ADMISSIBLE PARTITION FOR BL-GENERAL FUZZY AUTOMATON

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Abstract. In this note, we define the concepts of admissible relation and admissible partition for an arbitrary BL-general fuzzy automaton. In particular, a connection between the admissible partition and the quotient BL-general fuzzy automaton is presented. It is shown that if we use the maximal admissible partition, then we obtain a quotient BL-general fuzzy automaton and this quotient is minimal. Finally, we present some examples to clarify the notions and results of this paper.

1. Introduction

Theory of fuzzy automaton was introduced by Wee [23]. Thereafter, there were a considerable number of authors, such as Mordeson and Malik [9, 10], Topencharov and Peeva [21], Qiu et. al. and others having contributed to this field [7, 15, 16, 17, 18]. Fuzzy finite automaton has many important applications such as in learning system, pattern recognition, neural networks, database theory and fuzzy discrete event systems [2, 4, 5, 9, 11, 12, 19, 24]. Basic logic (BL) has been introduced by Hajek [6] in order to provide a general framework for formalizing statements of fuzzy nature. By considering the notions of BL-algebra and residuated lattice, every BL-algebra is a residuated lattice. The supervisory control of fuzzy discrete event systems was established first by Qiu [19].

The idea of studying fuzzy automaton with membership values in some structured abstract set comes back to Wechler [22], and in recent years, researcher's attention has been aimed mostly to fuzzy automaton with membership values in complete residuated lattices, lattice-ordered monoids, and other kinds of lattices. Fuzzy automaton taking membership values in a complete residuated lattice were first studied by Qiu in [15, 16], where some basic concepts were discussed, and later, Qiu and his coworkers have carried out extensive research of these fuzzy automata(cf. [17, 18, 25, 27, 26, 28, 29]). State reduction and equivalence of fuzzy automaton were studied by [8, 13, 14, 18, 25].

In 2004, Doostfatemeh and Kremer [3] extended the notion of the fuzzy automaton and gave the notion of the general fuzzy automaton. Their key motivation

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of introducing the notion of general fuzzy automaton was the insufficiency of the current literature to handle the applications which rely on fuzzy automaton as a modeling tool, assigning membership values to active states of a fuzzy automaton, resolve the multi -membership.

In 2012, Abolpour and Zahedi [1] extended the notion of the general fuzzy automaton and gave the notion of the BL-general fuzzy automaton (BL-GFA).

State reduction and equivalence of fuzzy automaton were studied by [8, 13, 14, 18, 25]. In this paper, we show that this methodology can be applied in a similar form to the BL-general fuzzy automaton. Nowadays, they are widely employed in the computer science, particularly in object-oriented languages, functional languages, verification tools, data types, domains, databases, program analysis, etc.

Now, in this paper, we follow the above publish papers and construct the quotient of a given BL-GFA, then we show that this quotient BL-GFA is a minimal BL-GFA.

2. **Preliminaries**

In this section, we review some notions which are needed in the next section.

Definition 2.1. [3] A general fuzzy automaton (GFA) $\tilde{F}$ is an eight-tuple machine denoted by $\tilde{F} = (Q, X, \tilde{R}, Z, \tilde{\delta}, \omega, F_1, F_2)$, where

- Q is a finite set of states,
- X is a finite set of input symbols,
- $\tilde{R}$ is the set of fuzzy start states, $\tilde{R} \subseteq \tilde{P}(Q)$,
- Z is a finite set of output symbols,
- $\tilde{\delta} : (Q \times [0, 1]) \times X \times Q \to [0, 1]$ is the augmented transition function,
- $\omega : Q \to Z$ is the output function,
- $F_1 : [0, 1] \times [0, 1] \to [0, 1]$ is called the membership assignment function.
$$\mu^{t+1}(q_j) = \tilde{\delta}((q_i, \mu^t(q_i)), a_k, q_j) = F_1(\mu^t(q_i), \delta(q_i, a_k, q_j)).$$
- $F_2 : [0, 1]^* \to [0, 1]$ is called the multi-membership resolution function.

Let Δ be the set of all transition of $\tilde{F}$. Now, suppose that $Q_{act}(t_i)$ be the set of all active state at time t_i, for all $i \geq 0$.

Definition 2.2. [6] A BL-algebra is an algebra $(L, \wedge, \vee, *, \to, 0, 1)$ with four binary operations $\wedge, \vee, *, \to$ and two constants $0, 1$ such that: (i) $(L, \wedge, \vee, 0, 1)$ is a bounded lattice, (ii) $(L, *, 1)$ is a commutative monoid, (iii) $*$ and $\to$ form an adjoint pair, i.e., $x \leq y \to z$ if and only if $x * y \leq z$ for all $x, y, z \in L$, (iv) $x \wedge y = x * (x \to y)$, (v) $(x \to y) \vee (y \to x) = 1$.

Definition 2.3. [20] Let $L = (L, \vee, \wedge, 0, 1)$ be a bounded complete lattice. A BL-general fuzzy automaton (BL-GFA) as a ten-tuple machine denoted by $\tilde{F}_l = (\bar{Q}, X, \tilde{R}, \bar{Z}, \omega_l, \delta_l, f_l, \tilde{\delta}_l, F_1, F_2)$, where

(i) $\bar{Q} = P(Q)$, where Q is a finite set and $\bar{Q}$ is the power set of Q,
(ii) X is a finite set of input symbols,
(iii) $\tilde{R}$ is the set of fuzzy start states,
(iv) $\bar{Z}$ is a finite set of output symbols, where $\bar{Z}$ is the power set of Z,

(v) $\omega_l : \bar{Q} \to \bar{Z}$ is the output function defined by: $\omega_l(Q_i) = \{\omega(q) \big| q \in Q_i\}$,

(vi) $\delta_l : \bar{Q} \times X \times \bar{Q} \to L$ is the transition function defined by: $\delta_l(\{p\}, a, \{q\}) = \delta(p, a, q)$ and $\delta_l(Q_i, a, Q_j) = \vee_{q_i \in Q_i, q_j \in Q_j} \delta(q_i, a, q_j)$, for all $Q_i, Q_j \in P(Q)$ and $a \in X$,

