FUZZY CONTROL CHARTS FOR VARIABLE AND ATTRIBUTE QUALITY CHARACTERISTICS

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ABSTRACT. This paper addresses the design of control charts for both variable ($\bar{x}$ chart) and attribute (u and c charts) quality characteristics, when there is uncertainty about the process parameters or sample data. Derived control charts are more flexible than the strict crisp case, due to the ability of encompassing the effects of vagueness in form of the degree of expert's presumption. We extend the use of proposed fuzzy control charts in case of linguistic data using a developed defuzzifier index, which is based on the metric distance between fuzzy sets.

1. Introduction

In processes, the causes of variations may be categorized as chance (un-assignable) causes and special (assignable) causes. The variations due to chance causes are inevitable, and difficult to detect and identify. On the other hand, the variations due to special causes prevent the process being stable and predictable. Such variations should be determined effectively and eliminated from the process by taking necessary corrective actions to maintain the process in control and improve the quality of the products as well.

The statistical process control (SPC) approach employs several powerful problem-solving tools such as histogram, Pareto diagram, cause-effect diagram, and control charts. This approach is useful in achieving process stability and improving capability through the reduction of variability. Control charts, also known as Shewhart [6] control charts, are usually preferred to the other tools. They have the ability to determine whether there are variations created by causes in the process.

In order to use control charts appropriately, first, a suitable approach for determining process characteristics, such as manufacturing type and volume, inspection strategy, and quality characteristics of the product being produced within the process, is selected. Then, by determining the parameters of the related control chart, control limits and control intervals are designed. Choosing the parameters of a control chart may be done by negotiation with experts who are familiar with the nature of the process, or may be done by sampling. In practice there are many situations that there is vagueness about the exact amount of data or parameters. When there is vague data, fuzzy set theory can be used to handle the vagueness. To deal with vagueness of human thought, Zadeh [11] introduced fuzzy set theory, which was oriented to the rationality of uncertainty due to imprecision or vagueness. A major contribution of fuzzy set theory is its capability to represent vague data. The theory also allows mathematical operators and programming to apply to

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the fuzzy domain. A fuzzy set is a class of objects with a continuum of grades of membership. Such a set is characterized by a membership function, which assigns to each object a grade of membership ranging between zero and one [3].

There are a number of researchers who have studied and developed control charts for fuzzy environments. Kahraman *et al.*[4] considered control charts as triangular fuzzy numbers. They proposed the corresponding degrees of membership of the traditional zones in control charts for tracking unnatural patterns. Tannock [7] provided a simple method for identifying a criterion representing the state of control of manufacturing processes having a single critical variable characteristic. Wang and Rowlands [9] proposed a new fuzzy-SPC evaluation and control method. This method uses fuzzy logic to create a fuzzy inference system, which represents SPC zone rules. Yongting [10] developed the concept of fuzzy process capability and proposed the fuzzy process capability index. Wang and Raz [8] considered the construction of control charts for linguistic data. They considered several methods for calculating the values representing sample means, for determining the control limits. Franceschini and Romano [2] proposed control charts for linguistic variables based on the use of Ordered Weight Average (OWA) operator, which does not require explicit information on the shape of the membership function of the level of the linguistic scale.

The main purpose of this paper is to propose control charts for both attributes and variable characteristics, when we are faced with the uncertainty, either in process parameters or in the sample data drawn from the process, using fuzzy logic. Here, for dealing with vagueness about the exact amount of some process parameters such as mean, standard deviation and mean number of defects, we use fuzzy numbers. We transform the degree of the belief of different experts about the amount of process parameters into its equivalent α-cuts. In other words, for example, the expert corresponding to the 0-cut may be the one who is the least certain about the characteristics of the parameters and his estimations are rather wide intervals and that corresponding to the 1-cut may be the one who is the most certain (has most information). We also develop a flexible defuzzifier index based on the metric distance between fuzzy sets when the process data are linguistic variables. The proposed index is capable of transforming a mean fuzzy set of sampling data into its representative value.

The rest of the paper is organized as follows: Section 2 gives a brief overview on the variable and attribute control charts. Section 3 addresses the design of $\overline{x}$ control charts when the mean and standard deviation of process are fuzzy numbers. Section 4 presents u and c control charts with fuzzy mean number of defects. Section 5 examines the case of linguistic data. We conclude the paper in section 6.

2. **Variable and Attribute Control Charts**

Control charts use two types of data, variables data and attributes data. In general, to use variables data, we have to take measurements in units such as length, temperature, etc. On the other hand, attributes data requires a good/bad or go/no-go decision and counting (for example, number of defects, percent late, etc.).

The $\bar{x}$ control chart is designed to be used primarily with variable data, which are usually measurements such as the length of an object, processing time, or the number of objects produced per period. When the mean value of the quality characteristic being studied changes, an "out-of-control" signal is reported. For process outputs which are normally distributed with a mean m and a variance σ^2, the $\bar{x}$ control limits are as follows:

$$\begin{aligned} UCL &= m + z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \cong m + 3\frac{\sigma}{\sqrt{n}} \\ CL &= m \\ LCL &= m - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \cong m - 3\frac{\sigma}{\sqrt{n}} \end{aligned} \tag{1}$$

In many cases we are unable to measure quality characteristics and report them in the form of digits. Quality characteristics that are classified by linguistic terms are called attribute characteristics. For example, number (ratio) of defective units in a product is an attribute characteristic. The most popular control charts for an attribute characteristic are *p, c,* and *u* control charts. A *p* chart is used for controlling the ratio of defective products. In *c* charts instead of the ratio of defective products, we use the number of defects per inspection unit of products. Finally, a *u* chart is used to control the mean number of defects of a product. Using the Poisson distribution, the control limits for a *u* chart are as follows:

$$\begin{aligned} UCL &= \lambda + 3\sqrt{\frac{\lambda}{n}} \\ CL &= \lambda \\ LCL &= \lambda - 3\sqrt{\frac{\lambda}{n}} \end{aligned} \tag{2}$$

where λ is the average number of defects per unit. Also if $LCL<0$, then we take $LCL=0$.

