FIXED POINT THEOREM ON INTUITIONISTIC FUZZY METRIC SPACES

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Abstract. In this paper, we introduce intuitionistic fuzzy contraction mapping and prove a fixed point theorem in intuitionistic fuzzy metric spaces.

1. Introduction

The notion of intuitionistic fuzzy metric spaces was introduced and studied by Park in [5]. Saadati and Park in [6], further developed the theory of intuitionistic fuzzy topology (both in metric and normed) spaces. In this paper, we introduce an intuitionistic fuzzy contraction mapping and prove a fixed point theorem in intuitionistic fuzzy metric spaces. For the basic notions and concepts, we refer to [1, 3, 4, 5, 6].

2. Preliminaries

We review some basic concepts in intuitionistic fuzzy metric spaces as well as the intuitionistic fuzzy topology due to Saadati and Park [6].

Definition 2.1. [5, 6] A binary operation $*: [0, 1] \times [0, 1] \rightarrow [0, 1]$ is a continuous t-norm if it satisfies the following conditions: (i) $*$ is associative and commutative; (ii) $*$ is continuous; (iii) $a * 1 = a$ for all $a \in [0, 1]$; (iv) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, for each $a, b, c, d \in [0, 1]$. Two typical examples of continuous t-norm are

\[ a * b = ab, \]
\[ a * b = \min(a, b). \]

Definition 2.2. [5, 6] A binary operation $\odot: [0, 1] \times [0, 1] \rightarrow [0, 1]$ is a continuous t-conorm if it satisfies the following conditions: (i) $\odot$ is associative and commutative; (ii) $\odot$ is continuous; (iii) $a \odot 0 = a$ for all $a \in [0, 1]$; (iv) $a \odot b \leq c \odot d$ whenever $a \leq c$ and $b \leq d$, for each $a, b, c, d \in [0, 1]$. Two typical examples of t-conorm are

\[ a \odot b = \min(a + b, 1), \]
\[ a \odot b = \max(a, b). \]

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Lemma 2.3. [5, 6] If \( * \) is a continuous t-norm and \( \odot \) is continuous t-conorm, then:

(i) For every \( a, b \in [0,1] \), if \( a > b \), there are \( c, d \in [0,1] \) such that \( a * c \geq b \) and \( a \geq b \odot d \).

(ii) If \( a \in [0,1] \), there are \( b, c \in [0,1] \) such that \( b * b \geq a \) and \( a \geq c \odot c \).

The following definition is obtained from Mihet in [4].

Definition 2.4. [4] A fuzzy metric space in the sense of Kramosil and Michalek is a triple \((X, M, \ast)\) where \( X \) is a nonempty set, \( \ast \) is a continuous t-norm and \( M : X^2 \times [0, \infty) \to [0,1] \) is a mapping which satisfies the following properties for every \( x, y, z \in X \):

- (FM-1) \( M(x, y, 0) = 0 \);
- (FM-2) \( M(x, y, t) = 1, \forall t > 0 \iff x = y \);
- (FM-3) \( M(x, y, t) = M(y, x, t), \forall t > 0 \);
- (FM-4) \( M(x, y, \cdot) : [0, \infty) \to [0,1] \) is left continuous;
- (FM-5) \( \lim_{t \to \infty} M(x, y, t) = 1 \);
- (FM-6) \( M(x, z, t + s) \geq M(x, y, t) * M(y, z, s), \forall t, s > 0 \).

In \((X, M, \ast)\), the open ball \( B_x(r, t) \) for \( t > 0 \) with center \( x \in X \) and radius \( r \in (0,1) \) is defined as

\[
B_x(r, t) = \{ y \in X \mid M(x, y, t) > 1 - r \}.
\]

The family \( \{ B_x(r, t) \mid x \in X, r \in (0,1), t > 0 \} \) is a neighborhood system for a Hausdorff topology on \( X \) induced by the fuzzy metric \( M \). In a similar fashion, the dual space of \((X, M, \ast)\) is the fuzzy metric space \((X, N, \odot)\) defined below:

Definition 2.5. (New) A fuzzy metric space \((X, N, \odot)\), where \( X \) is a nonempty set, \( \odot \) is a continuous t-conorm and \( N : X^2 \times [0, \infty) \to [0,1] \) is a mapping assumed to satisfies the following properties for all \( x, y, z \in X \):

- (FM-D1) \( N(x, y, 0) = 1 \);
- (FM-D2) \( N(x, y, t) = 0, \forall t > 0 \iff x = y \);
- (FM-D3) \( N(x, y, t) = N(y, x, t), \forall t > 0 \);
- (FM-D4) \( N(x, y, \cdot) : [0, \infty) \to [0,1] \) is left continuous;
- (FM-D5) \( \lim_{t \to \infty} N(x, y, t) = 0 \);
- (FM-D6) \( N(x, z, t + s) \leq N(x, y, t) \odot N(y, z, s), \forall t, s > 0 \).

