicity [31].

In what follows we only touch the third interesting streaming in aggregation theory, namely the generalizations of the additivity and maxitivity of aggregation functions. An aggregation function $F\colon [0,1]^n \to [0,1]$ is said to be additive whenever

$$F(\mathbf{x} + \mathbf{y}) = F(\mathbf{x}) + F(\mathbf{y}) \tag{1}$$

for all $\mathbf{x}, \mathbf{y} \in [0,1]^n$ such that $\mathbf{x}+\mathbf{y} \in [0,1]^n$. Functional equation (1) is a kind of the Cauchy equation and its solutions are just weighted arithmetic means $F = WAM_{\mathbf{w}}$, where the weighting vector $\mathbf{w} = (w_1, \ldots, w_n) \in [0,1]^n$ satisfies the normalization constraint $\sum\limits_{i=1}^{n} w_i = 1$. Then $F(\mathbf{x}) = \sum\limits_{i=1}^{n} w_i x_i$.

For any $k \in \mathbb{N}$ we can define the notion of k-additivity of an aggregation function $F\colon [0,1]^n \to [0,1]$ by requiring

$$\sum_{i=1}^{k+1} (-1)^{k+1-i} \left(\sum_{\substack{I \subseteq \{1,\ldots,k+1\} \\ |I|=i}} F\left(\sum_{j \in I} \mathbf{x}_j \right) \right) = 0.$$

for all $\mathbf{x}_1, \ldots, \mathbf{x}_{k+1} \in [0,1]^n$ such that also $\sum\limits_{i=1}^{k+1} \mathbf{x}_i \in [0,1]^n$.

It can be shown that k-additive aggregation functions are monotone polynomials whose degree is at most k [29]. Particular k-additive aggregation functions are linked to k-additive fuzzy measures [20] by means of the Owen extension [29, 43].

Similarly, k-maxitive aggregation functions can be introduced and studied. So, for example, considering any k-ary aggregation function $B\colon [0,1]^k \to [0,1]$ and $n \geq k$, the function $F\colon [0,1]^n \to [0,1]$ given by

$$F(x_1, \ldots, x_n) = B(x_{\sigma(1)}, \ldots, x_{\sigma(k)}),$$

where $\sigma\colon \{1, \ldots, n\} \to \{1, \ldots, n\}$ is a permutation such that $x_{\sigma(1)} \geq \cdots \geq x_{\sigma(n)}$, is a symmetric k-maxitive n-ary aggregation function. This result is not only a construction method, but also a representation. As an example, observe that the ternary median is 2-maxitive and $Med(x_1, x_2, x_3) = Min(x_{\sigma(1)}, x_{\sigma(2)})$. For more details we recommend [37, 38].

5. Some Applications of Aggregation Functions

Here we only recall several fields in which aggregation functions are substantially applied. Note that since early beginnings of mathematics, aggregation functions were considered and applied in various domains (recall, e.g., the construction of pyramides, see already mentioned Problem 14 from the Moscow papyrus [3], compare with the Heronian mean dated back to the ancient Greece, too). From among the most prominent application fields of the aggregation functions we emphasize the following ones:

- multi-valued logics;
- fuzzy rule-based systems;
- fuzzy games;
- expert systems;
- decision making under multiple criteria or attributes;
- multiobjective optimization;
- pattern recognition and classification;
- image processing.

For more information we recommend [22, 49].

6. **Conclusions**

We have discussed the role of aggregation functions in fuzzy set theory, some new developments and directions in aggregation theory, as well as a few major application fields of aggregation theory. The importance of aggregation theory has considerably grown since several new generalizations of Zadeh's fuzzy set theory have been introduced, where an important role is also played by specific properties of underlying lattice structures. For example, when considering interval-valued fuzzy sets, we have a trivial approach based on the so-called representable aggregation functions, where both end-points are considered independently. However, there are particular aggregation functions on intervals, e.g., triangular norms, which cannot be split into two separate parts acting independently on the left end-points and right end-points [14]. To mention some new developments of the interval fuzzy set theory, where aggregation plays a substantial role, we recall at least the recent paper [48]. Similar considerations are valid for L^*-valued aggregation functions, i.e., in intuitionistic fuzzy set theory.

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