INCOMPLETE INTERVAL-VALUED HESITANT FUZZY PREFERENCE RELATIONS IN DECISION MAKING

A. KHALID AND I. BEG

Abstract. In this article, we propose a method to deal with incomplete interval-valued hesitant fuzzy preference relations. For this purpose, an additive transitivity inspired technique for interval-valued hesitant fuzzy preference relations is formulated which assists in estimating missing preferences. First of all, we introduce a condition for decision makers providing incomplete information. Decision makers expressing incomplete data are expected to abide by the proposed condition. This ensures that the estimated preferences are well-defined intervals which otherwise may not be possible. Additionally, this condition eliminates the problem of outlying estimated preferences. After resolving the issue of incompleteness, this article proposes a ranking rule for reciprocal and non-reciprocal interval-valued hesitant fuzzy preference relations.

1. Introduction

An essential component of preference modeling is the representation format that is used to express preferences in a decision modeling framework. Over the past two decades, researchers have proposed several different domains for decision makers, in order to express their choices effectively. As a consequence, the process of decision making has evolved and improved in the sense that real world problems are represented in a more efficient and realistic manner.

Preference relations are used as an essential tool to model decision making and multiple attribute decision making problems [9]. For this purpose, literature proposes preference relations, fuzzy and multiplicative fuzzy preference relations [15], linguistic and multi granular linguistic preference relations [14]. To model subjective uncertainty in decision making models, interval fuzzy preference relations [29, 30] were proposed. To cater for ambiguity and vagueness, hesitant fuzzy sets (HFSs) were introduced by Torra [16, 17] and consequently hesitant fuzzy preference relations (HFPRs) were studied by Xia et al. [23]. Further generalization of HFPRs are interval valued hesitant fuzzy preference relations (IVHFPRs) proposed by Chen et al. [2] and aggregation operations for IVHFPRs were studied in [21].

It is well established in literature that consistency properties are an essential part of decision modeling. Consistency in decision making is based on transitivity properties. Consistency properties of fuzzy and intuitionistic fuzzy preference relations were studied by Xu et al. [32] and Liao et al. [10], respectively. Cardinal

Received: October 2016; Revised: August 2017; Accepted: March 2018

Key words and phrases: Hesitant fuzzy preference relations, Incomplete preference relations, Decision making, Preference modeling.

consistency of preference relations and consistency of two tuple linguistic sets was studied in [4, 5]. Consistency of interval valued intuitionistic fuzzy relations was proposed by Liao et al. [11]. Also, consistency properties of multiplicative hesitant fuzzy preference relations was studied by Liu et al. [12].

Expecting consistent and complete information from all experts in a decision making process is unrealistic. In this paper, it is asserted that incomplete information should not be discarded. The reason is that this may lead to loss of some important data. Literature proposes several methods to deal with incompleteness [1, 3, 8, 18, 35, 27]. Recent studies dealing with incompleteness can be studied in detail in [7, 6, 25, 31, 36, 33, 18, 24]. In this paper, in order to define consistency, we propose transitivity property for IVHFPRs and use this property to tackle incomplete information. We prove that use of transitivity for IVHFPRs alone is not appropriate to estimate preferences because transitivity may lead to estimations that are not well-defined. Moreover, transitivity property alone results in estimated preferences that surpass the defined domain. Literature proposes transformation functions for surpassed values but we stress that such functions void originality of the decision maker's original preferences.

The focus of this paper is on completing missing information without changing the information provided by the decision makers. It is stressed that if estimated preference outlies the settled domain, then the resultant matrix does not qualify as IVHFPR. We assert that if transformation functions are used, then the transformed estimated values are attained at the cost of voiding originality of the decision maker's original preferences. To resolve these problems, we propose an upper bound condition ($cihr$) for all decision makers presenting incomplete information. This condition is inspired by the additive transitivity condition for IVHFPRs. It has already been discussed that transitivity condition alone can not be used to tackle incompleteness in IVHFPRs. This is because it results in intervals that are not well-defined and hence are not applicable in the process of decision making. With the help of property ($cihr$) we can estimate missing preferences that are well defined and respect the given domain.

The other focus of this paper is on ranking of IVHFPRs. Zhu et al. [34] proposed a ranking method for hesitant fuzzy preference relations. Ranking method for hesitant fuzzy linguistic term set was presented in [19]. Score functions were used by Zhang et al. [37] to rank interval valued multiplicative hesitant fuzzy preference relations. Mandal et al. [13] studied a distance metric based ranking rule for IVHFPRs used in critical path problem. This paper proposes a ranking method for IVHFPRs that is less complicated and less time consuming as compared to other ranking methods in the literature. This ranking method finds scores of all interval valued hesitant fuzzy elements and then identifies the least score of each row. The least score represents an alternative that is preferred over all other alternatives by at least this score. Then we find the degree of possibility of an alternative to be preferred over the other. We set a lower cut of 0.5 and stress that an alternative is preferred over the other if the degree of possibility of an alternative preferred over the other alternative is greater than or equal to 0.5. This ranking method is applicable to transitive and non-transitive IVHFPRs.

The main contribution of this paper is the upper bound condition ($cihr$) for IVHFPRs. We split the case of incompleteness in to three categories which have not been considered in literature before. In literauture, it is presumed that number of intervals in each interval valued hesitant fuzzy element (IVHFE) is the same. This is a strong assumption. In our study, equal number of intervals for each IVHFE is one of the three possible cases. The other two cases cater for situations when cardinality of two IVHFE is not the same.

This paper is organized as follows: Section 2 is based on preliminaries that are used in the sequel. Section 3 states that using transitivity alone leads to estimated values that are not intervals. The reason is that they are not well-defined. The second problem caused by transitivity is exaggeration of estimated values from the defined domain. In this section we restate additive transitivity for IVHFPRs and instead of using transitivity in its crude form, we formulate a transitivity inspired method to estimate missing information. This results in estimated preferences that are well-defined. To cater for outliers, we propose condition ($cihr$) for decision makers with incomplete information. This condition ensures that the missing preferences do not surpass the domain $D[0,1]$, representing set of all subintervals of the unit interval. Section 3 uses [28] to develop a ranking method appropriate for IVHFPRs. Section 4 concludes the research article and proposes possible future directions.

2. **Preliminaries**

Definition 2.1. [16] Let X be a fixed non-empty set, a hesitant fuzzy set (HFS) on X is represented by a function h that when applied to X, returns a subset of $[0,1]$. Xia and Xu [22] proposed that HFS can be stated mathematically as follows:

$$E = \{< x, h_E(x) >: x \in X\}$$

where $h_E(x)$, hesitant fuzzy element, is the set of values in $[0,1]$ and it represents the probable membership degrees of the element $x \in X$ to the set E.

