FUZZY LOGISTIC DIFFERENCE EQUATION

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Abstract. In this study, we consider two different inequivalent formulations of the logistic difference equation $x_{n+1} = \beta x_n(1 - x_n)$, $n = 0, 1, ...$, where x_n is a sequence of fuzzy numbers and β is a positive fuzzy number. The major contribution of this paper is to study the existence, uniqueness and global behavior of the solutions for two corresponding equations, using the concept of Hukuhara difference for fuzzy numbers. Finally, some examples are given to illustrate our results.

1. Introduction

Difference equations appear as a natural way of evolution phenomena because most measurements of time evolving variables are discrete and these equations are important in mathematical models. More importantly, difference equations also appear in the study of discretization methods for differential equations. Furthermore, the application of the theory of difference equations is rapidly increasing to various fields such as numerical analysis, control theory, finance mathematics and computer science [8, 15]. The logistic difference equation was popularized in a seminal 1976 paper by the biologist Robert May [10], as a discrete-time demographic model analogous to the logistic equation first created by Pierre Franois Verhulst [1]. This model is continuous in time, but a modified version of the continuous equation to a discrete quadratic equation, is widely studied [12, 13]. The continuous version of the logistic model is described by the differential equation $\frac{dn}{dt} = an(1 - \frac{n}{k})$, where a is the rate of maximum population growth and k is the carrying capacity. By dividing both sides in k and defining $x = \frac{n}{k}$, then gives the differential equation $\frac{dx}{dt} = ax(1 - x)$. The discrete version of the logistic model is written as [6]

$$x_{n+1} = ax_n(1 - x_n), \quad n = 0, 1, ... \tag{1}$$

We mention that if $a < 1$, the model describes extinction of population. In [11], EI-Metwally et al. studied the behavior of the population model

$$x_{n+1} = \alpha + \beta x_{n-1}e^{-x_n}, \quad n = 0, 1, ..., \tag{2}$$

Received: June 2016; Revised: December 2016; Accepted: December 2016

Key words and phrases: Fuzzy difference equations, Logistic difference equation, Boundedness, Persistence, Equilibrium point.

This paper is the extended version of presented paper in "International Conference on Mathematics of Fuzziness (ICMF)" which was held during 27-29 April 2016 at Institute for Advanced Studies in Basic Sciences (IASBS), Zanjan, Iran.

where α is a immigration rate and β is the population growth rate. Subsequently, Zhang et al. in [16] studied the fuzzy analogues of (2) where x_n is a sequence of positive fuzzy numbers, α, β and initial values x_{-1}, x_0 are positive fuzzy numbers. Fuzzy set theory is a powerful tool for modeling uncertainty and for processing vague in mathematical model. One approach for considering uncertainty in a dynamical system to predict the behavior of imprecise real-world phenomena is the fuzzification of the corresponding crisp difference equations [3, 9]. Independently of the similar particular formulations of the equation in any model, we expect that the solution reflects accurately the real behavior of the system. Therefore getting different results through fuzzification of the unique crisp equation may seem unnatural. However, we can consider this fact as an advantage of fuzzy mathematics, due to the existence of several choices which can be examined [7]. Especially, the use of fuzzy difference equation is a natural way to study dynamical systems with uncertainty.

Fuzzy difference equation is a difference equation with fuzzy parameters and fuzzy initial values, and the solution is a sequence of the fuzzy numbers. Fuzzy difference equations is important due for analysis of real world phenomena. For example, fuzzy difference equations are suitable to study finance problems, time series and population model [4, 5]. In this paper we study the fuzzy difference equations

$$x_{n+1} = \beta x_n(1 - x_n), \tag{3}$$

and

$$x_{n+1} = \beta x_n \ominus \beta {x_n}^2, \quad n = 0, 1, ..., \tag{4}$$

where x_n is a sequence of positive fuzzy numbers, β and initial value x_0 are positive fuzzy numbers and $\ominus$ denotes the Hukuhara difference (H-difference) of two fuzzy numbers.

The paper is organized as follows. Section 2 introduces some preliminaries that is necessary in other sections. Two different inequivalent formulations of the logistic difference equation are studied in Section 3. Possible applications to logistic difference equation are briefly illustrated in Section 4.

2. Preliminaries

For the convenience of readers, we give the following preliminaries, see [2, 5].

Definition 2.1. [2] Consider a fuzzy subset of the real line $u : \mathbb{R} \to [0, 1]$. Then we say u is a fuzzy number if it satisfies the following properties

(i) u is normal, i.e., $\exists x_0 \in \mathbb{R}$ with $u(x_0) = 1$,
(ii) u is fuzzy convex, i.e., $u(tx+(1-t)y) \geq \min\{u(x), u(y)\}, \forall t \in [0, 1], x, y \in \mathbb{R}$,
(iii) u is upper semicontinuous on $\mathbb{R}$,
(iv) u is compactly supported i.e., $\overline{\{x \in \mathbb{R}; u(x) > 0\}}$, is compact.

Let us denote by $\mathbb{R}_{\mathcal{F}}$ the space of all fuzzy numbers. For $0 < \alpha \leq 1$ and $u \in \mathbb{R}_{\mathcal{F}}$, we denote $\alpha-$cuts of fuzzy number u by $[u]_\alpha = \{x \in \mathbb{R}; u(x) \geq \alpha\}$ and $[u]_0 = \overline{\{x \in \mathbb{R}; u(x) > 0\}}$. We call $[u]_0$, the support of fuzzy number u and denote it by $supp(u)$.

The fuzzy number A is called positive if *supp* $(A) \subset (0,\infty)$. We denote by $\mathbb{R}^+_{\mathcal{F}}$, the space of all positive fuzzy numbers.