(vii) $f_l : \bar{Q} \times X \to \bar{Q}$ is the next state map defined by: $f_l(Q_i, a) = \cup_{q_i \in Q_i} \{q_j \big| \delta(q_i, a, q_j) \in \Delta\}$,

(viii) $\tilde{\delta}_l : (\bar{Q} \times L) \times X \times \bar{Q} \to L$ is the augmented transition function defined $\tilde{\delta}_l((Q_i, \mu^t(Q_i)), a, Q_j) = F_1(\mu^t(Q_i), \delta_l(Q_i, a, Q_j))$,

(ix) $F_1 : L \times L \to L$ is called membership assignment function,

(x) $F_2 : L^* \to L$ is called multi-membership resolution function.

Definition 2.4. [1] Let $\tilde{F}_l = (\bar{Q}, X, (\{q_0\}, \mu^{t_0}(\{q_0\})), \bar{Z}, \omega_l, \delta_l, f_l, \tilde{\delta}_l, F_1, F_2)$ be a BL-GFA. The run map of the BL-GFA $\tilde{F}_l$ is the map $\rho : X^* \to \bar{Q}$ defined by the following induction: $\rho(\Lambda) = \{q_0\}$ and $\rho(a_1a_2...a_n) = Q_{i_n}, \rho(a_1a_2...a_na_{n+1}) = f_l(Q_{i_n}, a_{n+1})$, where $(Q_{i_n}, \mu^{t_0+n}(Q_{i_n})) \in Q_{act}(a_1a_2...a_n)$, for every $a_1, ..., a_n \in X$.

Definition 2.5. Let Q be a nonempty set and let H_1 and H_2 be partition of Q. Then $H_1 \leq H_2$, if for every $A \in H_1$, there exists $B \in H_2$ such that $A \subseteq B$.

In the rest of this note, F_1 always denotes a t-norm and F_2 always denotes a t-conorm. Also, $\tilde{F}_l$ always denotes a BL-GFA as follows: $\tilde{F}_l = (\bar{Q}, X, (\{q_0\}, \mu^{t_0}(\{q_0\})), \bar{Z}, \omega_l, \delta_l, f_l, \tilde{\delta}_l, F_1, F_2)$.

3. Admissible Partition for BL-general Fuzzy Automaton

In this section, first, we give the definitions of admissible partition and admissible relation for a BL-general fuzzy automaton. After that by using the definition of maximal admissible partition, we obtain a quotient BL-general fuzzy automaton and then we show that this quotient is a minimal BL-general fuzzy automaton. Finally, some examples are given to clarify these new notions and results.

Definition 3.1. Let $\tilde{F}_l = (\bar{Q}, X, (\{q_0\}, \mu^{t_0}(\{q_0\})), \bar{Z}, \omega_l, \delta_l, f_l, \tilde{\delta}_l, F_1, F_2)$ be a BL-GFA and let $\sim$ be an equivalence relation on $\bar{Q}$. Then $\sim$ is called an admissible relation on $\bar{Q}$ if the following hold:

(i) if $Q', Q'' \in Q_{act}(t_i), a \in X \cup \{\Lambda\}, P' \in \bar{Q}, Q' \sim Q''$ and $\tilde{\delta}_l((Q', \mu^{t_i}(Q')), a, P') > 0$, then there exists $P'' \in \bar{Q}$ such that

$$\tilde{\delta}_l((Q'', \mu^{t_i}(Q'')), a, P'') \geq \tilde{\delta}_l((Q', \mu^{t_i}(Q')), a, P'),$$

and $P' \sim P''$,

(ii) if $Q' \sim Q''$, then $\omega_l(Q') = \omega_l(Q'')$.

Theorem 3.2. *Let $\tilde{F}_l$ be a BL-GFA and let $\sim$ be an equivalence relation on $\bar{Q}$. Then $\sim$ is an admissible relation on $\bar{Q}$ if and only if the following hold:*

(i) if $Q', Q'' \in Q_{act}(t_i), x \in X^*, P' \in \bar{Q}, Q' \sim Q''$ and $\tilde{\delta}^*_l((Q', \mu^{t_i}(Q')), x, P') > 0$, then there exists $P'' \in \bar{Q}$ such that $\tilde{\delta}^*_l((Q'', \mu^{t_i}(Q'')), x, P'') \geq \tilde{\delta}^*_l((Q', \mu^{t_i}(Q')), x, P')$ and $P' \sim P''$,

(ii) if $Q' \sim Q''$, then $\omega_l(Q') = \omega_l(Q'')$.

Proof. First, let $\sim$ be an admissible relation on $\bar{Q}$. We prove the claim by induction on n. For $n = 1$, the claim is clear by Definition 3.1. Now, suppose (i) holds for any $x \in X^*$, in which $1 \leq |x| < n$. If $|x| = n$, $Q' \sim Q''$ and $\tilde{\delta}_l^*((Q', \mu^{t_i}(Q')), x, P') > 0$, for some $P' \in \bar{Q}$, then $x = ya$, where $|y| = n - 1, y \in X^*, a \in X, n > 1$ and $\tilde{\delta}_l^*((Q', \mu^{t_i}(Q')), x, P) > 0$. So, for some $R' \in \bar{Q}$, we have

$$\tilde{\delta}_l^*((Q', \mu^{t_i}(Q')), x, P) = \tilde{\delta}_l^*((Q', \mu^{t_i}(Q')), y, R') \wedge \tilde{\delta}_l^*((R', \mu^{t_{i+n-1}}(R')), a, P) > 0.$$