Definition 2.2. [9] Variance of a hesitant fuzzy element h is defined as

$$v(h) = \frac{1}{l_h}\sqrt{\sum_{\gamma_i,\gamma_j \in h} (\gamma_i - \gamma_j)^2}$$

where l_h is the cardinality of h and $v(h)$ is the deviation degree of h. This reflects the standard deviation among all pairs of elements in a hesitant fuzzy element of h. For two hesitant fuzzy elements h_1 and h_2 if $v(h_1) > v(h_2)$ then $h_1 < h_2$. Moreover, if $v(h_1) = v(h_2)$ then $h_1 = h_2$.

Definition 2.3. [23] A hesitant fuzzy preference relation H on X is represented in matrix form as $H = (h_{ij})_{n\times n} \subset X \times X$, where $h_{ij} = \{h_{ij}^s, s = 1,2,...,l_{h_{ij}}\}$ is a hesitant fuzzy element indicating all possible degrees to which x_i is preferred over x_j. Furthermore, the following conditions must be satisfied for $i,j = 1,2,...n$.

i: $h_{ii} = \{0.5\}$

ii: $h_{ij}^{\sigma(s)} + h_{ji}^{\sigma(l_{h_{ij}}-s+1)} = 1$

iii: $l_{h_{ij}} = l_{h_{ji}}$

Definition 2.4. [28] Let $\dot{a} = [a^L, a^U]$ and $\dot{b} = [b^L, b^U]$ be two interval numbers and $\ell \geq 0$, then,

i: $\dot{a} = \dot{b}$ if and only if $a^L = b^L$ and $a^U = b^U$
ii: $\dot{a} \mp \dot{b} = [a^L + b^L, a^U + b^U]$
iii: $\ell \dot{a} = [\ell a^L, \ell a^U]$

Definition 2.5. [28] Let $\dot{a} = [a^L, a^U]$ and $\dot{b} = [b^L, b^U]$ be two interval numbers and let $l_{\dot{a}} = a^U - a^L$ and $l_{\dot{b}} = b^U - b^L$ represent the length of $\dot{a}$ and $\dot{b}$; then to compare two interval numbers, the degree of possibility of $\widetilde{a} \succcurlyeq \widetilde{b}$ is defined as

$$p(\dot{a} \succcurlyeq \dot{b}) = \max\{1 - \max(\frac{b^U - a^L}{l_{\dot{a}} + l_{\dot{b}}}, 0), 0\}$$

Definition 2.6. [2] Let X be a reference set, and $D[0,1]$ be the set of all closed subintervals of $[0,1]$. An interval-valued hesitant fuzzy set IVHFS on X is defined as

$$\widetilde{A} = \{< x_i, \widetilde{h}_{\widetilde{A}}(x_i) >: x_i \in X, i = 1, 2, ..., n\}$$

where $\widetilde{h}_{\widetilde{A}}(x_i) : X \to D[0,1]$ represents all possible interval valued membership intensities of x_i to $\widetilde{A}$. Also, $\widetilde{h}_{\widetilde{A}}(x_i)$ is called an interval-valued hesitant fuzzy element which reads $\widetilde{h}_{\widetilde{A}}(x_i) = \{\gamma : \gamma \in \widetilde{h}_{\widetilde{A}}(x_i)\}$. Here, $\gamma = [\gamma^L, \gamma^U]$ is an interval number such that $\gamma^L = \inf \gamma$ and $\gamma^U = \sup \gamma$ is the lower and upper limit of γ, respectively.

Definition 2.7. [2] An interval-valued hesitant fuzzy preference relation (IVHFPR) on X is denoted by $R = (\widetilde{r}_{ij})_{n \times n} \subset X \times X$, where $\widetilde{r}_{ij} = \{\widetilde{r}^s_{ij}, s = 1, 2, .., l_{\widetilde{r}_{ij}}\}$ is an interval-valued hesitant fuzzy element (IVHFE) representing all possible degrees to which alternative x_i is preferred over x_j and $l_{\widetilde{r}_{ij}}$ denotes the number of intervals in an IVHFE. Also, $\widetilde{r}_{ij}$ should satisfy the following:

i: $\inf \widetilde{r}^{\sigma(s)}_{ij} + \sup \widetilde{r}^{\sigma(l_{\widetilde{r}_{ij}} - s + 1)}_{ji} = \sup \widetilde{r}^{\sigma(s)}_{ji} + \inf \widetilde{r}^{\sigma(l_{\widetilde{r}_{ji}} - s + 1)}_{ij} = 1$
ii: $\widetilde{r}_{ii} = \{[0.5, 0.5]\}$

where $\widetilde{r}^{\sigma(s)}_{ij}$ represents the smallest value in $\widetilde{r}_{ij}$. Moreover, $\inf \widetilde{r}^{\sigma(s)}_{ij}$ and $\sup \widetilde{r}^{\sigma(s)}_{ij}$ denote the lower and upper limits of $\widetilde{r}^{\sigma(s)}_{ij}$.

Definition 2.8. [2] Let $\widetilde{h}, \widetilde{h}_1$ and $\widetilde{h}_2$ be IVHFEs. Then the following operations are defined:

i: $\widetilde{h}^c = \{1 - \gamma^U, 1 - \gamma^L : \gamma \in \widetilde{h}\}$
ii: $\widetilde{h}_1 \cup \widetilde{h}_2 = \{[\max(\gamma^L_1, \gamma^L_2), \max(\gamma^U_1, \gamma^U_2)] : \gamma_1 \in \widetilde{h}_1, \gamma_2 \in \widetilde{h}_2\}$
iii: $\widetilde{h}_1 \cap \widetilde{h}_2 = \{[\min(\gamma^L_1, \gamma^L_2), \min(\gamma^U_1, \gamma^U_2)] : \gamma_1 \in \widetilde{h}_1, \gamma_2 \in \widetilde{h}_2\}$
iv: $\widetilde{h}_1 \oplus \widetilde{h}_2 = \{[\gamma^L_1 + \gamma^L_2 - \gamma^L_1\gamma^L_2, \gamma^U_1 + \gamma^U_2 - \gamma^U_1\gamma^U_2] : \gamma_1 \in \widetilde{h}_1, \gamma_2 \in \widetilde{h}_2\}$,
iii: $\widetilde{h}_1 \otimes \widetilde{h}_2 = \{[\gamma^L_1\gamma^L_2, \gamma^U_1\gamma^U_2] : \gamma_1 \in \widetilde{h}_1, \gamma_2 \in \widetilde{h}_2\}$

Definition 2.9. [2] Score function $\widetilde{h}$ of an IVHFE is defined as

$$s(\widetilde{h}) = \frac{1}{\left|\widetilde{h}\right|} \sum_{\gamma \in \widetilde{h}} \gamma$$

where $\left|\widetilde{h}\right|$ represents the number of intervals in $\widetilde{h}$ and $s(\widetilde{h})$ is an interval value which is a subset of $[0, 1]$.