For $u,v \in \mathbb{R}_{\mathcal{F}}$, $[u]_\alpha = [\underline{u}_\alpha, \overline{u}_\alpha]$, $[v]_\alpha = [\underline{v}_\alpha, \overline{v}_\alpha]$ and $\lambda \in \mathbb{R}$, the sum $u+v$, the scalar product $\lambda.u$ and multiplication uv in the standard interval arithmetic (SIA) setting are defined by

$$[u+v]_\alpha = [u]_\alpha + [v]_\alpha, \quad [\lambda.u]_\alpha = \lambda[u]_\alpha, \ \forall \alpha \in [0,1],$$

$$[uv]_\alpha = [\min\{\underline{u}_\alpha\underline{v}_\alpha, \underline{u}_\alpha\overline{v}_\alpha, \overline{u}_\alpha\underline{v}_\alpha, \overline{u}_\alpha\overline{v}_\alpha\}, \max\{\underline{u}_\alpha\underline{v}_\alpha, \underline{u}_\alpha\overline{v}_\alpha, \overline{u}_\alpha\underline{v}_\alpha, \overline{u}_\alpha\overline{v}_\alpha,\}].$$

Indeed, for $u,v \in \mathbb{R}^+_{\mathcal{F}}$, we have $[uv]_\alpha = [\underline{u}_\alpha\underline{v}_\alpha, \overline{u}_\alpha\overline{v}_\alpha]$. Also in this paper, we consider u^2 as the multiplication of u and u with $\alpha-$cuts $[u^2]_\alpha = [u.u]_\alpha = [\underline{u}^2, \overline{u}^2]_\alpha$.

Definition 2.2. [5] We refer to $\underline{u}$ and $\overline{u}$ as the lower and upper branches of u, respectively. For $u \in \mathbb{R}_{\mathcal{F}}$, we define the length of u as diam(u) $= \overline{u}_\alpha - \underline{u}_\alpha$.

Definition 2.3. Let $x,y \in \mathbb{R}_{\mathcal{F}}$. If there exists $z \in \mathbb{R}_{\mathcal{F}}$ such that $x = y+z$, then z is called the H-difference of x,y and it is denoted $x \ominus y$.

Theorem 2.4. [2] *(Stacking Theorem) If $A \in \mathbb{R}_{\mathcal{F}}$ and A_α are its level-cuts then*

(i) *A_α is a closed interval $A_\alpha = [\underline{A}_\alpha, \overline{A}_\alpha]$, for any $\alpha \in [0,1]$,*
(ii) *If $0 \le \alpha_1 \le \alpha_2 \le 1$, then $A_{\alpha_2} \subseteq A_{\alpha_1}$,*
(iii) *For any sequence α_n which converges from below to $\alpha \in [0,1]$, we have*

$$\bigcap_{n=1}^{\infty} A_{\alpha_n} = A_0,$$

(iv) *For any sequence α_n which converges from above to 0, we have*

$$\overline{\bigcup_{n=1}^{\infty} A_{\alpha_n}} = A_{\alpha_n}.$$

Theorem 2.5. [2] *Let us consider the functions*

$$\underline{A}_\alpha, \overline{A}_\alpha : [0,1] \to \mathbb{R},$$

satisfy the following conditions

(i) *$\underline{A}_\alpha \in \mathbb{R}$ is a bounded, non-decreasing, left-continuous function in $(0,1]$ and it is right-continuous at 0.*
(ii) *$\overline{A}_\alpha \in \mathbb{R}$ is a bounded, non-increasing, left-continuous function in $(0,1]$ and it is right-continuous at 0.*
(iii) *$\underline{A}_1 \le \overline{A}_1$.*

Then there is a fuzzy number $A \in \mathbb{R}_{\mathcal{F}}$ that has $\underline{A}_\alpha, \overline{A}_\alpha$ as endpoints of its α-cuts, A_α. Conversely let $A \in \mathbb{R}_{\mathcal{F}}$ with endpoints $\underline{A}_\alpha, \overline{A}_\alpha$, then conditions (i)-(iii) are satisfied.

Definition 2.6. [5] Let A,B be fuzzy numbers with $[A]_\alpha = [\underline{A}_\alpha, \overline{A}_\alpha]$, $[B]_\alpha = [\underline{B}_\alpha, \overline{B}_\alpha]$, $\alpha \in [0,1]$. Then the metric on the fuzzy numbers space is defined as follow

$$D(A,B) = \sup\max\{|\underline{A}_\alpha - \underline{B}_\alpha|, |\overline{A}_\alpha - \overline{B}_\alpha|\}$$

where sup is taken for all $\alpha \in [0,1]$.

Definition 2.7. [14] A sequence of positive fuzzy numbers $\{x_n\}$ is bounded and persists if there exist positive real numbers $M, N > 0$ such that

$$supp(x_n) \subset [M, N], \quad n = 1, 2, ...$$

3. Main Results

In this section, we study the fuzzy difference equations (3) and (4) where x_n is a sequence of positive fuzzy numbers and $x_0, \beta \in \mathbb{R}^+_{\mathcal{F}}$. We study the existence, uniqueness and global behavior of the solutions.