In \((X, N, \odot)\), the open ball \( D_x(r, t) \) for \( t > 0 \) with center \( x \in X \) and radius \( r \in (0,1) \) is defined as

\[
D_x(r, t) = \{ y \in X \mid N(x, y, t) < r \}.
\]

The family \( \{ D_x(r, t) \mid x \in X, r \in (0,1), t > 0 \} \) is a neighborhood’s system for a Hausdorff topology on \( X \) induced by the fuzzy metric \( N \). The following definition is introduced and studied by Park in [5].

Definition 2.6. [5] A 5-tuple \((X, M, N, \ast, \odot)\) is called an intuitionistic fuzzy metric space if \( X \) is an arbitrary nonempty set, \( \ast \) a continuous t-norm, \( \odot \) a continuous t-conorm and \( M, N \) are fuzzy sets on \( X^2 \times (0, \infty) \), satisfying the following conditions for each \( x, y, z \in X \) and \( t, s > 0 \):

- (a) \( M(x, y, t) + N(x, y, t) \leq 1 \);
The pair \((M, N)\) is called an intuitionistic fuzzy metric on \(X\). Every fuzzy metric space \((X, M, \ast)\) is an intuitionistic fuzzy metric space of the form \((X, M, 1 - M, \ast, \diamond)\) such that t-norm \(\ast\) and t-conorm \(\diamond\) are associated [6], i.e \(x \diamond y = 1 - [(1 - x) \ast (1 - y)]\) for any \(x, y \in X\).

Let \((X, M, N, \ast, \diamond)\) be an intuitionistic fuzzy metric space. For \(t > 0\), the open ball \(G_x(r, t)\) with center \(x \in X\) and radius \(r \in (0, 1)\) is defined by
\[
G_x(r, t) = \{ y \in X \mid M(x, y, t) > 1 - r, N(x, y, t) < r \}.
\]

Note that it can be easily seen that \(G_x = B_x \cap D_x\) where \(B_x\) and \(D_x\) as given by (1) and (2) respectively.

Since \(\ast\) and \(\diamond\) are respectively a continuous t-norm and t-conorm, the family \(\{G_x(r, t) \mid x \in X, r \in (0, 1), t > 0\}\) generates a topology \(T_{(M, N)}\), called the \((M, N)\)-topology (see [3, 6]). We have:

\[ A \in T_{(M, N)} \text{ if and only if } \forall x \in A, \exists t > 0, \exists r \in (0, 1) \text{ such that } G_x(r, t) \subset A. \]

We denote the \((M, N)\)-uniformity (or the uniformity generated by \(M, N\)) by \(U_{(M, N)}\). The family \(\{U_{r,t} \mid r \in (0, 1), t > 0\}\), where
\[
U_{r,t} = \{ (x, y) \in X^2 \mid M(x, y, t) > 1 - r, N(x, y, t) < r \},
\]
is a base for this uniformity.

**Definition 2.7.** [5, 6] Let \((X, M, N, \ast, \diamond)\) be the intuitionistic fuzzy metric space endowed with \((m, n)\)-topology and \(\{x_n\}\) in \(X\). Then

- \((i)\) \(x_n \to x \Leftrightarrow M(x_n, x, t) \to 1\) and \(N(x_n, x, t) \to 0\) as \(n \to \infty\), for each \(t > 0\).
- \((ii)\) \(\{x_n\}\) is called a \((M, N)\)-Cauchy sequence if for each \(r \in (0, 1)\) and \(t > 0\), there exists an integer \(n_0\) such that \(M(x_n, x_m, t) > 1 - r\) and \(N(x_n, x_m, t) < r\) for each \(n, m \geq n_0\).
- \((iii)\) The intuitionistic fuzzy metric space \((X, M, N, \ast, \diamond)\) is said to be \((M, N)\)-complete if every \((M, N)\)-Cauchy sequence is convergent.

3. **Main Results**

In the following sequel the letters \(\mathbb{N}\) and \(\mathbb{R}^+\) denote the sets of positive integer numbers and positive real numbers, respectively.