Then, there exists $R'' \in \bar{Q}$ such that $\tilde{\delta}_l^*((Q'', \mu^{t_i}(Q'')), y, R'') \geq \tilde{\delta}_l^*((Q', \mu^{t_i}(Q')), y, R')$ and $R'' \sim R'$. Also, by Definition 3.1, $\tilde{\delta}_l^*((R', \mu^{t_{i+n-1}}(R')), a, P) > 0$ implies that $\tilde{\delta}_l^*((R'', \mu^{t_{i+n-1}}(R'')), a, P'') \geq \tilde{\delta}_l^*((R', \mu^{t_{i+n-1}}(R')), a,\ P')$. Therefore,

$$\begin{aligned}\tilde{\delta}_l^*((Q'', \mu^{t_i}(Q'')), x, P'') &\geq \tilde{\delta}_l^*((Q'', \mu^{t_i}(Q'')), y, R'') \wedge \tilde{\delta}_l^*((R'', \mu^{t_{i+n-1}}(R'')), a, P'')\\ &\geq \tilde{\delta}_l^*((Q', \mu^{t_i}(Q')), y, R') \wedge \tilde{\delta}_l^*((R', \mu^{t_{i+n-1}}(R')), a, P')\\ &= \tilde{\delta}_l^*((Q', \mu^{t_i}(Q')), ya, P').\end{aligned}$$

Hence, the claim holds. The converse is trivial. □

Definition 3.3. Let $\tilde{F}_l$ be a BL-GFA and let $H = \{Q_1, Q_2, ..., Q_k\}$ be a partition of $\bar{Q}$. Then H is called an admissible partition of $\bar{Q}$ if the following hold:

(i) if $a \in X \cup \{\Lambda\}$, then for every l_1 there exists l_2, where $1 \leq l_1, l_2 \leq k$. Let $P_1, P_2 \in Q_{l_1}$. If $\tilde{\delta}_l((P_1, \mu^{t_i}(P_1)), a,\ R_1) > 0$, for some $R_1 \in \bar{Q}$, then there exists $R_2 \in \bar{Q}$ such that $\tilde{\delta}_l((P_2, \mu^{t_i}(P_2)), a, R_2) \geq \tilde{\delta}_l((P_1, \mu^{t_i}(P_1)), a,\ R_1)$, where $R_1, R_2 \in Q_{l_2}$.

(ii) if $Q', Q'' \in Q_l$, where $1 \leq l \leq k$, then $\omega_l(Q') = \omega_l(Q'')$.

Corollary 3.4. *Let $\tilde{F}_l$ be a BL-GFA. Then $1_{\bar{Q}}$ is an admissible partitions of $\bar{Q}$.*

Proof. Let $a \in X \cup \{\Lambda\}$ and $P_1, P_2 \in Q_{l_1} \in 1_{\bar{Q}}$. Since $1_{\bar{Q}} = \{P | P \in \bar{Q}\}$, then $P_1 = P_2$. Clearly, if $\tilde{\delta}_l((P_1, \mu^{t_i}(P_1)), a,\ R_1) > 0$, for some $R_1 \in \bar{Q}$, then there exists $R_2 = R_1 \in \bar{Q}$ such that $\tilde{\delta}_l((P_2, \mu^{t_i}(P_2)), a, R_2) \geq \tilde{\delta}_l((P_1, \mu^{t_i}(P_1)), a,\ R_1)$, where $R_1 = R_2 \in Q_{l_2} \in 1_{\bar{Q}}$. Also, we have $\omega_l(P_1) = \omega_l(P_2)$. □

Theorem 3.5. *Let $\tilde{F}_l$ be a BL-GFA and $H = \{Q_1, Q_2, ..., Q_k\}$ be a partition of $\bar{Q}$. Then H is an admissible partition of $\bar{Q}$ if and only if the following hold:*

(i) Let $x \in X^*$. Then for every l_1 there exists l_2, where $1 \leq l_1, l_2 \leq k$. Let $P_1, P_2 \in Q_{l_1}$. If $\tilde{\delta}_l^*((P_1, \mu^{t_i}(P_1)), x,\ R_1) > 0$, for some $R_1 \in \bar{Q}$, then there is $R_2 \in \bar{Q}$ such that $\tilde{\delta}_l^*((P_2, \mu^{t_i}(P_2)), x, R_2) \geq \tilde{\delta}_l^*((P_1, \mu^{t_i}(P_1)), x, R_1)$ and $R_1, R_2 \in Q_{l_2}$.

(ii) Let $Q', Q'' \in Q_l$, where $1 \leq l \leq k$. Then $\omega_l(Q') = \omega_l(Q'')$.

Proof. The proof is straightforward and similar to the proof of Theorem 3.2. □

Example 3.6. Let $(L, \wedge, \vee, 0, 1)$ be a complete lattice as in Figure 1. Consider BL-general fuzzy automaton $\tilde{F}_l = (\bar{Q}, X, (\{q_1\}, \mu^{t_0}(\{q_1\})), \bar{Z}, \omega_l, \delta_l, f_l, \tilde{\delta}_l, F_1, F_2)$, where $\bar{Q} = \{\emptyset, \{q_0\}, \{q_1\}, \{q_2\}, \{q_0, q_1\}, \{q_0, q_2\}, \{q_1, q_2\}, \{q_0, q_1, q_2\}\}$, $\bar{Z} = \{\emptyset, \{z_1\}, \{z_2\}, \{z_1, z_2\}\}$,

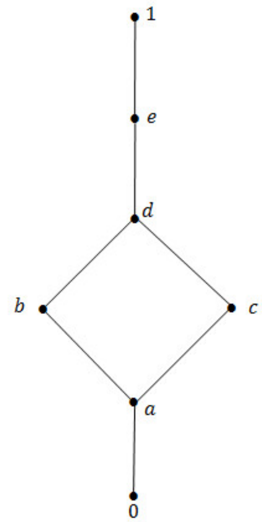

Figure 1. The Complete Lattice L of Example 3.6

$\omega_l(\{q_0\}) = \{z_1\}, \omega_l(\{q_1\}) = \{z_1\}, \omega_l(\{q_2\}) = \{z_2\}, \omega_l(\{q_0, q_1\}) = \{z_1\}, \omega_l(\{q_0, q_2\}) = \{z_1, z_2\}, \omega_l(\{q_1, q_2\}) = \{z_1, z_2\}, \omega_l(\{q_0, q_1, q_2\}) = \{z_1, z_2\}$, and