3. Incomplete Interval-valued Hesitant Fuzzy Preference Relations

Xu et al. [26] proposed definition of an additive transitive interval-valued fuzzy preference relations as follows. For all i, j, k such that $i < j < k$,

$$\widetilde{r}_{ij}\overline{+}\widetilde{r}_{jk}=\widetilde{r}_{ik}\overline{+}[0.5, 0.5] \tag{1}$$

However, an example was given in [26] which was contrary to equation (1). Wang [20] highlighted that this definition is dependent on alternative labels and is not robust to permutations of the decision maker's pairwise judgments. Consequently, for all $i, j, k = 1, 2, ..., n$, Wang stated additive transitivity for interval fuzzy preference relations as follows:

$$\widetilde{r}_{ij}\overline{+}\widetilde{r}_{jk}\overline{+}\widetilde{r}_{ki} = \widetilde{r}_{kj}\overline{+}\widetilde{r}_{ji}\overline{+}\widetilde{r}_{ik}$$

In this section, we first propose additive transitivity for IVHFPR. Our proposed definition is more convenient and useful for estimating missing information. It needs to be noted that operations used for IVHFPRs must be well defined. This means that when two interval-valued hesitant fuzzy sets are added, the resultant must be a subset of the unit interval. Secondly, literature portrays examples based on binary operations on IVHFPRs but the particular case where IVHFEs have different cardinalities is not presented.

We introduce additive transitivity for IVHFPRs to cater for incompleteness. Consider $\widetilde{r}_{ik}$ and $\widetilde{r}_{kj}$ to be IVHFPRs, then for $t \in \{1, 2, ..., p\}$, we have the following:

$$\begin{aligned}\widetilde{r}_{ij}^{(t)} &= \widetilde{r}_{ik}^{(t)}\widehat{\oplus}\widetilde{r}_{kj}^{(t)}\widehat{\ominus}[0.5, 0.5] = \\ &[\inf(\inf \widetilde{r}_{ik}^{(t)} + \sup \widetilde{r}_{kj}^{(t)} - 0.5, \sup \widetilde{r}_{ik}^{(t)} + \inf \widetilde{r}_{kj}^{(t)} - 0.5), \\ &\sup(\inf \widetilde{r}_{ik}^{(t)} + \sup \widetilde{r}_{kj}^{(t)} - 0.5, \sup \widetilde{r}_{ik}^{(t)} + \inf \widetilde{r}_{kj}^{(t)} - 0.5)]\end{aligned} \tag{2}$$

Additive transitivity property may be used to complete IVHFPRs where all known IVHFEs have the same cardinalities. However, number of intervals in corresponding IVHFEs may not be equal. For this purpose, we need to discuss the case where cardinalities of two IVHFEs are different. We propose the following algorithm which discusses the possibilities of two IVHFEs with the same cardinalities and also when cardinalities are not the same. In literature, these cases have not been addressed.

Consider $\widetilde{r}_{ik}$ and $\widetilde{r}_{kj}$ to be two IVHFPRs. Let $p = \min\{|\widetilde{r}_{ik}|, |\widetilde{r}_{kj}|\}$ where $|\widetilde{r}_{ik}|$ and $|\widetilde{r}_{kj}|$ represents the number of intervals in $\widetilde{r}_{ik}$ and $\widetilde{r}_{kj}$ respectively. In the following, we formulate an additive transitivity inspired algorithm to resolve incompleteness in IVHFPR. This algorithm also caters for cases when two IVHFEs have different cardinalities. We stress that this case should not be ignored.

If $|\widetilde{r}_{ik}| = |\widetilde{r}_{kj}|$ then for $t \in \{1, 2, ..., p\}$,

$$\begin{aligned}\widetilde{r}_{ij}^{(t)} \quad = \quad &[\inf(\inf \widetilde{r}_{ik}^{(t)} + \sup \widetilde{r}_{kj}^{(t)} - 0.5, \sup \widetilde{r}_{ik}^{(t)} + \inf \widetilde{r}_{kj}^{(t)} - 0.5), \sup(\inf \widetilde{r}_{ik}^{(t)} + \\ &\sup \widetilde{r}_{kj}^{(t)} - 0.5, \sup \widetilde{r}_{ik}^{(t)} + \inf \widetilde{r}_{kj}^{(t)} - 0.5)]\end{aligned}$$

Otherwise, if $|\widetilde{r}_{ik}| > |\widetilde{r}_{kj}|$ then for $t \in \{1, 2, .., p\}$,

$$\widetilde{r}_{ij}^{(t)} = [\inf(\inf \widetilde{r}_{ik}^{(t)} + \sup \widetilde{r}_{kj}^{(t)} - 0.5, \sup \widetilde{r}_{ik}^{(t)} + \inf \widetilde{r}_{kj}^{(t)} - 0.5), \sup(\inf \widetilde{r}_{ik}^{(t)} + \sup \widetilde{r}_{kj}^{(t)} - 0.5, \sup \widetilde{r}_{ik}^{(t)} + \inf \widetilde{r}_{kj}^{(t)} - 0.5)]$$

and for $t \in \{p+1, p+2, .., |\widetilde{r}_{ik}|\}$

$$\widetilde{r}_{ij}^{(t)} = \widetilde{r}_{ik}^{(t)}$$

Whereas, if $|\widetilde{r}_{ik}| < |\widetilde{r}_{kj}|$ then for $t \in \{1, 2, .., p\}$

$$\widetilde{r}_{ij}^{(t)} = [\inf(\inf \widetilde{r}_{ik}^{(t)} + \sup \widetilde{r}_{kj}^{(t)} - 0.5, \sup \widetilde{r}_{ik}^{(t)} + \inf \widetilde{r}_{kj}^{(t)} - 0.5), \sup(\inf \widetilde{r}_{ik}^{(t)} + \sup \widetilde{r}_{kj}^{(t)} - 0.5, \sup \widetilde{r}_{ik}^{(t)} + \inf \widetilde{r}_{kj}^{(t)} - 0.5)]$$

and for $t \in \{p+1, p+2, .., |\widetilde{r}_{kj}|\}$

$$\widetilde{r}_{ij}^{(t)} = \widetilde{r}_{kj}^{(t)}$$

Using this criteria, two of the above mentioned problems are resolved. This algorithm will estimate IVHFEs that are well-defined intervals. Secondly, this algorithm can be used effeciently when the number of intervals in two IVHFEs is not the same. The only persisting issue is that of outlying preferences. If we estimate missing preferences and then define a transformation function to bring back the outliers then the originality of decision maker's personal opinion will be voided. That is, the altered values will not be an amalgamation of the decision maker's provided preferences $\widetilde{r}_{ik}$ and $\widetilde{r}_{kj}$. Therefore, in order to resolve the problem of outlying preferences, we propose condition $(cihr)$ for decision makers delivering incomplete IVHFPRs. Given that the least element of an IVHFE is less than 0.5, the preference intensities are said to satisfy condition $(cihr)$ if the least and greatest members of IVHFEs of any fixed row i' satisfy the following condition.

$$\inf(\inf_{t \in \{1,2,..,n\}} \widetilde{r}_{i'k}^{(t)}) > \sup(\sup_{s \in \{1,2,..,n\}} \widetilde{r}_{i'j}^{(s)}) - 0.5$$

We assume that the least element in any IVHFE will be less than 0.5, otherwise the reciprocal IVHFE will have least element, that is, less than 0.5 anyway. Which means that this condition will still apply in one way or the other. We claim that if a decision maker satisfies this condition, then the estimated preferences will be expressible. However, if the decision maker does not satisfy this property, then additive transitivity for IVHFPRs will not help in estimating expressible preferences. We prove this statement in the next theorem. We refer to a missing preference as hesitant crucial preference if it can be estimated using the least and greatest member of all hesitant fuzzy elements provided by the expert. This definition of hesitant crucial preference is used in proving the following theorem.