3. Fuzzy $\bar{x}$ Control Charts with Fuzzy Parameters ($\tilde{m}, \tilde{\sigma}$)

Fuzzy logic is a method of common sense or inference based on natural language. Fuzzy logic starts with the concept of a fuzzy set. A fuzzy set is a set without a crisp, clearly defined boundary. It describes vague concept.

In this section we present a fuzzy $\bar{x}$ control chart when the mean and standard deviation of the process are fuzzy parameters. Here there is no apparent value for these parameters. Consider the case where, because of the unknown nature of the process or because of different degree of belief corresponding to various experts, we are unableto assign an exact value to the process parameters. For representing these uncertain situations, we write the process parameters in form of fuzzy numbers. Using *L*-type fuzzy numbers of Dubois and Prade [1], possibility distributions are:

$$\mu_{\mathrm{M}}(m) = L\{(c_m - m)/d_m\} \text{ and } \mu_{\Sigma}(\sigma) = L\{(c_\sigma - \sigma)/d_\sigma\} \tag{3}$$

where *L* is a function type. In case of triangular fuzzy numbers we have:

$$\mu_{\mathrm{M}}(m)=\begin{cases}1-\dfrac{|m-c_m|}{d_m} & c_m-d_m\le m\le c_m+d_m\\ 0 & otherwise\end{cases}\qquad \mu_{\Sigma}(\sigma)=\begin{cases}1-\dfrac{|\sigma-c_\sigma|}{d_\sigma} & c_\sigma-d_\sigma\le\sigma\le c_\sigma+d_\sigma, c_\sigma\ge d_\sigma\\ 0 & otherwise\end{cases} \quad (4)$$

where, $\mu_{\mathrm{M}}(m)$ and $\mu_{\Sigma}(\sigma)$ are the membership functions related to the process mean and standard deviation fuzzy sets represented by M and Σ, respectively. c_m and c_σ are the centers of the respective fuzzy numbers and, d_m and d_σ represent the width or spread around these centers. In case of fuzzy process parameters, the control limits are as follows:

$$\begin{cases}U\tilde{C}L_{\bar{x}}=\tilde{m}+A\tilde{\sigma}\\ \tilde{C}L_{\bar{x}}=\tilde{m}\\ L\tilde{C}L_{\bar{x}}=\tilde{m}-A\tilde{\sigma}\end{cases} \quad (5)$$

where $\tilde{m}$ and $\tilde{\sigma}$ are the triangular fuzzy numbers in equation (4). Each point in the $\bar{x}$ control interval can be represented in form of: $\bar{x}=m+k\sigma$, where, $-A\le k\le A$. It can be shown that for each point in the fuzzy control interval there is a set of membership functions. In other words, each point can belong to the interval with more than one degree of membership. We assume each point pertains to the interval with its maximum degree of membership. Suppose $\tilde{\bar{x}}_1=\tilde{m}+k_1\tilde{\sigma}$, and $\tilde{\bar{x}}_2=\tilde{m}+k_2\tilde{\sigma}$ are two fuzzy sets ($k_1<k_2$) shown in Figure 1. Using fuzzy arithmetic principles we can define the membership of fuzzy numbers $\tilde{\bar{x}}_1$ and $\tilde{\bar{x}}_2$ as follows:

$$\mu_{\tilde{\bar{x}}_{1(2)}}(\bar{x})=\begin{cases}1-\dfrac{\left|\bar{x}-(c_m+k_{1(2)}c_\sigma)\right|}{d_m+\left|k_{1(2)}\right|d_\sigma} & c_m+k_{1(2)}c_\sigma-(d_m+\left|k_{1(2)}\right|d_\sigma)\le\bar{x},\\ & \bar{x}\le c_m+k_{1(2)}c_\sigma+(d_m+\left|k_{1(2)}\right|d_\sigma)\\ 0 & otherwise\end{cases} \quad (6)$$

If $0\le k_1<k_2\le\dfrac{2d_m}{c_\sigma-d_\sigma}+k_1\dfrac{c_\sigma+d_\sigma}{c_\sigma-d_\sigma}$, $k_1<k_2\le\dfrac{2d_m}{c_\sigma+d_\sigma}+k_1\dfrac{c_\sigma-d_\sigma}{c_\sigma+d_\sigma}$ where, $k_1,k_2\le 0$, $k_2\le k_1+\dfrac{2d_m}{c_\sigma-d_\sigma}$ where, $k_1<,k_2>0$, then the two fuzzy sets overlap.