$\delta_l(\{q_0\}, \sigma, \{q_1\}) = b,$ $\delta_l(\{q_0\}, \sigma, \{q_0, q_1\}) = b,$

$\delta_l(\{q_0\}, \sigma, \{q_1, q_2\}) = b,$ $\delta_l(\{q_0\}, \sigma, \{q_0, q_1, q_2\}) = b,$

$\delta_l(\{q_1\}, \sigma, \{q_0\}) = a,$ $\delta_l(\{q_1\}, \sigma, \{q_0, q_1\}) = a,$

$\delta_l(\{q_1\}, \sigma, \{q_0, q_2\}) = a,$ $\delta_l(\{q_1\}, \sigma, \{q_0, q_1, q_2\}) = a,$

$\delta_l(\{q_2\}, \sigma, \{q_1\}) = e,$ $\delta_l(\{q_2\}, \sigma, \{q_0, q_1\}) = e,$

$\delta_l(\{q_2\}, \sigma, \{q_1, q_2\}) = e,$ $\delta_l(\{q_2\}, \sigma, \{q_0, q_1, q_2\}) = e,$

$\delta_l(\{q_0, q_1\}, \sigma, \{q_0\}) = a,$ $\delta_l(\{q_0, q_1\}, \sigma, \{q_1\}) = b,$

$\delta_l(\{q_0, q_1\}, \sigma, \{q_0, q_1\}) = b,$ $\delta_l(\{q_0, q_1\}, \sigma, \{q_0, q_2\}) = a,$

$\delta_l(\{q_0, q_1\}, \sigma, \{q_1, q_2\}) = b,$ $\delta_l(\{q_0, q_1\}, \sigma, \{q_0, q_1, q_2\}) = b,$

$\delta_l(\{q_0, q_2\}, \sigma, \{q_1\}) = e,$ $\delta_l(\{q_0, q_2\}, \sigma, \{q_0, q_1\}) = e,$

$\delta_l(\{q_0, q_2\}, \sigma, \{q_1, q_2\}) = e,$ $\delta_l(\{q_0, q_2\}, \sigma, \{q_0, q_1, q_2\}) = e,$

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$\delta_l(\{q_1, q_2\}, \sigma, \{q_0, q_1\}) = e,$ $\delta_l(\{q_1, q_2\}, \sigma, \{q_0, q_2\}) = a,$

$\delta_l(\{q_1, q_2\}, \sigma, \{q_1, q_2\}) = e,$ $\delta_l(\{q_1, q_2\}, \sigma, \{q_0, q_1, q_2\}) = e,$

$\delta_l(\{q_0, q_1, q_2\}, \sigma, \{q_0\}) = a,$ $\delta_l(\{q_0, q_1, q_2\}, \sigma, \{q_1\}) = e,$

$\delta_l(\{q_0, q_1, q_2\}, \sigma, \{q_0, q_1\}) = e,$ $\delta_l(\{q_0, q_1, q_2\}, \sigma, \{q_0, q_2\}) = a,$

$\delta_l(\{q_0, q_1, q_2\}, \sigma, \{q_1, q_2\}) = e,$ $\delta_l(\{q_0, q_1, q_2\}, \sigma, \{q_0, q_1, q_2\}) = e.$

By considering Definition 3.3, we obtain admissible partition on $\bar{Q}$ as follow:

$$\pi = \Big\{\big\{\{q_1\}\big\}, \big\{\{q_0\}, \{q_0, q_1\}\big\}, \big\{\{q_2\}, \{q_0, q_2\}, \{q_1, q_2\}, \{q_0, q_1, q_2\}\big\}\Big\}.$$

Theorem 3.7. *Let $\tilde{F}_l$ be a BL-GFA. Then every admissible partition H of $\bar{Q}$ gives an admissible relation $\sim$ on $\bar{Q}$.*

Proof. Consider set of all equivalence classes of $\sim$ is H. Let $Q' \sim Q''$ and $a \in X \cup \{\Lambda\}, P' \in \bar{Q}$ and $\tilde{\delta}_l((Q', \mu^{t_i}(Q')), a, P') > 0$. Then there exists $Q_{l_1} \in H$ such that $Q', Q'' \in Q_{l_1}$. By considering Definition 3.1, there exist $P'' \in \bar{Q}$ and $Q_{l_2} \in H$ such that $\tilde{\delta}_l((Q'', \mu^{t_i}(Q'')), a, P'') \geq \tilde{\delta}_l((Q', \mu^{t_i}(Q')), a, P')$ and $P', P'' \in Q_{l_2}$. Therefore, $P' \sim P''$. Also, if $Q' \sim Q''$, then $\omega_l(Q') = \omega_l(Q'')$. Hence, the claim holds. □

Theorem 3.8. *Let $\tilde{F}_l$ be a BL-GFA. Then the set of all equivalence classes of an admissible relation on $\bar{Q}$ gives an admissible partition $\bar{Q}$.*

Proof. Consider H is the set of all equivalence classes of the admissible relation. Let $\sim$ be an admissible relation of $\bar{Q}$. Let $Q', Q'' \in Q_{l_1}, Q_{l_1} \in H$, $a \in X \cup \{\Lambda\}, P' \in \bar{Q}$ and $\tilde{\delta}_l((Q', \mu^{t_i}(Q')), a, P') > 0$. Since $Q', Q'' \in Q_{l_1}$, then $Q' \sim Q''$. By considering Definition 3.3, there exist $P'' \in \bar{Q}$ such that $\tilde{\delta}_l((Q'', \mu^{t_i}(Q'')), a, P'') \geq \tilde{\delta}_l((Q', \mu^{t_i}(Q')), a, P')$ and $P' \sim P''$. Therefore, $P', P'' \in Q_{l_2}$, where $Q_{l_2} \in H$. If $Q', Q'' \in Q_{l_1}, Q_{l_1} \in H$, then $\omega_l(Q') = \omega_l(Q'')$. Hence, the claim holds. □

