FIXED POINT THEOREM ON INTUITIONISTIC FUZZY METRIC SPACES

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Abstract. In this paper, we introduce intuitionistic fuzzy contraction mapping and prove a fixed point theorem in intuitionistic fuzzy metric spaces.

1. **Introduction**

The notion of intuitionistic fuzzy metric spaces was introduced and studied by Park in [5]. Saadati and Park in [6], further developed the theory of intutionistic fuzzy topology (both in metric and normed) spaces. In this paper, we introduce an intuitionistic fuzzy contraction mapping and prove a fixed point theorem in intuitionistic fuzzy metric spaces. For the basic notions and concepts, we refer to [1, 3, 4, 5, 6].

2. **Preliminaries**

We review some basic concepts in intuitionistic fuzzy metric spaces as well as the intuitionistic fuzzy topology due to Saadati and Park [6].

Definition 2.1. [5, 6] A binary operation $*\colon [0,1]\times[0,1]\to[0,1]$ is a continuous t-norm if it satisfies the following conditions: (i) $*$ is associative and commutative; (ii) $*$ is continuous; (iii) $a*1=a$ for all $a\in[0,1]$; (iv) $a*b\le c*d$ whenever $a\le c$ and $b\le d$, for each $a,b,c,d\in[0,1]$. Two typical examples of continuous t-norm are

$$a*b=ab,$$

$$a*b=min(a,b).$$

Definition 2.2. [5, 6] A binary operation $\diamond\colon [0,1]\times[0,1]\to[0,1]$ is a continuous t-conorm if it satisfies the following conditions: (i) $\diamond$ is associative and commutative; (ii) $\diamond$ is continuous; (iii) $a\diamond 0=a$ for all $a\in[0,1]$; (iv) $a*b\le c*d$ whenever $a\le c$ and $b\le d$, for each $a,b,c,d\in[0,1]$. Two typical examples of t-conorm are

$$a\diamond b=min(a+b,1),$$

$$a\diamond b=max(a,b).$$

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Lemma 2.3. [5, 6] *If $*$ is a continuous t-norm and $\diamond$ is continuous t-conorm, then:*
(i) For every a,b $\in [0,1]$, if $a > b$, there are c, d $\in [0,1]$ such that $a * c \geq b$ and $a \geq b \diamond d$.
(ii) If $a \in [0,1]$, there are b, c$\in [0,1]$ such that $b * b \geq a$ and $a \geq c \diamond c$.

The following definition is obtained from Mihet in [4].

Definition 2.4. [4] A fuzzy metric space in the sense of Kramosil and Michalek is a triple $(X, M, *)$ where X is a nonempty set, $*$ is a continuous t-norm and $M\colon X^2 \times [0,\infty) \to [0,1]$ is a mapping which satisfies the following properties for every x, y, $z \in X$:
(FM-1) $M(x,y,0) = 0$;
(FM-2) $M(x,y,t) = 1, \forall t > 0 \Leftrightarrow x = y$;
(FM-3) $M(x,y,t) = M(y,x,t), \forall t > 0$;
(FM-4) $M(x,y,\cdot)\colon [0,\infty) \to [0,1]$ is left continuous;
(FM-5) $\lim_{t\to\infty} M(x,y,t) = 1$;
(FM-6) $M(x,z,t+s) \geq M(x,y,t) * M(y,z,s), \forall t,s > 0$.

In $(X, M, *)$, the open ball $B_x(r,t)$ for $t > 0$ with center $x \in X$ and radius $r \in (0,1)$ is defined as

$$B_x(r,t) = \{y \in X \mid M(x,y,t) > 1 - r\}. \tag{1}$$

The family $\{B_x(r,t) \mid x \in X, r \in (0,1), t > 0\}$ is a neighborhood system for a Hausdorff topology on X induced by the fuzzy metric M. In a similar fashion, the dual space of $(X, M, *)$ is the fuzzy metric space $(X, N, \diamond)$ defined below:

Definition 2.5. (New) A fuzzy metric space $(X, N, \diamond)$, where X is a nonempty set, $\diamond$ is a continuous t-conorm and $N\colon X^2 \times [0,\infty) \to [0,1]$ is a mapping assumed to satisfies the following properties for all x, y, $z \in X$:
(FM-D1) $N(x,y,0) = 1$;
(FM-D2) $N(x,y,t) = 0, \forall t > 0 \Leftrightarrow x = y$;
(FM-D3) $N(x,y,t) = N(y,x,t), \forall t > 0$;
(FM-D4) $N(x,y,\cdot)\colon [0,\infty) \to [0,1]$ is left continuous;
(FM-D5) $\lim_{t\to\infty} N(x,y,t) = 0$;
(FM-D6) $N(x,z,t+s) \leq N(x,y,t) \diamond N(y,z,s), \forall t,s > 0$.