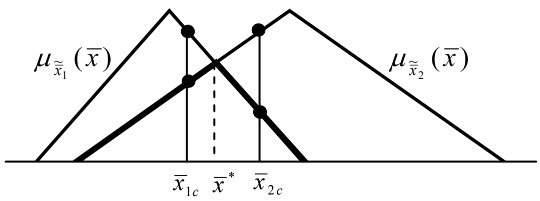

FIGURE 1. $\tilde{\bar{x}}_1$ and $\tilde{\bar{x}}_2$ Fuzzy Sets

$$\text{if} \quad \bar{x} = \bar{x}^* \Leftrightarrow \mu_{\tilde{\tilde{x}}_1}(\bar{x}) = \mu_{\tilde{\tilde{x}}_2}(\bar{x}) \Rightarrow \bar{x}^* = c_m + \frac{c_\sigma\left[(k_1 + k_2)d_m + (k_2|k_1| + k_1|k_2|)d_\sigma\right]}{2d_m + (|k_1| + |k_2|)d_\sigma}$$

$$i)\ c_m + k_2 c_\sigma - (d_m + |k_2| d_\sigma) \le \bar{x} \le c_m + k_1 c_\sigma \Rightarrow \mu_{\tilde{\tilde{x}}_1}(\bar{x}) - \mu_{\tilde{\tilde{x}}_2}(\bar{x}) = \frac{(\bar{x} - c_m)(|k_2| - |k_1|)d_\sigma + c_\sigma\left[d_m(k_2 - k_1) + d_\sigma(k_2|k_1| - k_1|k_2|)\right]}{(d_m + |k_1| d_\sigma)(d_m + |k_2| d_\sigma)}$$

$$\left.\begin{array}{l}
\text{if}\ k_1, k_2 \ge 0 \Rightarrow \bar{x} - c_m \ge k_2(c_\sigma - d_\sigma) - d_m \Rightarrow (\bar{x} - c_m)(k_2 - k_1)d_\sigma + c_\sigma d_m(k_2 - k_1) \ge \\
\qquad k_2(c_\sigma - d_\sigma)(k_2 - k_1)d_\sigma + d_m(k_2 - k_1)(c_\sigma - d_\sigma) \ge 0 \\[2ex]
\text{if}\ k_1, k_2 \le 0 \Rightarrow \bar{x} - c_m \le k_1 c_\sigma \Rightarrow (\bar{x} - c_m)(k_1 - k_2)d_\sigma + c_\sigma d_m(k_2 - k_1) \ge \\
\qquad k_1 c_\sigma(k_1 - k_2)d_\sigma + c_\sigma d_m(k_2 - k_1) \ge 0 \\[2ex]
\text{if}\ k_1 < 0, k_2 > 0, k_1 + k_2 > 0 \Rightarrow (\bar{x} - c_m)(k_2 + k_1)d_\sigma + c_\sigma\left[d_m(k_2 - k_1) - 2k_1 k_2 c_\sigma d_\sigma\right] \ge \\
\qquad k_2(c_\sigma - d_\sigma)(k_1 + k_2)d_\sigma + d_m(k_2(c_\sigma - d_\sigma) - k_1(c_\sigma + d_\sigma)) - 2k_1 k_2 c_\sigma d_\sigma \ge 0 \\[2ex]
\text{if}\ k_1 < 0, k_2 > 0, k_1 + k_2 < 0 \Rightarrow (\bar{x} - c_m)(k_2 + k_1)d_\sigma + c_\sigma\left[d_m(k_2 - k_1) - 2k_1 k_2 c_\sigma d_\sigma\right] \ge \\
\qquad k_1 c_\sigma(k_2 + k_1)d_\sigma + c_\sigma\left[d_m(k_2 - k_1) - 2k_1 k_2 c_\sigma d_\sigma\right] \ge 0
\end{array}\right\} \Rightarrow \mu_{\tilde{\tilde{x}}_1}(\bar{x}) \ge \mu_{\tilde{\tilde{x}}_2}(\bar{x})$$

$$ii)\ c_m + k_1 c_\sigma \le \bar{x} \le \bar{x}^* \Rightarrow \begin{cases} 1 - \dfrac{(k_2 - k_1)c_\sigma}{2d_m + (|k_1| + |k_2|)d_\sigma} \le 1 - \dfrac{\bar{x} - (c_m + k_1 c_\sigma)}{d_m + |k_1| d_\sigma} \le 1 \\[3ex] 1 - \dfrac{(k_2 - k_1)c_\sigma}{d_m + |k_2| d_\sigma} \le 1 + \dfrac{\bar{x} - (c_m + k_2 c_\sigma)}{d_m + |k_2| d_\sigma} \le 1 - \dfrac{(k_2 - k_1)c_\sigma}{2d_m + (|k_1| + |k_2|)d_\sigma} \end{cases} \Rightarrow \mu_{\tilde{\tilde{x}}_1}(\bar{x}) \ge \mu_{\tilde{\tilde{x}}_2}(\bar{x})$$

$$iii)\ \bar{x}^* \le \bar{x} \le c_m + k_2 c_\sigma \Rightarrow \begin{cases} 1 - \dfrac{(k_2 - k_1)c_\sigma}{2d_m + (|k_1| + |k_2|)d_\sigma} \le 1 + \dfrac{\bar{x} - (c_m + k_2 c_\sigma)}{d_m + |k_2| d_\sigma} \le 1 \\[3ex] 1 - \dfrac{(k_2 - k_1)c_\sigma}{d_m + |k_1| d_\sigma} \le 1 - \dfrac{\bar{x} - (c_m + k_1 c_\sigma)}{d_m + |k_1| d_\sigma} \le 1 - \dfrac{(k_2 - k_1)c_\sigma}{2d_m + (|k_1| + |k_2|)d_\sigma} \end{cases} \Rightarrow \mu_{\tilde{\tilde{x}}_2}(\bar{x}) \ge \mu_{\tilde{\tilde{x}}_1}(\bar{x})$$