Proposition 3.1. *Consider Eq.* (3) *where x_n is a sequence of positive fuzzy numbers and $x_0, \beta \in \mathbb{R}^+_{\mathcal{F}}$. If $\overline{x_{0,\alpha}}, \overline{\beta_\alpha} < 1, \quad \forall \alpha \in (0,1]$, then for every positive fuzzy number x_0, there exists a unique positive solution x_n of* (3) *with initial condition x_0.*

Proof. Suppose that there exists a sequence of fuzzy numbers x_n satisfying (3) with initial data x_0. Consider the $\alpha-$cuts, $\alpha \in (0,1]$

$$[x_n]_\alpha = [\underline{x_\alpha}, \overline{x_\alpha}], \quad n = 0, 1, ... \tag{5}$$

Then from (3) and (5), it follows that

$$[x_{n+1}]_\alpha = [\underline{x_{n+1,\alpha}}, \overline{x_{n+1,\alpha}}] = [\underline{\beta_\alpha}\underline{x_{n,\alpha}}(1 - \overline{x_{n,\alpha}}), \overline{\beta_\alpha}\overline{x_{n,\alpha}}(1 - \underline{x_{n,\alpha}})]. \tag{6}$$

Therefore we have for $n = 0, 1, ...$

$$\underline{x_{n+1,\alpha}} = \underline{\beta_\alpha}\underline{x_{n,\alpha}}(1 - \overline{x_{n,\alpha}}), \quad \overline{x_{n+1,\alpha}} = \overline{\beta_\alpha}\overline{x_{n,\alpha}}(1 - \underline{x_{n,\alpha}}), \quad \alpha \in (0,1]. \tag{7}$$

By Theorem 2.4 and since $\beta, x_0 \in \mathbb{R}^+_{\mathcal{F}}$ for any $\alpha_1 \le \alpha_2, \quad \alpha_1, \alpha_2 \in (0,1]$, we have

$$\begin{aligned} &0 < \underline{x_{0,\alpha_1}} \le \underline{x_{0,\alpha_2}} \le \overline{x_{0,\alpha_2}} \le \overline{x_{0,\alpha_1}}, \\ &0 < \underline{\beta_{\alpha_1}} \le \underline{\beta_{\alpha_2}} \le \overline{\beta_{\alpha_2}} \le \overline{\beta_{\alpha_1}}. \end{aligned} \tag{8}$$

We prove by induction that

$$0 < \underline{x_{n,\alpha_1}} \le \underline{x_{n,\alpha_2}} \le \overline{x_{n,\alpha_2}} \le \overline{x_{n,\alpha_1}}. \tag{9}$$

From (8) we see that (9) holds for $n = 0$. Suppose that (9) are true for $n \le k, \quad k \in \{1, 2, ...\}$. Then from (6), (8) and (9) it follows that

$$\begin{aligned} &\underline{x_{k+1,\alpha_1}} = \underline{\beta_{\alpha_1}}\underline{x_{k,\alpha_1}}(1 - \overline{x_{k,\alpha_1}}) \le \underline{\beta_{\alpha_2}}\underline{x_{k,\alpha_2}}(1 - \overline{x_{k,\alpha_2}}) = \underline{x_{k+1,\alpha_2}} \\ &\le \overline{\beta_{\alpha_2}}\overline{x_{k,\alpha_2}}(1 - \underline{x_{k,\alpha_2}}) = \overline{x_{k+1,\alpha_2}} \le \overline{\beta_{\alpha_1}}\overline{x_{k,\alpha_1}}(1 - \underline{x_{k,\alpha_1}}) = \overline{x_{k+1,\alpha_1}}. \end{aligned} \tag{10}$$

Therefore (9) are satisfied. Moreover from (7) we have,

$$\underline{x_{1,\alpha}} = \underline{\beta_\alpha}\underline{x_{0,\alpha}}(1 - \overline{x_{0,\alpha}}), \quad \overline{x_{1,\alpha}} = \overline{\beta_\alpha}\overline{x_{0,\alpha}}(1 - \underline{x_{0,\alpha}}), \quad \alpha \in (0,1]. \tag{11}$$

By Theorem 2.5, we see that $\underline{x_{0,\alpha}}, \overline{x_{0,\alpha}}$ are left continuous on $(0,1]$. So by (11) we see that $\underline{x_{1,\alpha}}, \overline{x_{1,\alpha}}$ are also left continuous. Also working inductively we can prove that $\underline{x_{n,\alpha}}, \overline{x_{n,\alpha}}$, $\quad n = 1,2,...$ are left continuous. By (10), $\underline{x_{n,\alpha}}$ is nondecreasing and $\overline{x_{n,\alpha}}$ is nonincreasing. Also by Theorem 2.5 $\underline{x_{n,\alpha}}, \overline{x_{n,\alpha}}$ are right continuous at 0. On the other hand by (9), we get $\underline{x_{n,1}} \leq \overline{x_{n,1}}$. Now we prove that supp$(x_n)$ is compact. It is sufficient to prove that $\bigcup_{\alpha\in(0,1]} [\underline{x_{n,\alpha}}, \overline{x_{n,\alpha}}]$ is bounded. Let $n = 1$. Since β and x_0 are positive fuzzy numbers and $\overline{\beta_\alpha}, \overline{x_{0,\alpha}} < 1$, there exist constants $0 < M_0, N_0, M_\beta, N_\beta \leq 1$ such that for all $\alpha \in (0,1]$,

$$\begin{aligned} &[\underline{x_{0,\alpha}}, \overline{x_{0,\alpha}}] \subset [M_0, N_0], \\ &[\underline{\beta_\alpha}, \overline{\beta_\alpha}] \subset [M_\beta, N_\beta]. \end{aligned} \tag{12}$$

Therefore from (11) and (12), we have

$$\bigcup_{\alpha\in(0,1]} [\underline{x_{1,\alpha}}, \overline{x_{1,\alpha}}] \subset [M_\beta M_0(1-N_0), N_\beta N_0(1-M_0)], \quad \alpha \in (0,1]. \tag{13}$$