In $(X, N, \diamond)$, the open ball $D_x(r,t)$ for $t > 0$ with center $x \in X$ and radius $r \in (0,1)$ is defined as

$$D_x(r,t) = \{y \in X \mid N(x,y,t) < r\}. \tag{2}$$

The family $\{D_x(r,t) \mid x \in X, r \in (0,1), t > 0\}$ is a neighborhood's system for a Hausdorff topology on X induced by the fuzzy metric N. The following definition is introduced and studied by Park in [5].

Definition 2.6. [5] A 5-tuple $(X, M, N, *, \diamond)$ is called a intuitionistic fuzzy metric space if X is an arbitrary nonempty set, $*$ a continuous t-norm, $\diamond$ a continuous t-conorm and M, N are fuzzy sets on $X^2 \times (0,\infty)$, satisfying the following conditions for each x, y, $z \in X$ and $t, s > 0$:
(a)$M(x,y,t) + N(x,y,t) \leq 1$;

(b)$M(x,y,t) > 0$;
(c)$M(x,y,t) = 1 \Leftrightarrow x = y$;
(d)$M(x,y,t) = M(y,x,t)$;
(e)$M(x,y,t) * M(y,z,s) \leq M(x,z,t+s)$;
(f)$M(x,y,\cdot)\colon [0,\infty) \to [0,1]$ is left continuous;
(g)$N(x,y,t) = N(y,x,t)$;
(h)$N(x,y,t) \diamond N(y,z,s) \geq N(x,z,t+s)$;
(i)$N(x,y,\cdot)\colon [0,\infty) \to [0,1]$ is continuous..

The pair (M,N) is called an intuitionistic fuzzy metric on X. Every fuzzy metric space $(X,M,*)$ is an intuitionistic fuzzy metric space of the form $(X,M,1-M,*,\diamond)$ such that t-norm $*$ and t-conorm $\diamond$ are associated [6], i.e $x \diamond y = 1-[(1-x)*(1-y)]$ for any x, $y \in X$.

Let $(X,M,N,*,\diamond)$ be a intuitionistic fuzzy metric space. For $t > 0$, the open ball $G_x(r,t)$ with center $x \in X$ and radius $r \in (0,1)$ is defined by

(3) $$G_x(r,t) = \{y \in X \mid M(x,y,t) > 1-r, N(x,y,t) < r\}.$$

Note that it can be easily seen that $G_x = B_x \cap D_x$ where B_x and D_x as given by (1) and (2) respectively.

Since $*$ and $\diamond$ are respectively a continuous t-norm and t-conorm, the family $\{G_x(r,t) \mid x \in X, r \in (0,1), t > 0\}$ generates a topology $T_{(M,N)}$, called the (M,N)-topology (see [3, 6]). We have:
$A \in T_{(M,N)}$ if and only if $\forall x \in A, \exists t > 0, \exists r \in (0,1)$ such that $G_x(r,t) \subset A$.

We denote the (M,N)-uniformity (or the uniformity generated by M, and N) by $U_{(M,N)}$. The family $\{U_{r,t}\}_{r\in(0,1),t>0}$, where

$$U_{r,t} = \{(x,y) \in X^2 \mid M(x,y,t) > 1-r, N(x,y,t) < r\},$$

is a base for this uniformity.

Definition 2.7. [5, 6] Let $(X,M,N,*,\diamond)$ be the intuitionistic fuzzy metric space endowed with (m,n)-topology and $\{x_n\}$ in X. Then
(i) $x_n \to x \Leftrightarrow M(x_n,x,t) \to 1$ and $N(x_n,x,t) \to 0$ as $n \to \infty$, for each $t > 0$.
(ii) $\{x_n\}$ is called a (M,N)-Cauchy sequence if for each $r \in (0,1)$ and $t > 0$, there exists an integer n_0 such that $M(x_n,x_m,t) > 1-r$ and $N(x_n,x_m,t) < r$ for each n,$m \geq n_0$.
(iii) The intuitionistic fuzzy metric space $(X,M,N,*,\diamond)$ is said to be (M,N)-complete if every (M,N)-Cauchy sequence is convergent.

3. Main Results

In the following sequel the letters $\mathbb{N}$ and $\mathbb{R}^+$ denote the sets of positive integer numbers and positive real numbers, respectively.