$$iv)\ c_m + k_2 c_\sigma \le \bar{x} \le c_m + k_1 c_\sigma + (d_m + |k_1| d_\sigma) \Rightarrow \mu_{\tilde{\tilde{x}}_2}(\bar{x}) - \mu_{\tilde{\tilde{x}}_1}(\bar{x}) = \frac{(\bar{x} - c_m)(|k_2| - |k_1|)d_\sigma + c_\sigma\left[d_m(k_2 - k_1) + d_\sigma(k_2|k_1| - k_1|k_2|)\right]}{(d_m + |k_1| d_\sigma)(d_m + |k_2| d_\sigma)}$$

$$\left.\begin{array}{l}
if\ \ k_1,k_2\geq 0 \Rightarrow \bar{x}-c_m \geq k_2 c_\sigma \geq 0 \Rightarrow (\bar{x}-c_m)(k_2-k_1)d_\sigma + c_\sigma d_m (k_2-k_1)\geq 0 \\[2ex]
if\ \ k_1,k_2\leq 0 \Rightarrow \bar{x}-c_m \leq k_1(c_\sigma-d_\sigma)+d_m \leq 0 \Rightarrow (\bar{x}-c_m)(k_1-k_2)d_\sigma + c_\sigma d_m (k_2-k_1)\geq 0 \\[2ex]
if\ \ k_1<0, k_2>0, k_1+k_2>0 \Rightarrow \bar{x}-c_m \geq k_2 c_\sigma \geq 0 \\
\quad \Rightarrow (\bar{x}-c_m)(k_2+k_1)d_\sigma + c_\sigma\left[d_m(k_2-k_1)-2k_1k_2c_\sigma d_\sigma\right]\geq 0 \\[2ex]
if\ \ k_1<0, k_2>0, k_1+k_2<0 \Rightarrow \bar{x}-c_m \leq k_1(c_\sigma-d_\sigma)+d_m \leq 0 \\
\quad \Rightarrow (\bar{x}-c_m)(k_1+k_2)d_\sigma + c_\sigma\left[d_m(k_2-k_1)-2k_1k_2c_\sigma d_\sigma\right]\geq 0
\end{array}\right\} \Rightarrow \mu_{\tilde{x}_2}(\bar{x}) \geq \mu_{\tilde{x}_1}(\bar{x})$$

From the above relations, it can be inferred that for points such as $\bar{x}_{1c}$ or $\bar{x}_{2c}$ in the fuzzy control interval, there is a set of membership grades. Consequently each point belongs to the interval with its maximum degree of membership. Finally we have:

$$if: \bar{x} \in \left[c_m - Ac_\sigma, c_m + Ac_\sigma\right] \qquad then\ \ \mu_{\tilde{x}} = 1.$$

$$if: \bar{x} \in \left[c_m + Ac_\sigma, c_m + d_m + A(c_\sigma + d_\sigma)\right]\ \ then\ \ \mu_{\tilde{x}} = \max_i \{\mu_{\tilde{x}_i}(\bar{x}) \,|\, 0 \leq \mu_{\tilde{x}_i}(\bar{x}) \leq 1 - \frac{\bar{x}-(c_m + Ac_\sigma)}{d_m + Ad_\sigma}\} = 1 - \frac{\bar{x}-(c_m+Ac_\sigma)}{d_m+Ad_\sigma}.$$

$$if: \bar{x} \in \left[c_m - d_m - A(c_\sigma + d_\sigma), c_m - Ac_\sigma\right]\ \ then\ \ \mu_{\tilde{x}} = \max_i \{\mu_{\tilde{x}_i}(\bar{x}) \,|\, 0 \leq \mu_{\tilde{x}_i}(\bar{x}) \leq 1 + \frac{\bar{x}-(c_m - Ac_\sigma)}{d_m + Ad_\sigma}\} = 1 + \frac{\bar{x}-(c_m-Ac_\sigma)}{d_m+Ad_\sigma}.$$

In other words, the possibility distribution of a fuzzy control interval is a symmetric trapezoidal fuzzy distribution with the parameters below:

$$(c_m - d_m - A(c_\sigma + d_\sigma),\ c_m - Ac_\sigma,\ c_m + Ac_\sigma,\ c_m + d_m + A(c_\sigma + d_\sigma)).$$

If $A = A_1 = 2/\sqrt{n}$, the warning interval is obtained. Figure 2 shows a fuzzy $\bar{x}$ control interval with fuzzy parameters.