Theorem 3.1. *Suppose that n preferences in an n by n interval-valued hesitant fuzzy preference relation are provided as $\widetilde{r}_{kj}^{(t)}$, $t \in N$ for fixed $k \neq j$ and $j \in \{1, 2.., n\}$. If interval-valued hesitant fuzzy preferences $\widetilde{r}_{kj}^{(t)}$ satisfy property $(cihr)$*

then the missing preference can be estimated. Moreover, the estimated preferences are expressible.

Proof. Since decision makers are consistent, therefore, $\widetilde{r}_{kj} = \{[0.5, 0.5]\}$ for $k = j$. Suppose that $(n-1)$ preference intensities provided by the decision maker satisfy condition $(cihr)$. This means that all interval-valued hesitant fuzzy set in the $k-th$ row $\widetilde{r}_{k1}^{(t)}, \widetilde{r}_{k2}^{(t)}, ..., \widetilde{r}_{kn}^{(t)}$ satisfy condition $(cihr)$. That is,

$$\inf(\inf_{t\in\{1,2,..,n\}} \widetilde{r}_{i'k}^{(t)}) > \sup(\sup_{s\in\{1,2,..,n\}} \widetilde{r}_{i'j}^{(s)}) - 0.5$$

and

$$\sup(\sup \widetilde{r}_{k1}^{(t)}, \sup \widetilde{r}_{k2}^{(t)}, ..., \sup \widetilde{r}_{kn}^{(t)}) < \inf(\inf \widetilde{r}_{k1}^{(t)}, \inf \widetilde{r}_{k2}^{(t)}, ..., \inf \widetilde{r}_{kn}^{(t)}) + 0.5.$$

Suppose that the least and the greatest element of the $k-th$ row in the IVHFPR is denoted by

$$\inf(\inf \widetilde{r}_{k1}^{(t)}, \inf \widetilde{r}_{k2}^{(t)}, ..., \inf \widetilde{r}_{kn}^{(t)}) + 0.5 = \inf \widetilde{r}_{ki}^{(t)}$$
$$\sup(\sup \widetilde{r}_{k1}^{(t)}, \sup \widetilde{r}_{k2}^{(t)}, ..., \sup \widetilde{r}_{kn}^{(t)}) = \sup \widetilde{r}_{kj}^{(t)}$$

respectively. Then according to property $(cihr)$ we have

$$0 \leq \sup \widetilde{r}_{kj}^{(t)} < \inf \widetilde{r}_{ki}^{(t)} + 0.5. \tag{4}$$

Also, $\inf \widetilde{r}_{ki}^{(t)} \leq \sup \widetilde{r}_{ki}^{(t)}$ and $\inf \widetilde{r}_{kj}^{(t)} \leq \sup \widetilde{r}_{kj}^{(t)}$ which further implies the following equations:

$$0 \leq \sup \widetilde{r}_{kj}^{(t)} < \sup \widetilde{r}_{ki}^{(t)} + 0.5 \tag{5}$$

$$0 \leq \inf \widetilde{r}_{kj}^{(t)} < \sup \widetilde{r}_{ki}^{(t)} + 0.5 \tag{6}$$

We state the proof in two steps. We first prove that if crucial preference is estimated to be expressible then all other unknown preferences will be expressible as well. We then prove that if $(cihr)$ is satisfied then the crucial preference will always be expressible. According to our assumption, the crucial preference is $\widetilde{r}_{ij}^{(t)} = \{ \widetilde{r}_{ij}^{(1)}, \widetilde{r}_{ij}^{(2)}, ..., \widetilde{r}_{ij}^{(t)}\}$.

For the first part of the proof, assume that $\widetilde{r}_{ij}^{(t)}$ is expressible, which means that $\widetilde{r}_{ij}^{(t)} \in D[0,1]$. Let $\widetilde{r}_{sj}^{(t)}, s \neq j$ be a missing preference other than the crucial value. Then, according to the definition of additive transitivity for IVHFPR, we have

$$\inf \widetilde{r}_{sj}^{(t)} = \inf \widetilde{r}_{sk}^{(t)} + \sup \widetilde{r}_{kj}^{(t)} - 0.5$$
$$\sup \widetilde{r}_{sj}^{(t)} = \sup \widetilde{r}_{sk}^{(t)} + \inf \widetilde{r}_{kj}^{(t)} - 0.5$$

We consider the above equation to prove that $\inf \widetilde{r}_{sj}^{(t)}$ is expressible. Consider $\inf \widetilde{r}_{sj}^{(t)} = \inf \widetilde{r}_{sk}^{(t)} + \sup \widetilde{r}_{kj}^{(t)} - 0.5 \leq (1 - \sup \widetilde{r}_{ks}^{(|\widetilde{r}_{ks}|-t+1)}) + \inf \widetilde{r}_{ki}^{(t)}$ since $\sup \widetilde{r}_{kj}^{(t)} - 0.5 < \inf \widetilde{r}_{ki}^{(t)}$ because of equation (5). We know that $\inf \widetilde{r}_{ki}^{(t)} \leq \sup \widetilde{r}_{ks}^{(|\widetilde{r}_{ks}|-t+1)} \leq \sup \widetilde{r}_{kj}^{(t)}$. Therefore, $\inf \widetilde{r}_{sj}^{(t)} \leq (1 - \sup \widetilde{r}_{ik}^{(|\widetilde{r}_{ki}|-t+1)}) + \sup \widetilde{r}_{ks}^{(|\widetilde{r}_{ks}|-t+1)}$, which is equal to 1. Therefore, $\inf \widetilde{r}_{sj}^{(t)} \in [0,1]$ and hence, $\widetilde{r}_{sj}^{(t)}$ is expressible.

Similarly, using equation (6) we can follow similar steps to prove that $\sup \widetilde{r}_{sj}^{(t)}$ is expressible. This proves that if hesitant crucial preference is expressible then so are other missing preferences.