Therefore (13) implies that $\bigcup_{\alpha\in(0,1]} [\underline{x_{1,\alpha}}, \overline{x_{1,\alpha}}] \subset (0,\infty)$ and is compact. By induction we can prove that $\bigcup_{\alpha\in(0,1]} [\underline{x_{n,\alpha}}, \overline{x_{n,\alpha}}]$ is compact and

$$\bigcup_{\alpha\in(0,1]} [\underline{x_{n,\alpha}}, \overline{x_{n,\alpha}}] \subset (0,\infty). \tag{14}$$

Therefore the conditions of Theorem 2.5 is holds. So $[\underline{x_{n,\alpha}}, \overline{x_{n,\alpha}}]$ determines a sequence of positive fuzzy numbers x_n such that (5) holds. Next, we prove the uniqueness of the solution. Suppose that there exists another solution $\bar{x}_n$ of (3) with initial data x_0. Then, arguing as above we can easily prove that

$$[\tilde{x}_n]_\alpha = [\underline{x_{n,\alpha}}, \overline{x_{n,\alpha}}], \quad \alpha \in (0,1], \quad n = 0,1,... \tag{15}$$

Then from (5) and (15), we get

$$[x_n]_\alpha = [\tilde{x}_n]_\alpha, \quad \alpha \in (0,1], \qquad n = 0,1,...,$$

i.e., $x_n = \tilde{x}_n, \quad n = 0,1,...$. Thus the proof is completed. □

Now, we study the existence, uniqueness and global behavior of solution for (4).

Lemma 3.2. *Let $\beta, x_0 \in \mathbb{R}^+_{\mathcal{F}}$ be such that $\overline{\beta_\alpha} < 1, \quad \overline{x_{0,\alpha}} \leq \frac{1}{2}, \quad \forall\alpha \in (0,1]$. Then the H-difference in* (4) *exists for any $n \geq 0$.*

Proof. For existence of H-difference in (4) we need to show that $\underline{\beta}(\underline{x_n} - \underline{x_n}^2) \leq \overline{\beta}(\overline{x_n} - \overline{x_n}^2)$ or equivalenyly $diam(\beta x_n{}^2) \leq diam(\beta x_n)$ and $\underline{x_{n,\alpha}}$ is nondecreasing and $\overline{x_{n,\alpha}}$ is nonincreasing in $\alpha \in (0,1]$, for $n = 0,1,...$ where $\underline{x_{n+1,\alpha}} = \underline{\beta_\alpha}\underline{x_{n,\alpha}} - \underline{\beta_\alpha}\underline{x^2_{n,\alpha}}$ and $\overline{x_{n+1,\alpha}} = \overline{\beta_\alpha}\overline{x_{n,\alpha}} - \overline{\beta_\alpha}\overline{x^2_{n,\alpha}}$. By induction it is easy to check that since $\underline{\beta_\alpha} \leq \overline{\beta_\alpha} < 1$ and $\underline{x_{0,\alpha}} \leq \overline{x_{0,\alpha}} \leq \frac{1}{2}$, we have

$$\underline{x_{n,\alpha}} \leq \overline{x_{n,\alpha}} \leq \frac{1}{2}, \quad \forall\alpha \in (0,1], \quad n \geq 0. \tag{16}$$

Therefore

$$\underline{x_{n,\alpha}} + \overline{x_{n,\alpha}} \le 1, \quad \forall \alpha \in (0,1]. \tag{17}$$

On the other hand, since $0 < \underline{\beta_\alpha} \le \overline{\beta_\alpha}$ and $\overline{x_{n,\alpha}} - \underline{x_{n,\alpha}} \ge 0$, by multiplication both sides of (17) in this value we obtain $\overline{x^2_{n,\alpha}} - \underline{x^2_{n,\alpha}} \le \overline{x_{n,\alpha}} - \underline{x_{n,\alpha}}$ or $\underline{x_{n,\alpha}} - \underline{x^2_{n,\alpha}} \le \overline{x_{n,\alpha}} - \overline{x^2_{n,\alpha}}$, so we have

$$\underline{\beta_\alpha}(\underline{x_{n,\alpha}} - \underline{x^2_{n,\alpha}}) \le \overline{\beta_\alpha}(\underline{x_{n,\alpha}} - \underline{x^2_{n,\alpha}}). \tag{18}$$

By inequality (18) we have $\overline{\beta_\alpha}\overline{x^2_{n,\alpha}} - \underline{\beta_\alpha}\underline{x^2_{n,\alpha}} \le \overline{\beta_\alpha}\overline{x_{n,\alpha}} - \underline{\beta_\alpha}\underline{x_{n,\alpha}}$. So

$$diam(\beta {x_n}^2) \le diam(\beta x_n). \tag{19}$$

Now, we show that $\underline{x_{n,\alpha}}$ is nondecreasing and $\overline{x_{n,\alpha}}$ is nonincreasing in $\alpha \in (0,1]$, i.e.,

$$\underline{x_{n,\alpha_1}} \le \underline{x_{n,\alpha_2}} \text{ and } \overline{x_{n,\alpha_2}} \le \overline{x_{n,\alpha_1}}, \quad \alpha_1 \le \alpha_2. \tag{20}$$

By Theorem 2.4 and since $\beta, x_0 \in \mathbb{R}^+_{\mathcal{F}}$, we have

$$\begin{aligned} &0 < \underline{x_{0,\alpha_1}} \le \underline{x_{0,\alpha_2}} \le \overline{x_{0,\alpha_2}} \le \overline{x_{0,\alpha_1}}, \\ &0 < \underline{\beta_{\alpha_1}} \le \underline{\beta_{\alpha_2}} \le \overline{\beta_{\alpha_2}} \le \overline{\beta_{\alpha_1}}. \end{aligned} \tag{21}$$