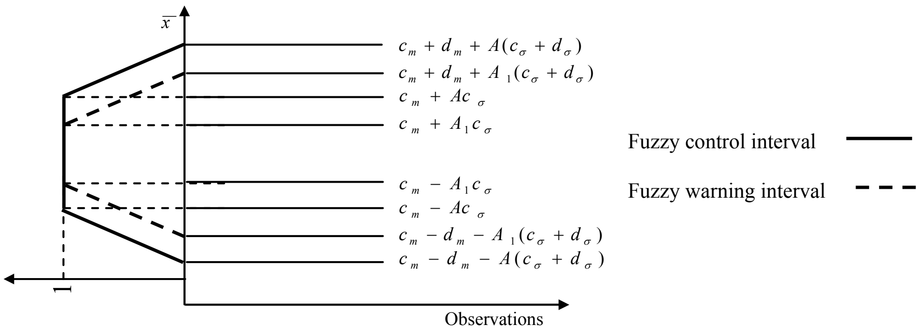

FIGURE 2. Fuzzy $\bar{x}$ Control Interval (Approximate Figure)

Using the α-cuts of a trapezoidal fuzzy number, the parametric control and warning intervals obtain as follows:

$$\begin{aligned} &UCL_{\bar{x}}(\alpha) = (d_m + Ad_\sigma)(1-\alpha) + c_m + Ac_\sigma \qquad && UWL_{\bar{x}}(\alpha) = (d_m + A_1 d_\sigma)(1-\alpha) + c_m + A_1 c_\sigma \\ &LCL_{\bar{x}}(\alpha) = -(d_m + Ad_\sigma)(1-\alpha) + c_m - Ac_\sigma && LWL_{\bar{x}}(\alpha) = -(d_m + A_1 d_\sigma)(1-\alpha) + c_m - A_1 c_\sigma \end{aligned} \tag{7}$$

4. Fuzzy u Control Charts with Fuzzy Mean Number of Defects (λ)

This section investigates the fuzzy *u* control chart with fuzzy mean number of defects. In this case the mean number of defects is not unknown but its exact value is unknown. Suppose that Λ is the fuzzy set related to the mean number of defects. The membership function of Λ is as follows:

$$\mu_\Lambda(\lambda) = \begin{cases} 1 - \dfrac{|\lambda - c_\lambda|}{d_\lambda} & c_\lambda - d_\lambda \le \lambda \le c_\lambda + d_\lambda \, , c_\lambda , d_\lambda \ge 0 \\ 0 & otherwise \end{cases} \tag{8}$$

where, $\mu_\Lambda(\lambda)$ is the membership function of the mean number of defects in the fuzzy set represented by Λ, c_λ is the center of the fuzzy number and d_λ is the width or spread around the center. Since λ is a positive value, we can infer that $c_\lambda \ge d_\lambda$. In case of crisp *u* control chart, when $\lambda \le 9/n$, we set the lower control limit to be zero. In the fuzzy case the conditions are a bit different. First consider the case $n(c_\lambda - d_\lambda) \ge 9$, where *n* is the sample size. Each point *x* in the *u* control interval can be represent as: $x = \lambda + 3k\sqrt{\lambda/n}, -1 \le k \le 1$, so for each value of *x*, the fuzzy equivalent is: $\tilde{x} = \tilde{\lambda} + 3k\sqrt{\tilde{\lambda}/n} \quad -1 \le k \le 1$. The fuzzy control limits are:

$$\begin{cases} U\tilde{C}L_x = \tilde{\lambda} + 3\sqrt{\tilde{\lambda}/n} \\ \tilde{C}L_x = \tilde{\lambda} \\ L\tilde{C}L_x = \tilde{\lambda} - 3\sqrt{\tilde{\lambda}/n} \end{cases} \tag{9}$$

By the extension principle, the membership function of $\tilde{x}$ will be obtained as follows:

$$x = \lambda + 3k\sqrt{\lambda/n}, -1 \le k \le 1 \;\Rightarrow n\lambda^2 - (2nx + 9k^2)\lambda + nx^2 = 0$$

$$\begin{cases} if \quad 0 \le k \le 1 \Rightarrow \lambda = \dfrac{(2nx + 9k^2) - k\sqrt{81k^2 + 36nx}}{2n} \\ if \quad -1 \le k < 0 \Rightarrow \lambda = \dfrac{(2nx + 9k^2) + |k|\sqrt{81k^2 + 36nx}}{2n} \end{cases} \Rightarrow \lambda = \frac{(2nx + 9k^2) - k\sqrt{81k^2 + 36nx}}{2n}$$

$$\mu_{\tilde{x}}(x) = 1 - \frac{\left|(2nx + 9k^2) - k\sqrt{81k^2 + 36nx} - 2nc_\lambda\right|}{2nd_\lambda} \tag{10}$$

The following relations result from equation (10):

$$1)\quad \mu_{\tilde{x}}(x)=0 \Leftrightarrow x=\begin{cases}(c_\lambda-d_\lambda)+3k\sqrt{\dfrac{c_\lambda-d_\lambda}{n}}\\ or\\ (c_\lambda+d_\lambda)+3k\sqrt{\dfrac{c_\lambda+d_\lambda}{n}}\end{cases} \tag{11}$$

$$2)\quad \mu_{\tilde{x}}(x)=1 \Leftrightarrow x=c_\lambda+3k\sqrt{\frac{c_\lambda}{n}}$$

When $n(c_\lambda-d_\lambda)\geq 9$ and $-1\leq k<0$, we have: $c_\lambda-d_\lambda+3k\sqrt{c_\lambda-d_\lambda/n}<c_\lambda+3k\sqrt{c_\lambda/n}<c_\lambda+d_\lambda+3k\sqrt{c_\lambda-d_\lambda/n}$. However, when $n(c_\lambda+d_\lambda)<9$ and $-1\leq k<0$, these inequalities may be false. Generally $\tilde{x}$ is not a symmetric fuzzy number. Also it is not quite a triangular fuzzy number but close to one. Moreover when $-1\leq k<0$ and $x<c_\lambda+3k\sqrt{c_\lambda/n}$, $\tilde{x}$ is convex. But when $-1\leq k<0$ and $x>c_\lambda+3k\sqrt{c_\lambda/n}$, $\tilde{x}$ is concave. Also when $0\leq k<1$ and $x<c_\lambda+3k\sqrt{c_\lambda/n}$, $\tilde{x}$ is concave and when $0\leq k<1$ and $x>c_\lambda+3k\sqrt{c_\lambda/n}$, $\tilde{x}$ is convex. Figure 3 shows the control limits for a fuzzy *u* control chart.