Definition 3.1. [2] A quasi-metric on a set X is a function $d\colon X^2 \to \mathbb{R}^+$ satisfying the following conditions for every x, y, $z \in X$:
(QM-1)$d(x,y) = 0$;
(QM-2)$d(x,y) = d(y,x)$;
(QM-3)$d(x,z) \leq d(x,y) + d(y,z)$.

Proposition 3.2. *Let $(X, M, N, *, \diamond)$ be the intuitionistic fuzzy metric space. For any $r \in (0, 1]$, we define $d \colon X^2 \to \mathbb{R}^+$ as follows:*

$$d_r(x, y) = inf\{t > 0 \mid M(x, y, t) > 1 - r, N(x, y, t) < r\} \tag{4}$$

Then,
(1) *$(X, d_r : r \in (0, 1])$ is a generating space of a quasi-metric family.*
(2) *the topology $T_{(d_r)}$ on $(X, d_r : r \in (0, 1])$ coincides with the (M, N)-topology on $(X, M, N, *, \diamond)$, (i.e., d_r is a compatible symmetric for $T_{(M,N)}$).*

Proof. (1)From the definition of $\{d_r : r \in (0, 1]\}$, it is easy to see that $\{d_r : r \in (0, 1]\}$ satisfies the condition (QM-1) and (QM-2) of Definiton 3.1. Now we prove that $\{d_r : r \in (0, 1]\}$ also satisfies the condition (QM-3). Since $*$ and $\diamond$ are continuous, by Lemma 2.3.(ii), for any given $r \in (0, 1)$, there exists $r' \in (0, r)$ such that

$$(1 - r') * (1 - r') > 1 - r$$

and

$$r' \diamond r' < r$$

. Setting $d_r(x, y) = a$ and $d_r(y, z) = b$, in equation (4),it follows that for any given $t > 0$,

$$M(x, y, a + t) > 1 - r', N(x, y, a + t) < r'$$

and

$$M(x, z, b + t) > 1 - r', N(y, z, b + t) < r'.$$

Whence

$$M(x, z, a + b + 2t) \geq M(x, y, a + t) * M(y, z, b + t) > (1 - r') * (1 - r') > 1 - r$$

and

$$N(x, z, a + b + 2t) \leq N(x, y, a + t) \diamond N(y, z, b + t) < r' \diamond r' < r.$$

Hence, we have

$$d_r(x, z) \leq a + b + 2t = d_r(x, y) + d_r(y, z) + 2t.$$

By the arbitrariness of $t > 0$, we have

$$d_r(x, z) \leq d_r(x, y) + d_r(y, z).$$

(2) To prove this condition, it is only necessary to show that for any $t > 0$ and $r \in (0, 1)$

$$d_r(x, y) < t \Leftrightarrow M(x, y, t) > 1 - r, N(x, y, t) < r.$$

In fact, if $d_r(x, y) < t$, then by (4), we have $M(x, y, t) > 1 - r$ and $N(x, y, t) < r$. Conversely, if $M(x, y, t) > 1 - r$ and $N(x, y, t) < r$, since M and N are continuous functions, there exists an $s > 0$ such that $M(x, y, t-s) > 1-r$ and $N(x, y, t-s) < r$ and so $d_r(x, y) \leq t - s < t$. This completes the proof. □

In fuzzy metric spaces $(X, M, *)$, the map $f\colon X \to X$ is said to be a fuzzy contraction if there exists $k \in (0, 1)$ such that

$$\frac{1}{M(f(x), f(y), t)} - 1 \leq k(\frac{1}{M(x, y, t) - 1}), \forall x, y \in X, \forall t > 0. \tag{5}$$

Several fixed point theorem has been proved by using (5) in a fuzzy metric spaces (see [1, 7]).
If the fuzzy metric space $(X, N, \diamond)$ is a dual space of $(X, M, *)$, the map $f\colon X \to X$ in $(X, N, \diamond)$ may also enjoy a fuzzy contractive condition. Since the definition of the dual space $(X, N, \diamond)$ is similar in the sense of metric space, we can consider the following new contractive condition for a self mapping f in $(X, N, \diamond)$.