Then we see that (20) holds for $n = 0$. Suppose that (20) is valid for $n \le k, \quad k \in \{1, 2, ...\}$. Then from (16) it follows that $\underline{x_{k,\alpha_1}} + \underline{x_{k,\alpha_2}} \le 1$. By multiplication both sides in positive value $\underline{x_{k,\alpha_2}} - \underline{x_{k,\alpha_1}}$, we have $\underline{x^2_{k,\alpha_2}} - \underline{x^2_{k,\alpha_1}} \le \underline{x_{k,\alpha_2}} - \underline{x_{k,\alpha_1}}$. Now, since $0 < \underline{\beta_{\alpha_1}} \le \underline{\beta_{\alpha_2}}$, we obtain

$$\underline{x_{k+1,\alpha_1}} = \underline{\beta_{\alpha_1}}(\underline{x_{k,\alpha_1}} - \underline{x^2_{k,\alpha_1}}) \le \underline{\beta_{\alpha_2}}(\underline{x_{k,\alpha_2}} - \underline{x^2_{k,\alpha_2}}) = \underline{x_{k+1,\alpha_2}}. \tag{22}$$

Next we show that $\overline{x_{n,\alpha}}$ is nonincreasing. Since by (16) we have $\overline{x_{k,\alpha_1}} + \overline{x_{k,\alpha_2}} \le 1$, then by multiplication both sides in positive value $\overline{x_{k,\alpha_1}} - \overline{x_{k,\alpha_2}}$, we obtain $\overline{x^2_{k,\alpha_1}} - \overline{x^2_{k,\alpha_2}} \le \overline{x_{k,\alpha_1}} - \overline{x_{k,\alpha_2}}$. On the other hand, since $0 < \overline{\beta_{\alpha_2}} \le \overline{\beta_{\alpha_1}}$, we conclude that

$$\overline{x_{k+1,\alpha_2}} = \overline{\beta_{\alpha_2}}(\overline{x_{k,\alpha_2}} - \overline{x^2_{k,\alpha_2}}) \le \overline{\beta_{\alpha_1}}(\overline{x_{k,\alpha_1}} - \overline{x^2_{k,\alpha_1}}) = \overline{x_{k+1,\alpha_1}}. \tag{23}$$

□

Remark 3.3. Using (19) in Lemma 3.2, it is easy to see that $\underline{x_{n,\alpha}} \le \overline{x_{n,\alpha}}, \quad \forall \alpha \in (0,1]$. Indeed, since $\underline{x_{k,\alpha}} + \overline{x_{k,\alpha}} \le 1$ and $\overline{x_{k,\alpha}} - \underline{x_{k,\alpha}} \ge 0$ then

$$\overline{x^2_{k,\alpha}} - \underline{x^2_{k,\alpha}} \le \overline{x_{k,\alpha}} - \underline{x_{k,\alpha}}.$$

Therefore using equation $0 < \underline{\beta_\alpha} \le \overline{\beta_\alpha}$, we get

$$\underline{x_{k+1,\alpha}} = \underline{\beta_\alpha}(\underline{x_{k,\alpha}} - \underline{x^2_{k,\alpha}}) \le \overline{\beta_\alpha}(\overline{x_{k,\alpha}} - \overline{x^2_{k,\alpha}}) = \overline{x_{k+1,\alpha}}. \tag{24}$$

Proposition 3.4. *Consider* (4) *where* x_n *is a sequence of positive fuzzy numbers and* $\beta, x_0 \in \mathbb{R}^+_{\mathcal{F}}$ *such that* $\overline{\beta_\alpha} < 1, \quad \overline{x}_{0,\alpha} \le \frac{1}{2}, \quad \forall \alpha \in (0,1]$. *Then for every positive fuzzy number* x_0*, there exists a unique positive solution* x_n *of* (4)*.*

Proof. Suppose that there exists a sequence of positive fuzzy numbers x_n satisfying (4) with initial data x_0. Consider the $\alpha-$cuts of x_n as

$$[x_n]_\alpha = [\underline{x_\alpha}, \overline{x_\alpha}], \quad n = 0, 1, ..., \ \alpha \in (0, 1]. \tag{25}$$

Then from (4) and (25) it follows that

$$[x_{n+1}]_\alpha = [\underline{x_{n+1,\alpha}}, \overline{x_{n+1,\alpha}}] = [\underline{\beta_\alpha}\underline{x_{n,\alpha}} - \underline{\beta_\alpha}\underline{x^2_{n,\alpha}}, \overline{\beta_\alpha}\overline{x_{n,\alpha}} - \overline{\beta_\alpha}\overline{x^2_{n,\alpha}}]. \tag{26}$$

Therefore for $n = 0, 1, ...,$ we have

$$\underline{x_{n+1,\alpha}} = \underline{\beta_\alpha}\underline{x_{n,\alpha}} - \underline{\beta_\alpha}\underline{x^2_{n,\alpha}}, \quad \overline{x_{n+1,\alpha}} = \overline{\beta_\alpha}\overline{x_{n,\alpha}} - \underline{\beta_\alpha}\overline{x^2_{n,\alpha}}, \quad \alpha \in (0, 1]. \tag{27}$$

Then by Lemma 3.2 and Remark 3.3 for any $\alpha_1, \alpha_2 \in (0, 1], \quad \alpha_1 \leq \alpha_2$, we have

$$\underline{x_{n,\alpha_1}} \leq \underline{x_{n,\alpha_2}} \leq \overline{x_{n,\alpha_2}} \leq \overline{x_{n,\alpha_1}}, \ n = 0, 1, ... \tag{28}$$

Moreover from (27) we obtain

$$\underline{x_{1,\alpha}} = \underline{\beta_\alpha}\underline{x_{0,\alpha}} - \underline{\beta_\alpha}\underline{x^2_{0,\alpha}}, \quad \overline{x_{1,\alpha}} = \overline{\beta_\alpha}\overline{x_{0,\alpha}} - \overline{\beta_\alpha}\overline{x^2_{0,\alpha}}, \quad \alpha \in (0, 1]. \tag{29}$$