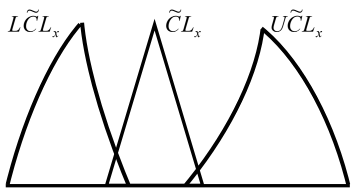

FIGURE 3. Fuzzy Control Limits for a *u* Chart (Approximate Figure)

$$\begin{cases}\mu_{U\tilde{C}L_x}(x)=1-\dfrac{|2nx+9-\sqrt{36nx+81}-2nc_\lambda|}{2nd_\lambda}\\ \mu_{L\tilde{C}L_x}(x)=1-\dfrac{|2nx+9+\sqrt{36x+81}-2nc_\lambda|}{2nd_\lambda}\end{cases}\qquad \begin{cases}\mu_{U\tilde{W}L_x}(x)=1-\dfrac{|2nx+4-4\sqrt{nx+1}-2nc_\lambda|}{2nd_\lambda}\\ \mu_{L\tilde{W}L_x}(x)=1-\dfrac{|2nx+4+4\sqrt{nx+1}-2nc_\lambda|}{2nd_\lambda}\end{cases} \tag{12}$$

In a manner similar to that of the previous section, it can be shown that for each point in the fuzzy control interval there is a set of membership functions and each point belongs to the interval with its maximum degree of membership. Consequently the distribution of fuzzy control interval is:

$$\mu(x)=\begin{cases}1-\dfrac{2nx+9-\sqrt{36nx+81}-2nc_\lambda}{2nd_\lambda} & c_\lambda+3\sqrt{\dfrac{c_\lambda}{n}}<x\le c_\lambda+d_\lambda+3\sqrt{\dfrac{c_\lambda+d_\lambda}{n}}\\ 1+\dfrac{2nx+9+\sqrt{36nx+81}-2nc_\lambda}{2nd_\lambda} & (c_\lambda-d_\lambda)-3\sqrt{\dfrac{c_\lambda-d_\lambda}{n}}\le x<c_\lambda-3\sqrt{\dfrac{c_\lambda}{n}}\\ 1 & c_\lambda-3\sqrt{c_\lambda}\le x\le c_\lambda+3\sqrt{\dfrac{c_\lambda}{n}}\end{cases} \tag{13}$$

The following relations show the parametric form of the fuzzy *u* control interval:

$$\begin{aligned}UCL_x(\alpha)&=c_\lambda+d_\lambda(1-\alpha)+3\sqrt{\frac{c_\lambda+d_\lambda(1-\alpha)}{n}}\\ LCL_x(\alpha)&=c_\lambda-d_\lambda(1-\alpha)-3\sqrt{\frac{c_\lambda-d_\lambda(1-\alpha)}{n}}\end{aligned} \tag{14}$$

Figure 4 shows the nonlinear trapezoidal distribution related to a fuzzy *u* control chart.

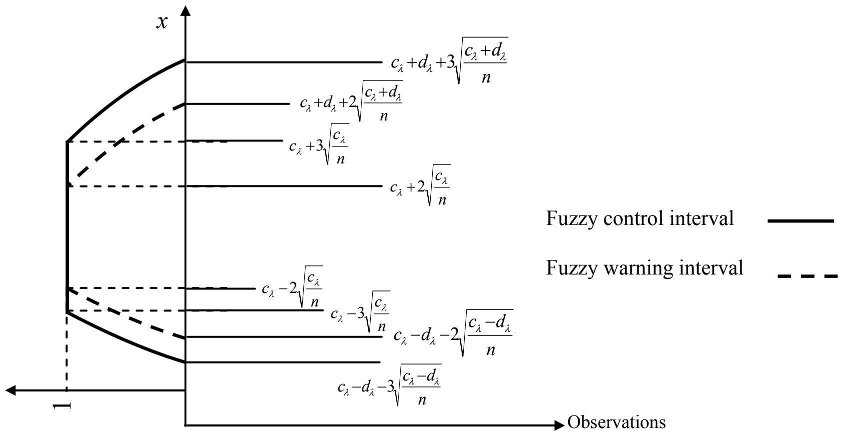

FIGURE 4. Fuzzy *u* Control Interval (Approximate Figure)

Until now, in the design of a fuzzy *u* control interval, we investigated the case with $n(c_\lambda-d_\lambda)\ge 9$. Now we want to design the fuzzy *u* control interval when $n(c_\lambda+d_\lambda)<9$ *or* $n(c_\lambda-d_\lambda)\le 9\le n(c_\lambda+d_\lambda)$. When $\tilde{\lambda}$ is strictly less than 9/*n*, or $n(c_\lambda+d_\lambda)<9$, the lower bound of fuzzy *u* control interval is equal to zero with degree of membership equal to 1 (Figure 5).

