

Category and subcategories of (L, M) -fuzzy convex spaces

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Abstract

In this paper, (L, M) -fuzzy domain finiteness and (L, M) -fuzzy restricted hull spaces are introduced, and several characterizations of the category (L, M) -CS of (L, M) -fuzzy convex spaces are obtained. Then, (L, M) -fuzzy stratified (resp. weakly induced, induced) convex spaces are introduced. It is proved that both categories, the category (L, M) -SCS of (L, M) -fuzzy stratified convex spaces and the category (L, M) -WICS of (L, M) -fuzzy weakly induced convex spaces, are coreflective subcategories of (L, M) -CS. It is also proved that three isomorphic categories, namely, the category M -CS of M -fuzzifying convex spaces, the category (L, M) -CGCS of (L, M) -fuzzy convex spaces induced by M -fuzzifying convex spaces and the category (L, M) -ICS of (L, M) -fuzzy induced convex spaces, are coreflective subcategories of both (L, M) -SCS and (L, M) -WICS.

Keywords: Domain finiteness, (L, M) -fuzzy (resp. stratified, weakly induced, induced) convex space, (L, M) -fuzzy restricted hull space, Galois correspondence, Coreflective.

1 Introduction

Convex Theory, being inspired originally by some simple geometric problems such as shapes of circles and polytopes in Euclidean spaces [4], has been developing along two directions since the last century. One was motivated by concrete problems such as existences of continuous selections and fixed points, or optimization problems [5, 13, 15, 23]; the other was based on an axiomatic point of view, where the notion of axiomatic convex structures was introduced and its theory was established [6, 12, 22, 34]. Later, it is found that some results of this theory, capturing many combinatorial features of convex structures, are quite useful for handling convex combinations [8, 14]. Now, this theory has been abstracted into many mathematical fields, where some new convex structures were discovered, including order convex structures [11], lattice convex structures [33], metric convex structures [34], graph convex structures [7] and median convex structures [17].

In the theory of convex spaces, domain finiteness is an important notion which distinguishes convex structures from closure structures and co-topologies. In addition, it is found that a closure space is a convex space iff it is stable for up-directed unions or its closure operator is domain finite. With the help of this fact, Vel showed that convex spaces and restricted hull operators are one-to-one corresponding, and that many properties of convex spaces can be characterized by finite sets [34].

Convex structures in fuzzy settings have been studied in several ways. The notion of fuzzy convex structures defined by Rosa [24] was further extended by Maruyama [16], who introduced the notion of L -convex structures. In fact, both notions were defined in a common way: each notion is actually a family of fuzzy or L -fuzzy sets satisfying certain set of axioms. However, from a totally different way, Shi and Xiu introduced the notion of M -fuzzifying convex structures, where each subset can be regarded as a convex set to some degree [31]. Now, many properties in M -fuzzifying convex

spaces have been studied [30, 31, 36, 38, 37, 41, 40, 42]. Later, Xiu and Shi introduced the notion of (L, M) -fuzzy convex structures [32], generalizing of both L -convex structures and M -fuzzifying convex structures. In fact, (L, M) -fuzzy convex structures can be generated by L -fuzzy Alexandroff topologies [9], L -fuzzy groups [3], L -fuzzy sublattices [2], L -fuzzy pre-ordered spaces [10], L -fuzzy vector spaces [16] and L -fuzzy ordered spaces [43], etc.

We will introduce and characterize domain finiteness of (L, M) -fuzzy convex spaces. We also introduce notions of (L, M) -fuzzy restricted hull (resp. stratified, weakly induced, induced, etc) convex spaces, and discuss their relations in a view of category aspect.

This paper is arranged as follows. In Section 2, we recall some notions of (L, M) -fuzzy convex spaces and M -fuzzifying convex spaces. In Sections 3 and 4, we introduce and characterize domain finiteness of both L -convex spaces and (L, M) -fuzzy convex spaces. Also, we characterize (L, M) -**CS** by the category (L, M) -**RHS** of (L, M) -fuzzy restricted hull spaces, the category (L, M) -**GCS** of (L, M) -fuzzy convex spaces generated by (L, M) -fuzzy closure spaces, the category (L, M) -**DFCOS** of (L, M) -fuzzy domain finite closure spaces and the category (L, M) -**CAS** of (L, M) -fuzzy concave spaces. We also show that the category L -**CS** of L -convex spaces is a coreflective subcategory of (L, M) -**CS**. In Sections 5 and 6, we introduce (L, M) -fuzzy stratified (resp. weakly induced) convex spaces. We show that both (L, M) -**SCS** and (L, M) -**WICS** are coreflective subcategories of (L, M) -**CS**. We also show that the category L -**SCS** of L -stratified convex spaces and the category L -**WICS** of L -weakly induced convex spaces are coreflective subcategories of (L, M) -**SCS** and (L, M) -**WICS**, respectively. In Section 7, we introduce (L, M) -fuzzy induced convex spaces and show that three isomorphic categories, the category M -**CS** of M -fuzzifying convex spaces, the category (L, M) -**CGCS** and the category (L, M) -**ICS** of (L, M) -fuzzy induced convex spaces, are coreflective subcategories of both (L, M) -**SCS** and (L, M) -**WICS**. In Section 8, we draw a diagram to show relations among (L, M) -**CS** and its subcategories.