We now prove that if the condition $(cihr)$ is satisfied by the given preferences then the crucial preference can be estimated and it is expressible. Using additive consistency for IVHFPR we state that for $i, j, k \in \{1, 2, ..., n\}$, $\inf \widetilde{r}_{ij}^{t} = \inf \widetilde{r}_{ik}^{t} + \sup \widetilde{r}_{kj}^{t} - 0.5$. Using equation (4) we have, $\inf \widetilde{r}_{ij}^{t} < \inf \widetilde{r}_{ik}^{t} + \inf \widetilde{r}_{ki}^{(t)} + 0.5 - 0.5 = \inf \widetilde{r}_{ik}^{t} + (1 - \sup \widetilde{r}_{ki}^{(|\widetilde{r}_{ik}|-t+1)}) = 1$. Which proves that $\inf \widetilde{r}_{ij}^{t}$ is expressible.

Similarly, $\sup \widetilde{r}_{ij}^{t} = \sup \widetilde{r}_{ik}^{t} + \inf \widetilde{r}_{kj}^{t} - 0.5 < \sup \widetilde{r}_{ik}^{t} + 1 - \sup \widetilde{r}_{ki}^{(|\widetilde{r}_{ik}|-t+1)} = 1$ implying, $\sup \widetilde{r}_{ij}^{t}$ is also expressible. Therefore, $\widetilde{r}_{ij}^{t}$ is written as $[\inf(\inf \widetilde{r}_{ik}^{(t)} + \sup \widetilde{r}_{kj}^{(t)} - 0.5, \sup \widetilde{r}_{ik}^{(t)} + \inf \widetilde{r}_{kj}^{(t)} - 0.5), \sup(\inf \widetilde{r}_{ik}^{(t)} + \sup \widetilde{r}_{kj}^{(t)} - 0.5, \sup \widetilde{r}_{ik}^{(t)} + \inf \widetilde{r}_{kj}^{(t)} - 0.5)]$. Otherwise, if cardinalities are not the same then they are written as mentioned earlier in this section.

Therefore, $\widetilde{r}_{ij}^{t} \in D[0, 1]$. Hence, the crucial interval-valued hesitant fuzzy preference is expressible. □

Example 3.2. Consider $X = \{hosp_1, hosp_2, hosp_3, hosp_4\}$ to be the set of hospitals in Lahore with the facility of hair transplant. A patient is to express his preferences over these hospitals based on the number of successful procedures proclaimed by these hospitals. The patient feels the need of more information to be able to express his opinion. Under the given circumstances, he is certain about his preferences of only the second hospital over others. However, the patient abides by condition $(cihr)$.

$$\begin{bmatrix} \{[0.5,0.5]\} & \{[0.2,0.3],[0.5,0.6]\} & - & - \\ \begin{matrix}\{[0.4,0.5],\\ [0.7,0.8]\}\end{matrix} & \{[0.5,0.5]\} & \{[0.7,0.8],[0.8,0.9]\} & \begin{matrix}\{[0.4,0.6],\\ [0.5,0.6],\\ [0.6,0.8]\}\end{matrix} \\ - & \{[0.1,0.2],[0.2,0.3]\} & \{[0.5,0.5]\} & - \\ - & \begin{matrix}\{[0.2,0.4],[0.4,0.5],\\ [0.4,0.6]\}\end{matrix} & - & \{[0.5,0.5]\} \end{bmatrix}$$

Here interval-valued hesitant fuzzy preferences intensities of the second column are estimated using definition 7. For example, to estimate $\widetilde{r}_{12}^{(1)}$, we have

$$\inf \widetilde{r}_{21}^{(1)} + \sup \widetilde{r}_{12}^{(2-1+1)} = 1, \sup \widetilde{r}_{21}^{(1)} + \inf \widetilde{r}_{12}^{(2-1+1)} = 1$$

and

$$\inf \widetilde{r}_{21}^{(2)} + \sup \widetilde{r}_{12}^{(2-2+1)} = 1, \sup \widetilde{r}_{21}^{(2)} + \inf \widetilde{r}_{12}^{(2-2+1)} = 1$$

That is, $\widetilde{r}_{12}^{(1)} = [0.2, 0.3]$ and $\sup \widetilde{r}_{12}^{(2)} = [0.5, 0.6]$. Preferences in the second column are stated using additive reciprocity for IVHFPRs.

Regarding other missing preferences, we do not use additive transitivity directly because that causes two issues; the problem of outliers and the issue of values that are not well-defined intervals. Therefore, we use algorithm mentioned earlier to estimate the missing preferences. In order to find $\widetilde{r}_{13}$ we need $\widetilde{r}_{12}$ and $\widetilde{r}_{23}$ and according to the above algorithm, we need to check cardinality of these interval-valued hesitant fuzzy sets. Since $|\widetilde{r}_{12}| = |\widetilde{r}_{23}|$, therefore,

$$\begin{aligned}\widetilde{r}_{13}^{(1)} &= \begin{array}{c}[\inf(\inf \widetilde{r}_{12}^{(1)} + \sup \widetilde{r}_{23}^{(1)} - 0.5, \sup \widetilde{r}_{12}^{(1)} + \inf \widetilde{r}_{23}^{(1)} - 0.5),\\ \sup(\inf \widetilde{r}_{12}^{(1)} + \sup \widetilde{r}_{23}^{(1)} - 0.5, \sup \widetilde{r}_{12}^{(1)} + \inf \widetilde{r}_{23}^{(1)} - 0.5)]\end{array} \quad \text{and} \\ \widetilde{r}_{13}^{(2)} &= \begin{array}{c}[\inf(\inf \widetilde{r}_{12}^{(2)} + \sup \widetilde{r}_{23}^{(2)} - 0.5, \sup \widetilde{r}_{12}^{(2)} + \inf \widetilde{r}_{23}^{(2)} - 0.5),\\ \sup(\inf \widetilde{r}_{12}^{(2)} + \sup \widetilde{r}_{23}^{(2)} - 0.5, \sup \widetilde{r}_{12}^{(2)} + \inf \widetilde{r}_{23}^{(2)} - 0.5)]\end{array}\end{aligned}$$

which implies that

$$\begin{aligned}\widetilde{r}_{13} &= \{[0.5, 0.5], [0.9, 0.9]\} \text{ and accordingly,} \\ \widetilde{r}_{31} &= \{[0.1, 0.1], [0.5, 0.5]\}\end{aligned}$$

Similarly, according to algorithm mentioned above, in order to estimate $\widetilde{r}_{14}$ we note that $|\widetilde{r}_{12}| < |\widetilde{r}_{24}|$. Therefore,

$$\widetilde{r}_{14}^{(1)} = \begin{array}{c}[\inf(\inf \widetilde{r}_{12}^{(1)} + \sup \widetilde{r}_{24}^{(1)} - 0.5, \sup \widetilde{r}_{12}^{(1)} + \inf \widetilde{r}_{24}^{(1)} - 0.5),\\ \sup(\inf \widetilde{r}_{12}^{(1)} + \sup \widetilde{r}_{24}^{(1)} - 0.5, \sup \widetilde{r}_{12}^{(1)} + \inf \widetilde{r}_{24}^{(1)} - 0.5)]\end{array}$$

which implies that $\widetilde{r}_{14}^{(1)} = [0.2, 0.3]$ and $\widetilde{r}_{14}^{(2)} = [0.6, 0.6]$. Also, $\widetilde{r}_{14}^{(3)} = \widetilde{r}_{24}^{(3)} = [0.6, 0.8]$. That is,

$$\begin{aligned}\widetilde{r}_{14} &= \{[0.2, 0.3], [0.6, 0.6], [0.6, 0.8]\} \text{ and} \\ \widetilde{r}_{41} &= \{[0.2, 0.4], [0.4, 0.4], [0.7, 0.8]\}.\end{aligned}$$