By Theorem 2.5, $\underline{x_{0,\alpha}}, \overline{x_{0,\alpha}}$ are left continuous on $(0, 1]$. So by (29) we see that $\underline{x_{1,\alpha}}, \overline{x_{1,\alpha}}$ are also left continuous. Working inductively we can prove that $\underline{x_{n,\alpha}}, \overline{x_{n,\alpha}}$, $n = 1, 2, ...$ are left continuous. By (28), $\underline{x_{n,\alpha}}$ is nondecreasing and $\overline{x_{n,\alpha}}$ is nonincreasing. By Theorem 2.5, $\underline{x_{n,\alpha}}, \overline{x_{n,\alpha}}$ are right continuous at 0. On the other hand by (28), we get $\underline{x_{n,1}} < \overline{x_{n,1}}$. It is easy to see that *supp*(x_n) is compact, i.e., $\bigcup_{\alpha\in(0,1]} [\underline{x_{n,\alpha}}, \overline{x_{n,\alpha}}]$ is bounded. Indeed, it is a direct consequence of (16).
Therefore the conditions of Theorem 2.5 is holds. So $[\underline{x_{n,\alpha}}, \overline{x_{n,\alpha}}]$ determines the $\alpha-$cuts of a sequence of positive fuzzy numbers x_n. Next, we prove the uniqueness of the solution. Suppose that there exists another solution $\bar{x}_n$ of (4) with initial data x_0. Then, arguing as above we can easily prove that

$$[\tilde{x}_n]_\alpha = [\underline{x_{n,\alpha}}, \overline{x_{n,\alpha}}], \quad \alpha \in (0, 1], \quad n = 0, 1, ... \tag{30}$$

Then from (25) and (30), we get

$$[x_n]_\alpha = [\tilde{x}_n]_\alpha, \ \ \alpha \in (0, 1], \qquad n = 0, 1, ...,$$

i.e., $x_n = \tilde{x}_n, \ \ n = 0, 1, ...$. Thus the proof is completed. □

Proposition 3.5. *Consider* (3) *where* $\beta, x_0 \in \mathbb{R}^+_\mathcal{F}$ *such that* $\overline{x_{0,\alpha}}, \overline{\beta_\alpha} < 1, \ \forall\alpha \in (0, 1]$*, then every positive solution of* (3) *is bounded and persists.*

Proof. Let x_n be a positive solution of (3). Since $\overline{x_{0,\alpha}}, \overline{\beta_\alpha} < 1, \ \ \forall\alpha \in (0, 1]$ and $0 < M_0, N_0, M_\beta, N_\beta \leq 1$, by (13) we obtain $[\underline{x_{n,\alpha}}, \overline{x_{n,\alpha}}] \subset [0, 1]$, $\forall\alpha \in (0, 1]$. Therefore $\bigcup_{\alpha\in(0,1]} [\underline{x_{n,\alpha}}, \overline{x_{n,\alpha}}] \subset [0, 1]$. Thus positive solution is bounded and persists. □

Similarly, we have the following result.

Proposition 3.6. *Consider* (4)*, where* β *and* x_0 *satisfy in the hypothesis of Lemma 3.2. Then every positive solution of* (4) *is bounded and persists.*

Definition 3.7. We say that fuzzy number x is a equilibrium for (3), if $x = \beta x(1-x)$. Similarly, the fuzzy number x is called a equilibrium for (4) if $x = \beta x \ominus \beta x^2$, provided the H-difference exists.

Theorem 3.8. *Consider* (3) *where* $\beta, x_0 \in \mathbb{R}^+_{\mathcal{F}}$ *such that* $\overline{\beta_\alpha}, \overline{x_{0,\alpha}} < 1$. *Then the following statements are true.*
(i) (3) *has a unique equilibrium point.*
(ii) Every positive solution x_n *of* (3) *converges to the unique equilibrium* x *with respect to* D *as* $n \to \infty$.

Proof. (i) By Definition 3.7, if x is a equilibrium point of (3), then

$$\underline{x_\alpha} = \underline{\beta_\alpha}\underline{x_\alpha}(1-\overline{x_\alpha}), \quad \overline{x_\alpha} = \overline{\beta_\alpha}\overline{x_\alpha}(1-\underline{x_\alpha}). \tag{31}$$

The system (31) has two solutions $\underline{x_\alpha} = \dfrac{\overline{\beta_\alpha}-1}{\overline{\beta_\alpha}}$, $\overline{x_\alpha} = \dfrac{\underline{\beta_\alpha}-1}{\underline{\beta_\alpha}}]$ and $\underline{x_\alpha} = \overline{x_\alpha} = 0$. It is easy to see that $[\dfrac{\overline{\beta_\alpha}-1}{\overline{\beta_\alpha}}, \dfrac{\underline{\beta_\alpha}-1}{\underline{\beta_\alpha}}]$ is not the valid $\alpha-$cuts of a fuzzy number. So $[\underline{x_\alpha}, \overline{x_\alpha}] = [0,0]$ is an equilibrium point of (3). For uniqueness, let there exists another equilibrium point $\hat{x} \in \mathbb{R}_{\mathcal{F}}$ for (3). Then $\underline{\hat{x}_\alpha} = \underline{\beta_\alpha}\underline{\hat{x}_\alpha}(1-\overline{\hat{x}_\alpha})$, $\overline{\hat{x}_\alpha} = \overline{\beta}_\alpha\overline{\hat{x}_\alpha}(1-\underline{\hat{x}_\alpha})$, $\alpha \in (0,1]$. So we have $\underline{x_\alpha} = \underline{\hat{x}_\alpha} = 0$, $\overline{x_\alpha} = \overline{\hat{x}_\alpha} = 0$, $\alpha \in (0,1]$.
(ii) From (7) and since $\overline{\beta}, \overline{x_{0,\alpha}} < 1$, we have $\lim_{n\to\infty} \underline{x_{n,\alpha}} = \underline{x_\alpha} = 0$ and $\lim_{n\to\infty} \overline{x_{n,\alpha}} = \overline{x_\alpha} = 0$. So we have

$$\lim_{n\to\infty} D(x_n, x) = \lim_{n\to\infty} \sup \max \left\{ \left| \underline{x_{n,\alpha}} - \underline{x_\alpha} \right|, |\overline{x_{n,\alpha}} - \overline{x_\alpha}| \right\} = 0.$$