Corollary 3.9. *Let $\tilde{F}_l$ be a BL-GFA. Then there exists a one-to-one corresponded between the set of admissible partition of $\bar{Q}$ and the set of admissible relation.*

Proof. It immediately follows follows Theorems 3.7 and 3.8. □

Theorem 3.10. *Let $\tilde{F}_l$ be a BL-GFA and $\pi = \{H_l \big| l \in I\}$ be an admissible partition of $\bar{Q}$ and $l_1, l_2 \in I$. Then for every $Q', Q'' \in H_{l_1}$ and $a \in X$ we have*

$$\vee\{\tilde{\delta}_l((Q', \mu^{t_i}(Q')), a, P') \big| P' \in H_{l_2}\} = \vee\{\tilde{\delta}_l((Q'', \mu^{t_i}(Q'')), a, P') \big| P' \in H_{l_2}\}.$$

Proof. Let $Q', Q'' \in H_{l_1}, a \in X$ and

$$A = \vee\{\tilde{\delta}_l((Q', \mu^{t_i}(Q')), a, P') \big| P' \in H_{l_2}\}, B = \vee\{\tilde{\delta}_l((Q'', \mu^{t_i}(Q'')), a, P') \big| P' \in H_{l_2}\}.$$

First, suppose that $\tilde{\delta}_l((Q', \mu^{t_i}(Q')), a, P') > 0$, for some $P' \in \bar{Q}$. Then by Definition 3.3 there exists $P'' \in \bar{Q}$ such that $\tilde{\delta}_l((Q'', \mu^{t_i}(Q'')), a, P'') \geq \tilde{\delta}_l((Q', \mu^{t_i}(Q')), a, P')$, where $P', P'' \in H_{l_2}$. Therefore, $B \geq A$. Similarly, $B \leq A$. Also, by using Definition 3.3 we get that: $\tilde{\delta}_l((Q', \mu^{t_i}(Q')), a, P') = 0$ if and only if $\tilde{\delta}_l((Q'', \mu^{t_i}(Q'')), a, P') = 0$, for every $P' \in H_{l_2}$. Then the proof is complete. □

Theorem 3.11. *Let $\tilde{F}_l$ be a BL-GFA and $\pi = \{H_l \big| l \in I\}$ be a partition of $\bar{Q}$ and $l_1, l_2 \in I$. If for every $Q', Q'' \in H_{l_1}$ and $a \in X$ we have*

$$\vee\{\tilde{\delta}_l((Q', \mu^{t_i}(Q')), a, P') \big| P' \in H_{l_2}\} = \vee\{\tilde{\delta}_l((Q'', \mu^{t_i}(Q'')), a, P') \big| P' \in H_{l_2}\}, \quad (1)$$

and $\omega_l(Q') = \omega_l(Q'')$, then π is an admissible partition of $\bar{Q}$.

Proof. Let $a \in X, l_1 \in I, Q', Q'' \in Q_{l_1}$ and $\tilde{\delta}_l((Q', \mu^{t_i}(Q')), a, P') > 0$, where $P' \in H_{l_2}$. Then by considering (1), there exists $P'' \in H_{l_2}$, such that

$$\begin{aligned}\tilde{\delta}_l((Q'', \mu^{t_i}(Q'')), a, P'') &= \vee\{\tilde{\delta}_l((Q'', \mu^{t_i}(Q'')), a, P') \big| P' \in H_{l_2}\} \\ &\geq \tilde{\delta}_l((Q', \mu^{t_i}(Q')), a, P'),\end{aligned}$$

and $P'', P' \in H_{l_2}$. Hence, π is an admissible partition of $\bar{Q}$. □

Definition 3.12. Let $\tilde{F}_l = (\bar{Q}, X, (\{q_0\}, \mu^{t_0}(\{q_0\})), \bar{Z}, \omega_l, \delta_l, f_l, \tilde{\delta}_l, F_1, F_2)$ be a BL-GFA and $\pi = \{H_l \big| l \in I\}$ be an admissible partition of $\bar{Q}$.

(1) Define $\tilde{\delta}_{l\pi} : (\pi \times L) \times X \times \pi \to L$ by

$$\tilde{\delta}_{l\pi}((H_{l_1}, \mu^{t_i}(H_{l_1})), a, H_{l_2}) = \vee\{\tilde{\delta}_l^*((Q', \mu^{t_i}(Q')), a, P') \big| P' \in H_{l_2}\},$$

for every $i \geq 0, H_{l_1}, H_{l_2} \in \pi, a \in X \cup \{\Lambda\}$ and $Q' \in H_{l_1}$.

(2) Define $\omega_{l\pi}(H_l) = \cup\{\omega_l(Q') \big| Q' \in H_l\}$ and consider $\tilde{R}_\pi = H_l$, where $\{q_0\} \in H_l$.

(3) Consider $\dfrac{\tilde{F}_l}{\pi} = (\pi, X, \tilde{R}_\pi = (H_l, \mu^{t_0}(H_l)), \bar{Z}, \omega_{l\pi}, \delta_{l\pi}, f_{l\pi}, \tilde{\delta}_{l\pi}, F_1, F_2)$.

Corollary 3.13. $\dfrac{\tilde{F}_l}{\pi}$ *is a BL-GFA and is called the quotient BL-GFA with respect to* π.