Definition 3.3. (New) Let $(X, N, \diamond)$ be a fuzzy metric space. The map $f\colon X \to X$ is a fuzzy contraction in $(X, N, \diamond)$ if there exists $k \in (0, 1)$ such that

$$N(f(x), f(y), t) \leq kN(x, y, t), \forall x, y \in X, \forall t > 0. \tag{6}$$

By using the contractive conditions (5) and (6), now we are able to define an intuitionistic fuzzy contractive map f as follow:

Definition 3.4. (New) Let $(X, M, N, *, \diamond)$ be an intuitionistic fuzzy metric space. We say that the mapping $f\colon X \to X$ is intuitionistic fuzzy contractive if there exists $k \in (0, 1)$ such that

$$\frac{1}{M(f(x), f(y), t)} - 1 \leq k(\frac{1}{M(x, y, t) - 1})$$

and

$$N(f(x), f(y), t) \leq kN(x, y, t),$$

for each x, $y \in X$ and $t > 0$.

We give a definition for a intuitionistic fixed point theorem in intuitionistic fuzzy metric space $(X, M, N, *, \diamond)$.

Definition 3.5. (New) Let $(X, M, N, *, \diamond)$ be an intuitionistic fuzzy metric space and let $f\colon X \to X$ be an intuitionistic fuzzy contractive mapping. Then there exists $z \in X$ such that $z = f(z)$ (We call z an intuitionistic fuzzy fixed point of f).

Now, we prove the following theorem.

Theorem 3.6. *Let $(X, M, N, *, \diamond)$ be a complete intuitionistic fuzzy metric space. Let $f\colon X \to X$ be an intuitionistic fuzzy contractive mapping. Then f has a unique intuitionistic fixed point.*

Proof. We fix $x_0 \in X$. Let $x_{n+1} = f(x_n), n \in \mathbb{N}$. We have for $n > m$ and $t > 0$,

$$\begin{aligned}
\frac{1}{M(x_n, x_{n+m}, t)} - 1 &= \frac{1}{M(f(x_{n-1}), f(x_{n+m-1}), t)} - 1 \\
&\leq k\left(\frac{1}{M(x_{n-1}, x_{n+m-1}, t)} - 1\right) \\
&= k\left(\frac{1}{M(f(x_{n-2}), f(x_{n+m-2}), t)} - 1\right) \\
&\vdots \\
&\leq k^n\left(\frac{1}{M(x_0, x_m, t)} - 1\right).
\end{aligned}$$

Whence, for $n > m, \frac{1}{M(x_n, x_{n+m}, t)} - 1 \to 0$ as $n \to \infty$, that is $M(x_n, x_{n+m}, t) \to 1$ as $n \to \infty$. Also, for $n > m$ and $t > 0$, by (6) we have

$$\begin{aligned}
N(x_n, x_{n+m}, t) &= N(f(x_{n-1}), f(x_{n+m-1}), t) \\
&\leq kN(x_{n-1}, x_{n+m-1}, t) \\
&= kN(f(x_{n-2}), f(x_{n+m-2}), t) \\
&\vdots \\
&\leq k^n N(x_0, x_m, t).
\end{aligned}$$

Whence, for $n > m$, $N(x_n, x_{n+m}, t) \to 0$ as $n \to \infty$. Therefore, we can conclude that $\{x_n\}$ is a Cauchy sequence in X. Since $(X, M, N, *, \diamond)$ is complete, the sequence $\{x_n\}$ converges to some $y \in X$. We show that y is an intuitionistic fixed point of f, i.e $y = f(y)$. By the contractive condition (5) of f, we have

$$\frac{1}{M(f(y), f(x_n), t)} - 1 \leq k\left(\frac{1}{M(y, x_n, t)} - 1\right) \to 0, asn \to \infty.$$

Hence,$\lim_{n\to\infty} M(f(y), f(x_n), t) = 1$ for every $t > 0$. And by (6), we have:

$$N(f(y), f(x_n), t) \leq kN(y, x_n, t) \to 0asn \to \infty$$

. Thus, $\lim_{n\to\infty} N(f(y), f(x_n), t) = 0$ for every $t > 0$. In both cases, we have $\lim_{n\to\infty} f(x_n) = f(y)$, i.e., $\lim_{n\to\infty} x_{n+1} = f(y)$, therefore $y = f(y)$. For uniqueness, assume $z = f(z)$ for some $z \in X$. Then for $t > 0$, we have

$$\begin{aligned}
\frac{1}{M(y, z, t)} - 1 = \frac{1}{M(f(y), f(z), t)} - 1 & \\
leq \quad & k\left(\frac{1}{M(y, z, t)} - 1\right) \\
&\vdots \\
leq \quad & k^n\left(\frac{1}{M(y, z, t)} - 1\right) \to 0asn \to \infty.
\end{aligned}$$

whence, for every $t > 0$ we have $M(y, z, t) = 1$, it follows that $z = y$. This completes the proof. $\square$

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