**Definition 3.1.** [2] A quasi-metric on a set \(X\) is a function \(d: X^2 \to \mathbb{R}^+\) satisfying the following conditions for every \(x, y, z \in X\):

- \((QM-1)d(x, y) = 0\);
- \((QM-2)d(x, y) = d(y, x)\);
- \((QM-3)d(x, z) \leq d(x, y) + d(y, z)\).
Proposition 3.2. Let \( (X, M, N, *, \odot) \) be the intuitionistic fuzzy metric space. For any \( r \in (0, 1] \), we define \( d_r : X^2 \rightarrow \mathbb{R}^+ \) as follows:

\[
d_r(x, y) = \inf \{ t > 0 \mid M(x, y, t) > 1 - r, N(x, y, t) < r \}
\]

Then,

1. \( (X, d_r : r \in (0, 1]) \) is a generating space of a quasi-metric family,
2. the topology \( T(d_r) \) on \( (X, d_r : r \in (0, 1]) \) coincides with the \( (M, N) \)-topology on \( (X, M, N, *, \odot) \), (i.e., \( d_r \) is a compatible symmetric for \( T(M, N) \)).

Proof. (1) From the definition of \( \{ d_r : r \in (0, 1] \} \), it is easy to see that \( \{ d_r : r \in (0, 1] \} \) satisfies the condition (QM-1) and (QM-2) of Definition 3.1. Now we prove that \( \{ d_r : r \in (0, 1] \} \) also satisfies the condition (QM-3). Since \( * \) and \( \odot \) are continuous, by Lemma 2.3.(ii), for any given \( r \in (0, 1) \), there exists \( r' \in (0, r) \) such that

\[
(1 - r') \odot (1 - r') > 1 - r
\]

and

\[
r' \odot r' < r
\]

. Setting \( d_r(x, y) = a \) and \( d_r(y, z) = b \), in equation (4), it follows that for any given \( t > 0 \),

\[
M(x, y, a + t) > 1 - r', N(x, y, a + t) < r'
\]

and

\[
M(x, z, b + t) > 1 - r', N(y, z, b + t) < r'.
\]

Whence

\[
M(x, z, a + b + 2t) \geq M(x, y, a + t) * M(y, z, b + t) > (1 - r') \odot (1 - r') > 1 - r
\]

and

\[
N(x, z, a + b + 2t) \leq N(x, y, a + t) \odot N(y, z, b + t) < r' \odot r' < r.
\]

Hence, we have

\[
d_r(x, z) = a + b + 2t = d_r(x, y) + d_r(y, z) + 2t.
\]

By the arbitrariness of \( t > 0 \), we have

\[
d_r(x, z) \leq d_r(x, y) + d_r(y, z).
\]

(2) To prove this condition, it is only necessary to show that for any \( t > 0 \) and \( r \in (0, 1) \)

\[
d_r(x, y) < t \Leftrightarrow M(x, y, t) > 1 - r, N(x, y, t) < r.
\]

In fact, if \( d_r(x, y) < t \), then by (4), we have \( M(x, y, t) > 1 - r \) and \( N(x, y, t) < r \). Conversely, if \( M(x, y, t) > 1 - r \) and \( N(x, y, t) < r \), since \( M \) and \( N \) are continuous functions, there exists an \( s > 0 \) such that \( M(x, y, t - s) > 1 - r \) and \( N(x, y, t - s) < r \) and so \( d_r(x, y) \leq t - s < t \). This completes the proof. \( \square \)
In fuzzy metric spaces \((X, M, *)\), the map \(f : X \to X\) is said to be a fuzzy contraction if there exists \(k \in (0, 1)\) such that
\[
\frac{1}{M(f(x), f(y), t)} - 1 \leq k\left(\frac{1}{M(x, y, t)} - 1\right), \forall x, y \in X, \forall t > 0.
\]
Several fixed point theorem has been proved by using (5) in a fuzzy metric spaces (see [1, 7]).

If the fuzzy metric space \((X, N, \diamond)\) is a dual space of \((X, M, *)\), the map \(f : X \to X\) in \((X, N, \diamond)\) may also enjoy a fuzzy contractive condition. Since the definition of the dual space \((X, N, \diamond)\) is similar in the sense of metric space, we can consider the following new contractive condition for a self mapping \(f\) in \((X, N, \diamond)\).