Proof. The proof is clear because $\tilde{\delta}_{l\pi}$ and $\omega_{l\pi}$ are well defined by Theorem 3.11 and Definition 3.3, respectively. □

Theorem 3.14. *Let* $\tilde{F}_l$ *be a BL-GFA and* $\pi = \{H_l \big| l \in I\}$ *be an admissible partition of* $\bar{Q}$. *Then for every* $Q' \in H_{l_1}$ *and* $x \in X^*$ *we have*

$$\tilde{\delta}_{l\pi}^*((H_{l_1}, \mu^{t_0}(H_{l_1})), x, H_{l_2}) = \vee\{\tilde{\delta}_l^*((Q', \mu^{t_0}(Q')), x, R') \big| R' \in H_{l_2}\}.$$

Proof. Let $H_{l_1}, H_{l_2} \in \pi$ and $x \in X^*$. We prove the claim by induction on $|x| = n$. For $n = 1$, the claim is clear by Definition 3.12. Now, let the claim holds for every $y \in X^*$ such that $|y| = n - 1, n \geq 1$. Suppose that $x = ya$, where $y \in X^*, a \in X, |y| = n - 1$ and $n > 1$. Then for every $Q' \in H_{l_1}$ and $S' \in H_k$ we have

$$\begin{aligned}\tilde{\delta}_{l\pi}^*((H_{l_1}, \mu^{t_0}(H_{l_1})), x, H_{l_2}) &= \vee\{\tilde{\delta}_{l\pi}^*((H_{l_1}, \mu^{t_0}(H_{l_1})), y, H_k) \\ &\qquad \wedge \tilde{\delta}_{l\pi}((H_k, \mu^{t_0+|y|}(H_k)), a, H_{l_2}) | H_k \in \pi\} \\ &= \bigvee\{\vee\{\tilde{\delta}_l^*((Q', \mu^{t_0}(Q')), y, R') \big| R' \in H_k\} \\ &\qquad \wedge \vee\{\tilde{\delta}_l((S', \mu^{t_0+|y|}(S')), a, P') \big| P' \in H_{l_2}\} H_k \in \pi\} \\ &\leq \bigvee\{\vee\{\tilde{\delta}_l^*((Q', \mu^{t_0}(Q')), y, R') \\ &\qquad \wedge \tilde{\delta}_l((R', \mu^{t_0+|y|}(R')), a, P') \big| P' \in H_{l_2}, R' \in H_k\} H_k \in \pi\} \\ &= \vee\{\tilde{\delta}_l^*((Q', \mu^{t_0}(Q')), ya, P') \big| P' \in H_{l_2}\}.\end{aligned}$$

Also, we have

$$\begin{aligned}\tilde{\delta}_l^*((Q',\mu^{t_0}(Q')),x,P') &= \tilde{\delta}_l^*((Q',\mu^{t_0}(Q')),ya,P')\\
&= \vee\{\tilde{\delta}_l^*((Q',\mu^{t_0}(Q')),y,R')\\
&\qquad \wedge \tilde{\delta}_l((R',\mu^{t_0+|y|}(R')),a,P')|R'\in\bar{Q}\}\\
&= \bigvee\{\vee\{\tilde{\delta}_l^*((Q',\mu^{t_0}(Q')),y,R')\\
&\qquad \wedge \tilde{\delta}_l((R',\mu^{t_0+|y|}(R')),a,P')\big|R'\in H_k\}H_k\in\pi\}\\
&\leq \bigvee\{\vee\{\tilde{\delta}_{l\pi}^*((H_{l_1},\mu^{t_0}(H_{l_1})),y,H_k)\\
&\qquad \wedge \tilde{\delta}_{l\pi}((H_k,\mu^{t_0+|y|}(H_k)),a,H_{l_2})\big|R'\in H_k\}H_k\in\pi\}\\
&= \tilde{\delta}_{l\pi}^*((H_{l_1},\mu^{t_0}(H_{l_1})),ya,H_{l_2}).\end{aligned}$$

Hence, the claim holds. □

Definition 3.15. Let $\tilde{F}_l$ be a BL-GFA and $\pi = \{H_l\big|l \in I\}$ be an admissible partition of $\bar{Q}$ and π_1 be nontrivial. We say that π_1 is maximal if for any admissible partition π_2 of $\bar{Q}$, $\pi_1 \leq \pi_2 \leq \{\bar{Q}\}$, implies that either $\pi_2 = \pi_1$ or $\pi_2 = \{\bar{Q}\}$.

Example 3.16. Let $(L,\wedge,\vee,0,1)$ be a complete lattice as in Figure 1. Let general fuzzy automaton $\tilde{F} = (Q,X,\tilde{\delta},\tilde{R},Z,\omega,F_1,F_2)$ as follow: $Q = \{q_0,q_1\}$ $\tilde{R} = \{(q_0,1)\}, X = \{\sigma_1,\sigma_2\}, Z = \{z\}, \omega(q_0) = \omega(q_1) = z$ and

$$\begin{aligned}&\delta(q_0,\sigma_1,q_0) = a, && \delta(q_0,\sigma_1,q_1) = b,\\
&\delta(q_1,\sigma_1,q_0) = d, && \delta(q_1,\sigma_1,q_1) = e,\\
&\delta(q_1,\sigma_2,q_0) = d, && \delta(q_1,\sigma_2,q_1) = e.\end{aligned}$$

Then by considering Definition 2.3, we have BL-general fuzzy automaton $\tilde{F}_l = (\bar{Q},X,\ (\{q_0\},\mu^{t_0}(\{q_0\})),\bar{Z},\omega_l,\delta_l,f_l,\ \tilde{\delta}_l,F_1,F_2)$, where $\bar{Q} = \{\emptyset,\{q_0\},\{q_1\},\{q_0,q_1\}\}$, $\bar{Z} = \{\emptyset,\{z\}\}$, $\omega_l(\{q_0\}) = \omega_l(\{q_1\}) = \omega_l(\{q_0,q_1\}) = \{z\}$ and