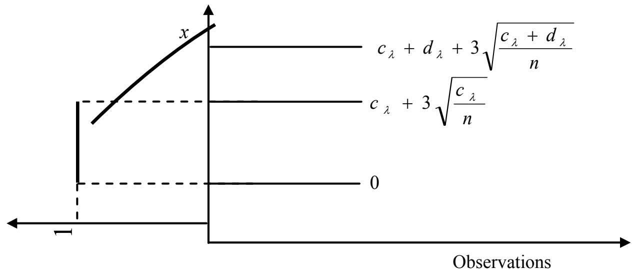

FIGURE 5. Fuzzy *u* Control Interval when $\tilde{\lambda} < 9/n$

When $n(c_\lambda - d_\lambda) \le 9 \le n(c_\lambda + d_\lambda)$, we have two cases:

i) $c_\lambda - d_\lambda < 9/n < c_\lambda$

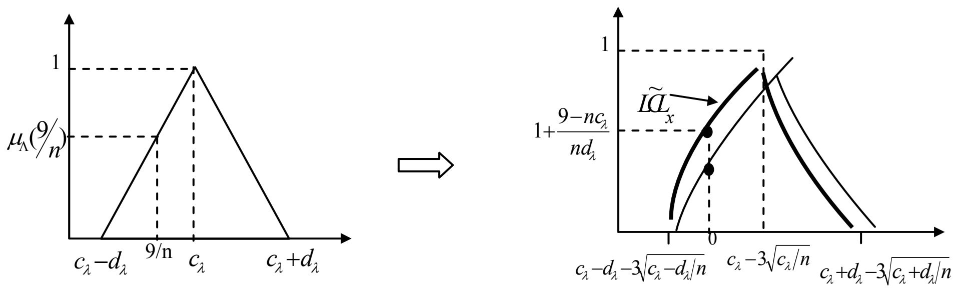

FIGURE 6. Fuzzy *LCL* in Case of $c_\lambda - d_\lambda < 9/n < c_\lambda$

Figure 7. shows the related fuzzy *u* control interval.

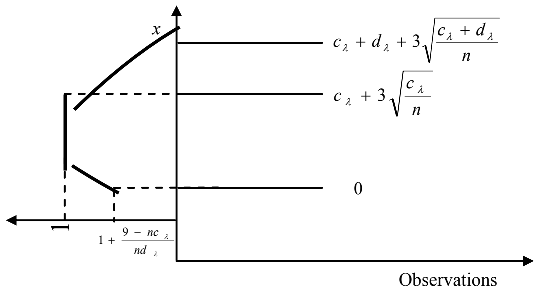

FIGURE 7. Fuzzy *u* Control Interval when $c_\lambda - d_\lambda < 9/n < c_\lambda$

ii) $c_\lambda \le 9/n < c_\lambda + d_\lambda$

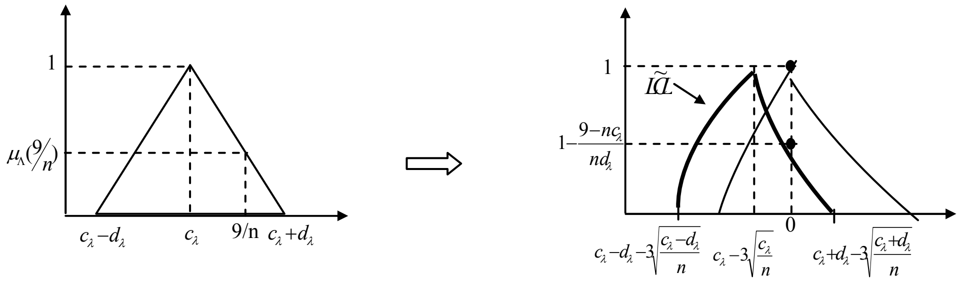

FIGURE 8. Fuzzy LCL in Case of $c_\lambda \le 9/n < c_\lambda + d_\lambda$

Figure 9 shows the related fuzzy *u* control interval.

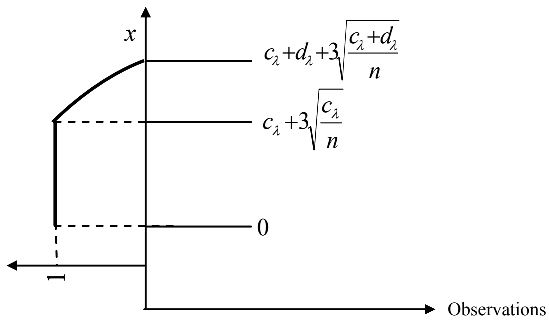

FIGURE 9. Fuzzy *u* Control Interval when $c_\lambda \le 9/n < c_\lambda + d_\lambda$

It should be noted in the formulation proposed for fuzzy *u* control intervals and other related relations, when *n* (number of inspection units) equals 1, we have a fuzzy *c* control chart.