**Definition 3.3.** (New) Let \((X, N, \diamond)\) be a fuzzy metric space. The map \(f : X \to X\) is a fuzzy contraction in \((X, N, \diamond)\) if there exists \(k \in (0, 1)\) such that
\[
N(f(x), f(y), t) \leq kN(x, y, t), \forall x, y \in X, \forall t > 0.
\]

By using the contractive conditions (5) and (6), now we are able to define an intuitionistic fuzzy contractive map \(f\) as follow:

**Definition 3.4.** (New) Let \((X, M, N, *, \diamond)\) be an intuitionistic fuzzy metric space. We say that the mapping \(f : X \to X\) is intuitionistic fuzzy contractive if there exists \(k \in (0, 1)\) such that
\[
\frac{1}{M(f(x), f(y), t)} - 1 \leq k\left(\frac{1}{M(x, y, t)} - 1\right)
\]
and
\[
N(f(x), f(y), t) \leq kN(x, y, t),
\]
for each \(x, y \in X\) and \(t > 0\).

We give a definition for a intuitionistic fixed point theorem in intuitionistic fuzzy metric space \((X, M, N, *, \diamond)\).

**Definition 3.5.** (New) Let \((X, M, N, *, \diamond)\) be an intuitionistic fuzzy metric space and let \(f : X \to X\) be an intuitionistic fuzzy contractive mapping. Then there exists \(z \in X\) such that \(z = f(z)\) (We call \(z\) an intuitionistic fuzzy fixed point of \(f\)).

Now, we prove the following theorem.

**Theorem 3.6.** Let \((X, M, N, *, \diamond)\) be a complete intuitionistic fuzzy metric space. Let \(f : X \to X\) be an intuitionistic fuzzy contractive mapping. Then \(f\) has a unique intuitionistic fixed point.
Proof. We fix $x_0 \in X$. Let $x_{n+1} = f(x_n), n \in \mathbb{N}$. We have for $n > m$ and $t > 0$,
\[
\frac{1}{M(x_n, x_{n+m}, t)} - 1 = \frac{1}{M(f(x_{n-1}), f(x_{n+m-1}), t)} - 1 \\
\leq k \left( \frac{1}{M(x_{n-1}, x_{n+m-1}, t)} - 1 \right) \\
\leq k \left( \frac{1}{M(f(x_{n-2}), f(x_{n+m-2}), t)} - 1 \right) \\
\vdots \\
\leq k^n \left( \frac{1}{M(x_0, x_m, t)} - 1 \right),
\]
whence, for $n > m$, $M(x_n, x_{n+m}, t) \to 0$ as $n \to \infty$, that is $M(x_n, x_{n+m}, t) \to 1$ as $n \to \infty$. Also, for $n > m$ and $t > 0$, by (6) we have
\[
N(x_n, x_{n+m}, t) = N(f(x_{n-1}), f(x_{n+m-1}), t) \\
\leq kN(x_{n-1}, x_{n+m-1}, t) \\
= kN(f(x_{n-2}), f(x_{n+m-2}), t) \\
\vdots \\
\leq k^n N(x_0, x_m, t).
\]
Whence, for $n > m$, $N(x_n, x_{n+m}, t) \to 0$ as $n \to \infty$. Therefore, we can conclude that $\{x_n\}$ is a Cauchy sequence in $X$. Since $(X, M, N, *, \odot)$ is complete, the sequence $\{x_n\}$ converges to some $y \in X$. We show that $y$ is an intuitionistic fixed point of $f$, i.e., $y = f(y)$.

By the contractive condition (5) of $f$, we have
\[
\frac{1}{M(y, x_n, t)} - 1 \leq k \left( \frac{1}{M(x_0, x_m, t)} - 1 \right) \to 0, \text{ as } n \to \infty.
\]

Hence, $\lim_{n \to \infty} M(f(y), f(x_n), t) = 1$ for every $t > 0$. And by (6), we have:

$N(f(y), f(x_n), t) \leq kN(y, x_n, t) \to 0$ as $n \to \infty$.

Thus, $\lim_{n \to \infty} N(f(y), f(x_n), t) = 0$ for every $t > 0$. In both cases, we have $\lim_{n \to \infty} f(x_n) = f(y)$, i.e., $\lim_{n \to \infty} x_{n+1} = f(y)$, therefore $y = f(y)$. For uniqueness, assume $z = f(z)$ for some $z \in X$. Then for $t > 0$, we have
\[
\frac{1}{M(y, z, t)} - 1 = \frac{1}{M(f(y), f(z), t)} - 1 \\
\leq k \left( \frac{1}{M(y, z, t)} - 1 \right) \\
\vdots \\
\leq k^n \left( \frac{1}{M(y, z, t)} - 1 \right) \to 0, \text{ as } n \to \infty.
\]

whence, for every $t > 0$ we have $M(y, z, t) = 1$, it follows that $z = y$. This completes the proof. \qed
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References


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