$$\begin{aligned}&\delta_l(\{q_0\},\sigma_1,\{q_0\}) = a, && \delta_l(\{q_0\},\sigma_1,\{q_1\}) = b,\\
&\delta_l(\{q_0\},\sigma_1,\{q_0,q_1\}) = b, && \delta_l(\{q_1\},\sigma_1,\{q_0\}) = d,\\
&\delta_l(\{q_1\},\sigma_1,\{q_1\}) = e, && \delta_l(\{q_1\},\sigma_1,\{q_0,q_1\}) = e,\\
&\delta_l(\{q_0,q_1\},\sigma_1,\{q_0\}) = d, && \delta_l(\{q_0,q_1\},\sigma_1,\{q_1\}) = e,\\
&\delta_l(\{q_0,q_1\},\sigma_1,\{q_0,q_1\}) = e, && \delta_l(\{q_1\},\sigma_2,\{q_0\}) = d,\\
&\delta_l(\{q_1\},\sigma_2,\{q_1\}) = e, && \delta_l(\{q_1\},\sigma_2,\{q_0,q_1\}) = e,\\
&\delta_l(\{q_0,q_1\},\sigma_2,\{q_0\}) = d, && \delta_l(\{q_0,q_1\},\sigma_2,\{q_1\}) = e,\\
&\delta_l(\{q_0,q_1\},\sigma_2,\{q_0,q_1\}) = e.\end{aligned}$$

By considering Definition 3.3, we obtain admissible partition on $\bar{Q}$ as follow: $\pi = \big\{\big\{\{q_0\}\big\},\ \big\{\{q_1\},\{q_0,q_1\}\big\}\big\}$. By Definition 3.15, π is maximal admissible partition.

Definition 3.17. Let $\tilde{F}_l$ be a BL-GFA. We say that $\tilde{F}_l$ is minimal if $|\bar{Q}| > 1$ and for every admissible partition π of $\bar{Q}$, where $1_{\bar{Q}} \leq \pi \leq \{\bar{Q}\}$, we have either $\pi = 1_{\bar{Q}}$ or $\pi = \{\bar{Q}\}$.

Theorem 3.18. *Let $\tilde{F}_l$ be a BL-GFA and $\pi = \{H_l \big| l \in I\}$ be an admissible partition of $\bar{Q}$. Then π is maximal if and only if $\dfrac{\tilde{F}_l}{\pi}$ is minimal.*

Proof. Let π ba a maximal admissible partition of $\bar{Q}$. Then $\pi \neq \{\bar{Q}\}$ and $\pi \neq 1_{\bar{Q}}$. Then $|\pi| > 1$. Let $\bar{\pi}$ be an admissible partition of π and $\bar{\pi} \neq 1_\pi$. Then there exists $\tau \subseteq \pi$ such that $\tau \in \bar{\pi}$ and $|\tau| > 1$. Let $\tau \neq \pi$. without loss of generality, we may assume that $\tau = \{H_1, H_2, ..., H_m\}$, where $1 < m < n$. Now, suppose that $\pi' = \{H_1 \cup ... \cup H_m, H_{m+1} \cup ... \cup H_n\}$. Then $\pi < \pi'$ and π' is a partition of $\bar{Q}$. Now, we show that π' is an admissible partition of $\bar{Q}$. Let $Q', Q'' \in H_1 \cup ... \cup H_m$, where $Q' \in H_1$ and $Q'' \in H_2$. Suppose that $\tilde{\delta}_l((Q'', \mu^{t_i}(Q'')), a, R) > 0$, where $R \in H_l$. Then $\tilde{\delta}_{l\pi}((H_2, \mu^{t_i}(H_2)), a, H_l) = \vee\{\tilde{\delta}_l((Q'', \mu^{t_i}(Q'')), a, R) \big| R \in H_l\}$. Since $\bar{\pi}$ is an admissible partition, then there exists $K \in \bar{\pi}$ such that $\tilde{\delta}_{l\pi}((H_1, \mu^{t_i}(H_1)), a, K) \geq \tilde{\delta}_{l\pi}((H_2, \mu^{t_i}(H_2)), a, H_l)$, where K and H_l belong to the same element of $\bar{\pi}$. Therefore,

$$\vee\{\tilde{\delta}_l((Q', \mu^{t_i}(Q')), a, R') \big| R' \in K\} \geq \vee\{\tilde{\delta}_l((Q'', \mu^{t_i}(Q'')), a, R) \big| R \in H_l\}.$$

Hence, there exists $R' \in K$ such that $\tilde{\delta}_l((Q', \mu^{t_i}(Q')), a, R') \geq \tilde{\delta}_l((Q'', \mu^{t_i}(Q'')), a, R)$. Also, we have $K \in \tau$ if and only if $H_l \in \tau$. If $K, H_l \in \tau$, then $R, R' \in H_1 \cup ...H_m$. So, if $K, H_l \notin \tau$, then $R, R' \in H_{m+1} \cup ...H_n$. Hence, π' is an admissible partition of $\bar{Q}$. Since π is maximal, it follows that $\pi' = \{\bar{Q}\}$, then $\tau = \{\bar{Q}\}$. This implies that $\bar{\pi} = \{\pi\}$. Then $\dfrac{\tilde{F}_l}{\pi}$ is minimal. Conversely, suppose that $\dfrac{\tilde{F}_l}{\pi}$ is minimal. Let τ be an admissible partition of $\bar{Q}$ such that $\pi \leq \tau \leq \{\bar{Q}\}$ and $\pi \neq \tau$. Suppose $\tau = \{H_1 \cup ... \cup H_m, H_{m+1}, ..., H_n\}, 1 \leq m \leq n$. Without loss of generality, suppose that $\bar{\tau} = \{\{H_1, ..., H_m\}, \{H_{m+1}\}, ..., \{H_n\}\} \neq 1_\pi$. Now, we show that $\bar{\tau}$ is an admissible partition of π. Suppose that $\tilde{\delta}_{l\pi}((H_1, \mu^{t_i}(H_1)), a, H_l) > 0$, for some $H_l \in \pi$. Therefore, $\vee\{\tilde{\delta}_l((Q', \mu^{t_i}(Q')), a, R) \big| R \in H_l\} > 0$, where $Q' \in H_1$. Let there exists $R \in H_l$ such that

$$\tilde{\delta}_l((Q', \mu^{t_i}(Q')), a, R) = \vee\{\tilde{\delta}_l((Q', \mu^{t_i}(Q')), a, R) \big| R \in H_l\} > 0.$$