This completes the proof. □

Theorem 3.9. *Consider* (4) *where* β *and the initial values* x_0 *satisfy the hypothesis of Lemma 3.2. Then the following statements are true.*
(i) (4) *has a unique equilibrium.*
(ii) Every positive solution x_n *of* (4) *converges to the unique equilibrium* x *with respect to* D *as* $n \to \infty$.

Proof. By Definition 3.7, if x is a equilibrium point of (4), then we obtain

$$\underline{x_\alpha} = \underline{\beta_\alpha}\underline{x_\alpha} - \underline{\beta_\alpha}\underline{x_\alpha}^2, \quad \overline{x_\alpha} = \overline{\beta_\alpha}\overline{x_\alpha} - \overline{\beta}_\alpha\overline{x_\alpha}^2. \tag{32}$$

The system (32) has two solutions $\underline{x_\alpha} = \dfrac{\underline{\beta_\alpha}-1}{\underline{\beta_\alpha}}$, $\overline{x_\alpha} = \dfrac{\overline{\beta_\alpha}-1}{\overline{\beta_\alpha}}$ and $\underline{x_\alpha} = \overline{x_\alpha} = 0$. It is easy to see that $[\dfrac{\underline{\beta_\alpha}-1}{\underline{\beta_\alpha}}, \dfrac{\overline{\beta_\alpha}-1}{\overline{\beta_\alpha}}]$ is not the valid $\alpha-$cuts of a fuzzy number. So $[\underline{x_\alpha}, \overline{x_\alpha}] = [0,0]$ is an equilibrium point of (4). Proof of the uniqueness is similar to the previous case.
(ii) From (26) and since $\overline{\beta_\alpha} < 1$ and $\underline{x_{n,\alpha}} \le \overline{x_{n,\alpha}} \le \dfrac{1}{2}$, we see $\lim_{n\to\infty} \underline{x_{n,\alpha}} = \underline{x_\alpha} = 0$ and $\lim_{n\to\infty} \overline{x_{n,\alpha}} = \overline{x_\alpha} = 0$. So we have

$$\lim_{n\to\infty} D(x_n, x) = \lim_{n\to\infty} \sup \max \left\{ \left| \underline{x_{n,\alpha}} - \underline{x_\alpha} \right|, |\overline{x_{n,\alpha}} - \overline{x_\alpha}| \right\} = 0.$$

This completes the proof. □

4. **Examples**

In this section, we present some examples to illustrate our results. We denote the trapezoidal fuzzy number u with $[u]_\alpha = [a+\alpha(b-a), d-\alpha(d-c)]$ by the quadruple $u = (a, b, c, d) \in \mathbb{R}^4$, $a \le b \le c \le d$. If we have $b = c$, then the fuzzy number u is called a triangular fuzzy number. Then a triplet $(a, b, c) \in \mathbb{R}^3, a \le b \le c$ represents a triangular fuzzy number. We plot the left and right branches of solution x_n for $\alpha = 0, 1$ namely $(\underline{x_n}_0, \overline{x_n}_0) = (L_{n,0}, , R_{n,0})$ and $(\underline{x_n}_1, \overline{x_n}_1,) = (L_{n,1}, R_{n,1})$.

Example 4.1. Consider the fuzzy difference equation (3) where β and initial value x_0 are triangular fuzzy numbers with membership functions as

$$\beta(x) = \begin{cases} 10x - 3, & 0.3 \le x \le 0.4, \\ -5x + 3, & 0.4 < x \le 0.6, \end{cases} \quad x_0 = \begin{cases} 5x - 0.5, & 0.1 \le x \le 0.3, \\ -5x + 2.5, & 0.3 < x \le 0.5, \end{cases}$$

Then we have $[\beta]_\alpha = [0.1\alpha + 0.3, 0.6 - 0.2\alpha]$, $[x_0]_\alpha = [0.2\alpha + 0.1, 0.5 - 0.2\alpha]$, $\alpha \in [0, 1]$. By Propositions 3.1 and 3.5, there exists a unique solution and it is bounded and persists. By Theorem 3.8 it has a unique equilibrium point $x = 0$ and every positive solution x_n converges to the equilibrium x (See Figure 1).