Similarly,

$$\begin{aligned}\widetilde{r}_{34} &= \{[0.1, 0.2], [0.3, 0.3], [0.6, 0.8]\} \text{ and} \\ \widetilde{r}_{43} &= \{[0.8, 0.9], [0.7, 0.7], [0.2, 0.4]\}.\end{aligned}$$

We have catered for incompleteness with well-defined interval-valued hesitant fuzzy elements. The completed relation is as follows:-

$$\begin{bmatrix} \{[0.5,0.5]\} & \{[0.2,0.3],[0.5,0.6]\} & \{[0.5,0.5],[0.9,0.9]\} & \{[0.2,0.3],[0.6,0.6],[0.6,0.8]\} \\ \{[0.4,0.5],[0.7,0.8]\} & \{[0.5,0.5]\} & \{[0.7,0.8],[0.8,0.9]\} & \{[0.4,0.6],[0.5,0.6],[0.6,0.8]\} \\ \{[0.1,0.1],[0.5,0.5]\} & \{[0.1,0.2],[0.2,0.3]\} & \{[0.5,0.5]\} & \{[0.1,0.2],[0.3,0.3],[0.6,0.8]\} \\ \{[0.2,0.4],[0.4,0.4],[0.7,0.8]\} & \{[0.2,0.4],[0.4,0.5],[0.4,0.6]\} & \{[0.8,0.9],[0.7,0.7],[0.2,0.4]\} & \{[0.5,0.5]\} \end{bmatrix}$$

The resultant relation comprises of expressible and well-defined preferences.

Using limited information provided by the patient, we have completed his preference relation. With complete information, the relation can be ranked in order to identify the most preferred hospital for hair transplant.

4. Ranking of Interval-valued Hesitant Fuzzy Preference Relations

Algorithm defined in section 3 ensures that the estimated preferences are well-defined. Moreover, condition (*cihr*) promises expressibility of the missing preferences. Once the problem of incompleteness is catered, we hereby discuss ranking of IVHFPRs. Ranking a preference relation is mandatory as it allows us to realize the importance of each alternative. Ranking of alternatives may deduce possible ties that may arise during the process. This situation can direct the decision makers to choose some other methods to dissolve ties.

Mandal et al [13] suggest a distance based ranking method for IVHFPRs used in critical path problem. This method is applicable in the presence of several IVHFPRs. In this case, a distance metric is defined to form difference matrices based on different IVHFEs. Eventually score functions are used on these difference matrices and preferred alternatives are discovered based on the corresponding scores. In this paper, we assume just one collective IVHFPR which may be transitive or non-transitive. Due to this assumption, we no longer require to go through distance metric based process that is relatively tedious.

We use the definition of degree of possibility and score function to define rank of IVHFPRs. Given an IVHFPR, for each row $i \in \{1, 2, .., n\}$, we first identify an IVHFE $\widetilde{h}_{ik'}, i \neq k' \neq j$, such that

$$\widetilde{h}_{ij} \not\succcurlyeq \widetilde{h}_{ik'} \text{ if } p(s(\widetilde{h}_{ij}) \not\succcurlyeq s(\widetilde{h}_{ik'})) \geq 0.5$$

This results in a collection of IVHFEs from each row $\{\widetilde{h}_{1k'}, \widetilde{h}_{2k'}, \widetilde{h}_{3k'}, ..., \widetilde{h}_{nk'}\}$, with low degree of possibility as compared to other IVHFEs of the row. Now, in order to compare alternative x_i over x_j, members of the set $\{\widetilde{h}_{1k'}, \widetilde{h}_{2k'}, ..., \widetilde{h}_{nk'}\}$ are compared pair-wise with the help of the following: $x_i \succcurlyeq x_j$ if and only if

$$p(s(\widetilde{h}_{ik'}), s(\widetilde{h}_{jk'})) \geq 0.5$$

According to definitions 5 and 9, this implies that alternative x_i is ranked higher than alternative x_j if and only if degree of possibility of the score of IVHFE $\widetilde{h}_{ik'}$ over $\widetilde{h}_{jk'}$ is greater than or equal to 0.5. This ranking rule is applicable to any transitive or non-transitive IVHFPR.

In diagram 1, we summarize and explain the method suggested in this paper.

This ranking method is applied to a non-transitive IVHFPR in the following example.

Example 4.1. Let $X = \{x_1, x_2, x_3\}$ represent three medicines to control the activity of Hepatitis B virus in patients. A doctor is requested to express her preferences over these medicines based on her experience of how well each medicine has served over years for patients of this disease. Following is the IVHFPR that is presented by the doctor.

$$\begin{bmatrix} \{[0.5, 0.5]\} & \{[0.2, 0.4], [0.3, 0.5]\} & \begin{matrix}\{[0.3, 0.4], [0.5, 0.6], \\ [0.4, 0.7]\}\end{matrix} \\ \{[0.5, 0.7], [0.6, 0.8]\} & \{[0.5, 0.5]\} & \{[0.7, 0.9], [0.8, 1]\} \\ \begin{matrix}\{[0.3, 0.6], [0.4, 0.5], \\ [0.6, 0.7]\}\end{matrix} & \{[0, 0.2], [0.1, 0.3]\} & \{[0.5, 0.5]\} \end{bmatrix}$$

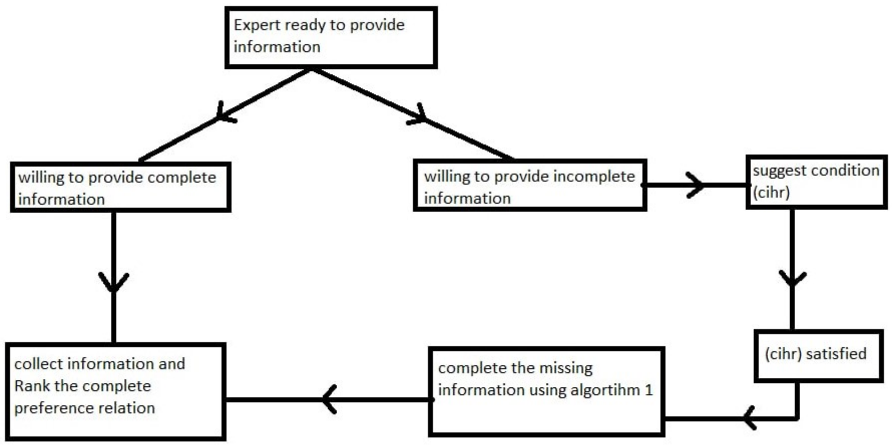

FIGURE 1

We rank this IVHFPR according to the ranking rule defined above. Let us first find score of each interval valued hesitant fuzzy element. We have the following scores.