Since τ is an admissible partition of $\bar{Q}$, then for every $Q'' \in H_2$ there exists $R' \in \bar{Q}$ such that $\tilde{\delta}_l((Q'', \mu^{t_i}(Q'')), a, R') \geq \tilde{\delta}_l((Q', \mu^{t_i}(Q')), a, R)$ and R, R' are in the same element of τ. If $H_l \notin \{H_1, ..., H_m\}$, then $R, R' \in H_l$

$$\begin{aligned}\tilde{\delta}_{l\pi}((H_2, \mu^{t_i}(H_2)), a, H_l) &\geq \tilde{\delta}_l((Q'', \mu^{t_i}(Q'')), a, R') \\ &\geq \tilde{\delta}_l((Q', \mu^{t_i}(Q')), a, R) \\ &= \tilde{\delta}_{l\pi}((H_1, \mu^{t_i}(H_1)), a, H_l).\end{aligned}$$

If $H_l \in \{H_1, ..., H_m\}$, then $R, R' \in H_1 \cup ... \cup H_m$. Suppose that $R' \in H_{l_1}, 1 \leq l_1 \leq m$, then

$$\begin{aligned}\tilde{\delta}_{l\pi}((H_2, \mu^{t_i}(H_2)), a, H_{l_1}) &\geq \tilde{\delta}_l((Q'', \mu^{t_i}(Q'')), a, R') \\ &\geq \tilde{\delta}_l((Q', \mu^{t_i}(Q')), a, R) \\ &= \tilde{\delta}_{l\pi}((H_1, \mu^{t_i}(H_1)), a, H_l).\end{aligned}$$

Then $\bar{\tau}$ is an admissible partition of π. Hence, π is maximal. □

Example 3.19. Let BL-general fuzzy automaton as in Example 3.16. Then by Definition 3.15, π is the maximal admissible partition and by considering Theorem 3.18, $\frac{\tilde{F}_l}{\pi}$ is a minimal quotient BL-GFA.

Definition 3.20. Let $\tilde{F}_l$ be a BL-GFA. Then the behavior of $\tilde{F}_l$ is the map $\beta_{\tilde{F}_l} = \omega_l \circ \rho : \mathcal{L}(\tilde{F}_l) \to \bar{Z}$, where

$$\mathcal{L}(\tilde{F}_l) = \{x \in X^* \big| \tilde{\delta}_l^*((\{q_0\}, \mu^{t_0}(\{q_0\})), x, P) > 0, \text{for some } \ P \in \bar{Q}\}.$$

Theorem 3.21. *Let $\tilde{F}_l$ be a BL-GFA and let $\pi = \{H_l \big| l \in I\}$ be a maximal admissible partition of $\bar{Q}$. Then $\beta_{\tilde{F}_l} = \beta_{\frac{\tilde{F}_l}{\pi}}$*

Proof. First, we show that $\mathcal{L}(\tilde{F}_l) = \mathcal{L}(\frac{\tilde{F}_l}{\pi})$. Let $x \in \mathcal{L}(\tilde{F}_l)$. Then there exists $P' \in \bar{Q}$ such that $\tilde{\delta}_l((\{q_0\}, \mu^{t_0}(\{q_0\})), x, P') > 0$. Since π is an admissible partition of $\bar{Q}$, then by Theorem 3.14 there are $l_1, l_2 \in I$ such that $\{q_0\} \in H_{l_1}, P' \in H_{l_2}$ and $\tilde{\delta}_{l\pi}((H_{l_1}, \mu^{t_0}(H_{l_1})), x, H_{l_2}) > 0$. Therefore, $x \in \mathcal{L}(\frac{\tilde{F}_l}{\pi})$. Now, suppose that $x \in \mathcal{L}(\frac{\tilde{F}_l}{\pi})$. Then there exist $H_{l_1}, H_{l_2} \in \pi$ such that $\tilde{\delta}_{l\pi}((H_{l_1}, \mu^{t_0}(H_{l_1})), x, H_{l_2}) > 0$. So, by Theorem 3.14 there exists $P' \in H_{l_2}$ such that $\tilde{\delta}_l((\{q_0\}, \mu^{t_0}(\{q_0\})), x, P') > 0$. Hence, $x \in \mathcal{L}(\tilde{F}_l)$. Now, let ρ_1 and ρ_2 be the run maps of $\tilde{F}_l$ and $\frac{\tilde{F}_l}{\pi}$, respectively. It is obvious that $\rho_1(x) \subseteq \rho_2(x)$, for every $x \in \mathcal{L}(\tilde{F}_l) = \mathcal{L}(\frac{\tilde{F}_l}{\pi})$. Now, let $Q' = \rho_1(x) \subseteq \rho_2(x) = H_l$. By Definition 3.3 we have $\omega_l(Q') = \omega_l(H_l)$. Hence, $\beta_{\tilde{F}_l} = \beta_{\frac{\tilde{F}_l}{\pi}}$. □

Example 3.22. Let BL-general fuzzy automaton as in Example 3.16. For $\sigma_1 \in X$ we have $\beta(\sigma_1) = \omega_l(\rho(\sigma_1)) = \{z\}$, where $\rho(\sigma_1) = f_l(\{q_0\}, \sigma_1) = \{q_0, q_1\}$ and $\omega_l(\{q_0, q_1\}) = \{z\}$. Also, for every $\sigma_1^n \in X^*, n \geq 1$ we have $\beta(x) = \{z\}$.

4. **Conclusion**

We show that a given admissible partition of states of a BL-general fuzzy automaton is maximal if and only if the quotient of the BL-general fuzzy automaton is minimal. Also, by considering the notion of admissible partition we obtain a minimal quotient BL-general fuzzy automaton. Finally, the Authors have shown that any quotient of a given BL-general fuzzy automaton and the BL-general fuzzy automaton itself have the same behavior.

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