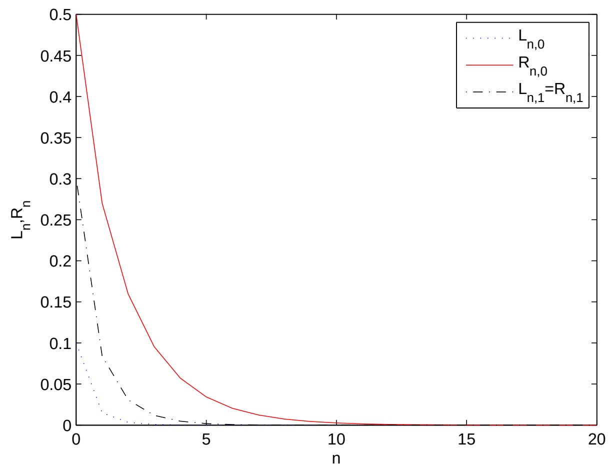

Figure 1. The Solution of Example 4.1

Example 4.2. Consider the fuzzy difference equation (3) where β and initial value x_0 are triangular fuzzy numbers with membership functions as

$$\beta(x) = \begin{cases} 10x - 5, & 0.5 \le x \le 0.6, \\ -5x + 4, & 0.6 < x \le 0.8, \end{cases} \quad x_0 = \begin{cases} 10x - 2, & 0.2 \le x \le 0.3, \\ -\dfrac{10x}{3} + 2, & 0.3 < x \le 0.6, \end{cases}$$

Then we have $[\beta]_\alpha = [0.1\alpha + 0.5, 0.8 - 0.2\alpha]$, $[x_0]_\alpha = [0.1\alpha + 0.2, 0.6 - 0.3\alpha]$, $\alpha \in [0, 1]$. By Propositions 3.1 and 3.5, there exists a unique solution and it is bounded and persists. By Theorem 3.8 it has a unique equilibrium $x = 0$ and every positive solution x_n converges to the equilibrium x (See Figure 2).

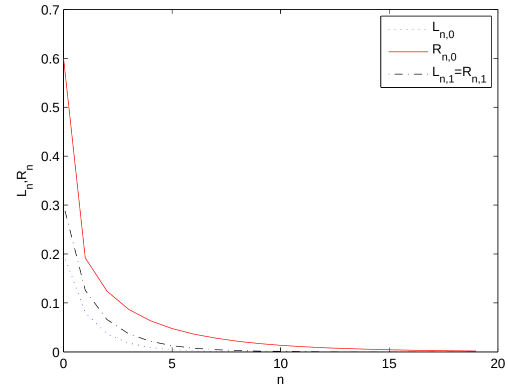

Figure 2. The Solution of Example 4.2

Example 4.3. Consider the fuzzy difference equation (4) where β and initial value x_0 are triangular fuzzy numbers with membership functions as

$$\beta(x) = \begin{cases} 10x - 3, & 0.3 \le x \le 0.4, \\ -5x + 3, & 0.4 < x \le 0.6, \end{cases} \quad x_0 = \begin{cases} 5x - 0.5, & 0.1 \le x \le 0.3, \\ -5x + 2.5, & 0.3 < x \le 0.5, \end{cases}$$

Then we have $[\beta]_\alpha = [0.1\alpha + 0.3, 0.6 - 0.2\alpha]$, $[x_0]_\alpha = [0.2\alpha + 0.1, 0.5 - 0.2\alpha]$, $\alpha \in [0, 1]$. By Propositions 3.4 and 3.6, there exists a unique solution and it is bounded and persists. By Theorem 3.9 it has a unique equilibrium $x = 0$ and every positive solution x_n converges to the equilibrium x (See Figure 3).

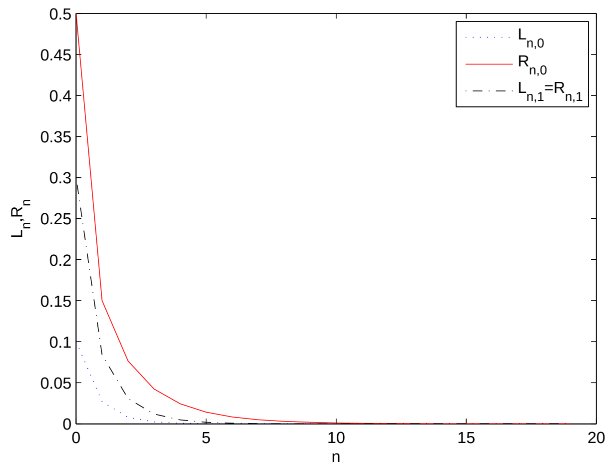

Figure 3. The Solution of Example 4.3

The following example shows that if the conditions of Lemma 3.2 are not satisfied, then the fuzzy solution may not exist.

Example 4.4. Consider the fuzzy difference equation (4) where β and initial value x_0 are triangular fuzzy numbers with membership functions as
$$\beta(x) = \begin{cases} 112x - 14, & 0.125 \leq x \leq 0.1339, \\ -112x + 16, & 0.1339 < x \leq 0.1429, \end{cases} \quad x_0 = \begin{cases} 8x - 4, & 0.5 \leq x \leq 0.625, \\ -8x + 6, & 0.625 < x \leq 0.75. \end{cases}$$
Then we have $[\beta]_\alpha = [0.0089\alpha + 0.125, 0.1429 - 0.0089\alpha]$, $[x_0]_\alpha = [0.125\alpha + 0.5, 0.75 - 0.125\alpha]$, $\alpha \in [0, 1]$. It is easy to see that the $\alpha-$cuts of x_n do not define a valid fuzzy number for any $n > 1$. (See Figure 4).

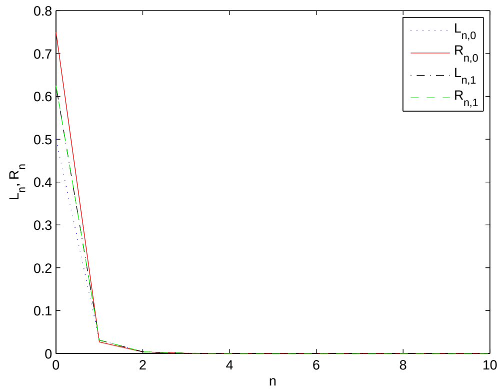

Figure 4. The Solution of Example 4.4

Acknowledgements. The author is grateful to the Editor in Chief, Editor and the anonymous Reviewers for their interesting and valuable comments.

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