$$\begin{aligned} s(\widetilde{h}_{12}) &= [\frac{0.2+0.3}{2}, \frac{0.9}{2}] = [0.25, 0.45] \\ s(\widetilde{h}_{21}) &= [\frac{0.5+0.6}{2}, \frac{1.5}{2}] = [0.55, 0.75] \\ s(\widetilde{h}_{13}) &= [0.4, 0.56]; s(\widetilde{h}_{31}) = [0.43, 0.6] \\ s(\widetilde{h}_{23}) &= [0.75, 0.95] \text{ and } s(\widetilde{h}_{32}) = [0.05, 0.25] \end{aligned}$$

Now let us determine the most desirable alternative among each row. For this purpose, we restrict to one row at a time. So, from the first row we notice that

$$p(s(\widetilde{h}_{13}), s(\widetilde{h}_{12})) = \max\{1 - \max(\frac{0.45-0.4}{0.2+0.16}, 0), 0\} = 0.861 \geq 0.5$$

which implies that $\widetilde{h}_{12} \succcurlyeq \widetilde{h}_{13}$ according to our ranking rule. Similarly,

$$p(s(\widetilde{h}_{21}), s(\widetilde{h}_{23})) = \max\{1 - \max(\frac{0.95-0.55}{0.2+0.2}, 0), 0\} \geq 0.5.$$

That is, $\widetilde{h}_{23} \succcurlyeq \widetilde{h}_{21}$. Also,

$$p(s(\widetilde{h}_{31}), s(\widetilde{h}_{32})) = \max\{1 - \max(\frac{0.25-0.43}{0.515+0.15}, 0), 0\} \geq 0.5$$

which means that $\widetilde{h}_{32} \succcurlyeq \widetilde{h}_{31}$.

Therefore, the set of interval-valued hesitant fuzzy element from each row whose score has low possibility as compared to others in the row is $\{\widetilde{h}_{12}, \widetilde{h}_{23}, \widetilde{h}_{32}\}$. Now, we compare the three alternatives x_1, x_2 and x_3 by pair-wise comparison of elements in the set $\{\widetilde{h}_{12}, \widetilde{h}_{23}, \widetilde{h}_{32}\}$. Accordingly, $p(s(\widetilde{h}_{12}), s(\widetilde{h}_{23})) = 0 \not\geq 0.5$ which implies that $x_2 \succcurlyeq x_1$. Also, $p(s(\widetilde{h}_{23}), s(\widetilde{h}_{32})) = 1 \geq 0.5$ which implies that $x_2 \succcurlyeq x_3$. And $p(s(\widetilde{h}_{12}), s(\widetilde{h}_{32})) \geq 0.5$ implying $x_1 \succcurlyeq x_3$.

Therefore, according to this example we have, $x_2 \succcurlyeq x_1 \succcurlyeq x_3$. That is, the second medicine has controlled the Hepatitis B virus better as compared to the first medicine which is still better than medicine number three.

Similarly, additive reciprocal IVHFPRs can be ranked using this method. Moreover, in case of incomplete IVHFPRs, we can first cater to the problem of incompleteness according to theorem 1 and then rank it using the proposed method.

5. **Conclusion and Future Direction**

As number of alternatives in a decision making problem exceeds a certain level, the complexity of the problem increases. With increasing level of difficulty, issues of vagueness and uncertainty arise. In such situations expecting complete information from all decision makers in the panel is an unrealistic assumption.

IVHFPRs are more flexible and realistic for decision modeling because they give the expert flexibility to choose several subintervals of $[0, 1]$ as IVHFEs. It has been mentioned in the literature that additive transitivity is not a realistic methodology to estimate missing information. In this article, we stress that transitivity can not be used in its original form as it may lead to estimating preferences that are not expressible; exceeding $D[0, 1]$. For this reason we propose additive transitivity inspired method to estimate missing preferences. In order to make sure that these intervals are members of $D[0, 1]$, we define condition $(cihr)$ that is presented to decision makers who intend to propose incomplete information set. If the decision maker with incomplete information abides by $(cihr)$ then the missing preferences are estimated and the estimated preferences are expressible which means that they do not out lie the defined domain.

In literature, methods have been proposed that use additive transitivity directly to estimate preferences. Once the estimated values outlie the domain, transformation functions are used to bring the outliers back to the domain. We stress that this method voids originality of the decision maker's opinions. It is better to estimate preferences based on decision maker's original information. The method can be further improved by discussing options to consider when an expert fails to satisfy condition $(cihr)$.The other focus of this paper is on ranking of IVHFPRs. Once incompleteness has been taken care of, we propose a method to rank alternatives. With this ranking method, alternatives of both reciprocal and non-reciprocal IVHFPRs can be ranked. However, problem of acyclicity may arise with this ranking method and it still needs to be studied.

The drawback of this method is that it does not discuss situation of possible ties among alternatives. For future directions, this work can be extended to interval-valued multiplicative hesitant fuzzy preference relations. Condition $(cihr)$ can be relaxed and studied further for IVHFPRs. Ranking of these relations can be extended to include situations of possible ties among alternatives and methods can be discussed to dissolve these ties. Moreover, applications of this work can be studied in medical diagnosis and image processing.

Acknowledgements. Authors would like to express their sincere thanks to the anonymous reviewers for their valuable suggestions.

References

[1] S. Alonso, F. Chiclana, F. Herrera, E. Herrera-Viedma, J. Alcalá-Fdez and C. Porcel, *A consistency-based procedure to estimate missing pairwise preference values*, Int J. Intell. Syst., **23(2)**(2008), 155–175.

[2] N. Chen, Z. Xu and M. Xia, *Interval-valued hesitant preference relations and their applications to group decision making*, Knowledge-Based Syst., **37** (2013), 528–540.

[3] F. Chiclana, E. Herrera-Viedma and J. Wu, *Trust based consensus model for social network in an incomplete linguistic Information context*, Appl. Soft Comput., **35** (2013), 827–839.

[4] F. Chiclana, E. Herrera-Viedma, S. Alonso and F. Herrera, *Cardinal consistency of reciprocal preference relations: a characterization of multiplicative transitivity*, IEEE Trans Fuzzy Syst., **17(1)** (2009), 14–23.

[5] Y. Dong and E. Herrera-Viedma, *Consistency-driven automatic methodology to set interval numerical scales of 2-tuple linguistic term sets and its use in the linguistic GDM with preference relation*, IEEE Trans Fuzzy Cybernetics, **45(4)** (2015), 780-792.