Example 4.1. The control limits for a *c* control chart with $\tilde{\lambda} = (440,600,760)$, and degrees of presumption equal to 0, 0.5, and 1 are:

$$LCL(\alpha) = 600 - 160(1-\alpha) - 3\sqrt{600-160(1-\alpha)} \qquad LWL(\alpha) = 600 - 160(1-\alpha) - 2\sqrt{600-160(1-\alpha)}$$

$$UCL(\alpha) = 600 + 160(1-\alpha) + 3\sqrt{600+160(1-\alpha)} \qquad UWL(\alpha) = 600 + 160(1-\alpha) + 2\sqrt{600+160(1-\alpha)}$$

$$\alpha = 0 \Rightarrow \begin{cases} LCL = 377.07 \\ LWL = 398.04 \\ UWL = 398.04 \\ UCL = 815.13 \end{cases} \qquad \alpha = 0.5 \Rightarrow \begin{cases} LCL = 451.58 \\ LWL = 474.39 \\ UWL = 732.15 \\ UCL = 758.23 \end{cases} \qquad \alpha = 1 \Rightarrow \begin{cases} LCL = 526.51 \\ LWL = 551.01 \\ UWL = 648.98 \\ UCL = 673.48 \end{cases}$$

5. **Control Charts for Linguistic Data**

In this section we shall extend the use of proposed fuzzy control charts to handle the case with linguistic data. A linguistic variable differs from a numerical variable in that its values are not numbers but words or phrases in some languages. For example in case of p control charts the binary classification into conforming and nonconforming might not be appropriate in many situations where product quality does not change abruptly from satisfactory to worthless, and there might be a number of intermediate levels. To supplement the binary classification, several intermediate levels may be expressed in the form of linguistic terms. For example, the quality of a product can be classified by one of the following terms: "perfect," "good," "medium," "poor," or "bad", depending on its deviation from specifications; appropriately selected continuous functions can then be used to describe the quality characteristic associated with each linguistic term [8].

Wang and Raz [8] proposed an approach based on fuzzy set theory by assigning fuzzy sets to each linguistic term, and then combining them for each sample using rules of fuzzy arithmetic. The result is a single fuzzy set. A measure of centrality of this aggregate fuzzy set is then plotted on a Shewhart type control chart. Instead of using the measure of centrality for linguistic terms to calculate the values representing sample mean we introduce a defuzzifier index based on the metric distance between fuzzy sets [5]. In this approach we can take into account the degree of expert presumption to calculate the values representing sample mean.

Suppose in the construction of a fuzzy control chart a preliminary sample of observations has been taken from the process, where each observation is classified by a linguistic value with a known membership function. Adding these linguistic values in the sample and then dividing by the number of observations in the sample, according to rules of fuzzy arithmetic, the fuzzy set equal to the mean of linguistic terms is obtained. Now we can use the fuzzy control limits described in previous sections. Figure 10 shows supposed control limits. To define the deffuzifier index, which gives the representative value corresponding to the sample mean fuzzy set, we use the following definition:

Definition 5.1. For arbitrary fuzzy numbers $u = (\underline{u}(\alpha), \overline{u}(\alpha))$, $v = (\underline{v}(\alpha), \overline{v}(\alpha))$, the quantity

$$D(u,v) = \left[\int_0^1 (\underline{u}(\alpha) - \underline{v}(\alpha))^2 \, d\alpha + \int_0^1 (\overline{u}(\alpha) - \overline{v}(\alpha))^2 \, d\alpha \right]^{\frac{1}{2}} \tag{15}$$

is the distance between u and v, where $(\underline{z}(\alpha), \overline{z}(\alpha))$ is α-cuts of fuzzy number z [5].

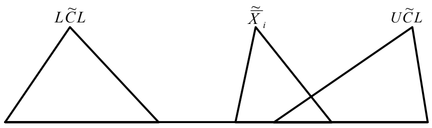

FIGURE 10. Supposed Fuzzy Control Limits (Approximate Figure)

Let $\tilde{\tilde{x}}_i$ be the mean fuzzy set related to the *i* th sample and $\bar{x}_i$ be its representative value. Considering $U\tilde{C}L$ and $\tilde{\tilde{x}}_i$, we have:

$$\frac{D(U\tilde{C}L, \tilde{\tilde{x}}_i)}{D(L\tilde{C}L, U\tilde{C}L)} \equiv \frac{\left|\overline{UCL}(\alpha) - \bar{x}_i\right|}{\overline{UCL}(\alpha) - \underline{LCL}(\alpha)} \tag{16}$$

where $D(u,v)$ is the distance between the fuzzy sets *u, v*. The above index efficiently uses the information encompassed by the possibility distribution of the mean fuzzy set $\tilde{\tilde{x}}_i$, whereas other approaches of computing the representative value of a given fuzzy set such as fuzzy mode, do not consider this information. For example, the fuzzy mod does not consider the spread around the fuzzy sets. We can also form a relation similar to that above, for the case of LCL and $\tilde{\tilde{x}}_i$. So, for each fuzzy number $\tilde{\tilde{x}}_i$, we can get two equivalent representatives for $\bar{x}_i$. Using the average operator or some weighting approaches, the representative value of samples mean fuzzy set will be yielded.

6. **Conclusions**

Design of control charts regarding the uncertain process parameters for both variables and attributes quality characteristic was investigated. Derived control intervals are more flexible than the similar crisp case because they are a function of degree of expert presumption. In case of fuzzy data, we develop a deffuzifier index based on the metric distance between fuzzy sets, which is flexible and easy to compute and efficiently uses the information encompassed by the possibility distribution of sampling fuzzy sets.

Future work may be in several directions as follows: the procedures used to develop the fuzzy control intervals for both variables and attributes can be simply extended to cover the cases of non-symmetric fuzzy numbers as process parameters. Also it can be simply extended our approach to include other types of attribute control charts such as *p* and *np* control charts.

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