[6] E. Herrera-Viedma, S. Alonso, F. Chiclana and F. Herrera, *A consensus model for group decision making with incomplete fuzzy preference relations*, IEEE Trans Fuzzy Syst., **15(5)** (2007), 863-877.

[7] E. Herrera-Viedma, F. Chiclana, F. Herrera and S. Alonso, *Group decision-making model with incomplete fuzzy preference relations based on additive consistency*, IEEE Trans Fuzzy Syst., Man, and Cybernetics, Part B (Cybernetics)., **37(1)** (2007), 176-189.

[8] A. Khalid and M. M. Awais, *Incomplete preference relations: An upper bound condition*, J Intell Fuzzy Syst., **26(3)** (2014), 1433-1438.

[9] H. Liao, Z. Xu and M. Xia, *Multiplicative consistency of hesitant fuzzy preference relation and its application in group decision making*, Int. J. Inf.Tech. & Decision Making, **13(01)** (2014), 47-76.

[10] H. Liao and Z. Xu, *Priorities of intuitionistic fuzzy preference relation based on multiplicative consistency* IEEE Trans. Fuzzy Syst., **22(6)** (2014), 1669-1681.

[11] H. Liao, Z. Xu and M. Xia, *Multiplicative consistency of interval-valued intuitionistic fuzzy preference relation*, J Intell Fuzzy Syst., **27(6)** (2014), 2969-2985.

[12] H. Liu, Z. Xu and H. Liao, *The multiplicative consistency index of hesitant fuzzy preference relation*, IEEE Trans Fuzzy Syst., **24(1)**(2016), 82-93.

[13] K. Mandal and K. Basu, *Simpler solution to group decision making problems based on interval valued hesitant fuzzy sets and a new approach to find critical path using hesitant fuzzy* Int. J. Pure and Appl. Math., **101(4)** (2015), 445-461.

[14] J. A. Morente-Molinera, I. J. Pérez, M. R. Ureña and E. Herrera-Viedma, *On multi-granular fuzzy linguistic modeling in group decision making problems: a systematic review and future trends*, Knowledge-Based Syst., **74** (2015), 49-60.

[15] S. A. Orlovsky, *Decision-making with a fuzzy preference relation*, Fuzzy Sets Syst., **1(3)** (1978), 155-167.

[16] V. Torra, *Hesitant fuzzy sets*, Int. J. Intell. Syst., **25(6)** (2010), 529-539.

[17] V. Torra,Y. Narukawa, *On hesitant fuzzy sets and decision* In: IEEE International Conference on Fuzzy Systems (FUZZ-IEEE) 2009, 1378–1382.

[18] R. Urena, F.Chiclana, J. A. Morente-Molinera and E. Herrera-Viedma, *Managing incomplete preference relations in decision making: a review and future trends* Inf. Sci., **302** (2015), 14-32.

[19] J. Q.Wang,J. Wang, Q. H. Chen, H. Y. Zhang and X. H. Chen, *An outranking approach for multi-criteria decision-making with hesitant fuzzy linguistic term sets*, Inf. Sci., **280** (2014), 338-351.

[20] Z. J. Wang, *A note on "Incomplete interval fuzzy preference relations and their applications*, Computers & Industrial Engineering, **77** (2014), 65–69.

[21] G. Wei, X. Zhao and R. Lin, *Some hesitant interval-valued fuzzy aggregation operators and their applications to multiple attribute decision making*, Knowledge-Based Syst., **46** (2013), 43–53.

[22] M. Xia and Z. Xu, *Hesitant fuzzy information aggregation in decision making*, Int. J. Approx. Reasoning, **52**(3) (2011), 395-407.

[23] M. Xia and Z. Xu, *Managing hesitant information in GDM problems under fuzzy and multiplicative preference relations*, International Journal of Uncertainty, Fuzziness and Knowledge-Based Syst., **21(06)** (2013), 865–897.

[24] Y. Xu, L. Chen, K. W.Li and H. Wang, *A chi-square method for priority derivation in group decision making with incomplete reciprocal preference relations*, Inf. Sci., **306** (2015), 166–179.

[25] Y. Xu, R. Patnayakuni and H. Wang, *Logarithmic least squares method to priority for group decision making with incomplete fuzzy preference relations*, Appl. Math. Modell., **37(4)** (2013), 2139–2152.

[26] Y. Xu, K. W. Li and H. Wang, *Incomplete interval fuzzy preference relations and their applications. Computers & Industrial Engineering*, **67** (2014), 93-103.

[27] Y. Xu, L. Chen, R. M Rodríguez, F. Herrera and H. Wang, *Deriving the priority weights from incomplete hesitant fuzzy preference relations in group decision making*, Knowledge-Based Syst., **99** (2016), 71-78.

[28] Z. S. Xu and Q. L.Da, *The uncertain OWA operator*, Int. J. Intell. Syst., **17(6)** (2002), 569–575.

[29] Z. Xu, *On compatibility of interval fuzzy preference relations*, Fuzzy Optim. Decision Making, **3(3)**(2004), 217–225.

[30] Z. Xu and J. Chen, *Some models for deriving the priority weights from interval fuzzy preference relations*, Eur J Oper Res., **184(1)** (2008), 266–280.

[31] Y. Xu,, J. N. Gupta and H. Wang, *The ordinal consistency of an incomplete reciprocal preference relation*, Fuzzy Sets Syst., **246** (2014), 62–77.

[32] Z. Xu and H. Liao, *A survey of approaches to decision making with intuitionistic fuzzy preference relations*, Knowledge-based syst., **80** (2015), 131–142.

[33] Y. Xu, F. Ma, F. Tao and H. Wang, *Some methods to deal with unacceptable incomplete 2-tuple fuzzy linguistic preference relations in group decision making*, Knowledge-Based Syst., **56** (2014), 179–190.

[34] B. Zhu, Z. Xu and J. Xu, *Deriving a ranking from hesitant fuzzy preference relations under group decision making*, IEEE Trans. Cybernetics, **44(8)** (2014), 1328–1337.

[35] G. Zhang, Y. Dong and Y. Xu, *Linear optimization modeling of consistency issues in group decision making based on fuzzy preference relations*, Expert Systems with Appl., **39(3)** (2012), 2415–2420.

[36] Y. Zhang, H. Ma, Q. Li, B. Liu and J. Liu, *Conditions of two methods for estimating missing preference information*, Inf. Sci., **279** (2014), 186–198.

[37] Z. Zhang and C. Wu, *Interval-valued hesitant multiplicative preference relations and their application to multi-criteria decision making*, British J. Math. Comput. Sc., **4(10)** (2014), 1390–1426.

A. Khalid*, Lahore School of Economics, Centre for Mathematics and Statistical Sciences, Lahore, Paksitan
E-mail address: `asmak@lahoreschool.edu.pk`

Ismat Beg, Centre for Mathematics and Statistical Sciences, Lahore School of Economics, Lahore, 54000, Pakistan
E-mail address: `ibeg@lahoreschool.edu.pk`

∗Corresponding author