2 Preliminaries

In this paper, X, Y are nonempty sets. 2^X is the power set of X and 2_{fin}^X is the set of all finite subsets of X . Both L and M are completely distributive lattices and M has an inverse involution $'$. The least element and the greatest element of L (resp. M) are denoted by \perp and \top . In particular, we write $\mathbf{2}$ for the lattice $\{\perp, \top\}$. If $\varphi \subseteq M$, we denote $\bigvee \varphi = \bigvee_{b \in \varphi} b$ and $\bigwedge \varphi = \bigwedge_{b \in \varphi} b$. Also, we adopt the convention that $\bigvee \emptyset = \perp$ and $\bigwedge \emptyset = \top$. An element $a \in M$ is called a prime, if for all $b, c \in M$, $b \wedge c \leq a$ implies $b \leq a$ or $c \leq a$. We denote $P(M)$ the set of all primes in $M \setminus \{\top\}$, and $J(M) = \{a \in M : a' \in P(M)\}$ [35].

A binary relation \prec on M is defined as: for all $a, b \in M$, $a \prec b$ iff for all $\varphi \subseteq M$, $b \leq \bigvee \varphi$ always implies the existence of $d \in \varphi$ such that $a \leq d$. Further, a binary relation \prec^{op} is defined as: for all $a, b \in M$, $a \prec^{op} b$ iff $b' \prec a'$. Clearly, $\beta(\bigvee_{i \in \Omega} a_i) = \bigcup_{i \in \Omega} \beta(a_i)$ and $\alpha(\bigwedge_{i \in \Omega} a_i) = \bigcup_{i \in \Omega} \alpha(a_i)$ for all $\{a_i\}_{i \in \Omega} \subseteq M$, where $\beta(a) = \{b : b \prec a\}$ and $\alpha(a) = \{b : a \prec^{op} b\}$ for all $a \in M$. We have $\beta(\perp) = \alpha(\top) = \emptyset$ and $a = \bigvee \beta(a) = \bigvee \beta^*(a) = \bigwedge \alpha(a) = \bigwedge \alpha^*(a)$ for all $a \in M$, where $\beta^*(a) = \beta(a) \cap J(M)$ and $\alpha^*(a) = \alpha(a) \cap P(M)$ [26, 35].

L^X is the set of all L -fuzzy sets on X , whose greatest element and least element are denoted by $\overline{\perp}$ and $\underline{\perp}$. The characterization function of a subset $U \in 2^X$ is an L -fuzzy set denoted by χ_U . The L -fuzzy set on X with a constant value $\lambda \in L$ is denoted by $\underline{\lambda}$. An L -fuzzy point with a support x and a value $r \in L$ is also an L -fuzzy set which is denoted by x_r . Also, we will denote $J(L^X) = \{x_r \in L^X : r \in J(L)\}$, $\beta(A) = \{x_a \in L^X : a \prec A(x)\}$ and $\beta^*(A) = \beta(A) \cap J(L^X)$ for each $A \in L^X$. Clearly, $A = \bigvee \beta(A) = \bigvee \beta^*(A)$. For convenience, we will write $x_\lambda \not\leq^* A$ for $x_\lambda \in J(L^X)$ and $x_\lambda \not\leq A$. We denote $A_{[a]} = \{x \in X : A(x) \geq a\}$ and $A^{[a]} = \{x \in X : a \notin \alpha(A(x))\}$ for each $A \in L^X$ and each $a \in L$ [26, 27, 28, 29, 31]. For convenience, if we write $\bigvee_{i \in \Omega}^{dir} A_i$, we mean that $\{A_i\}_{i \in \Omega} \subseteq L^X$ is up-directed.

A mapping $f_L^{\rightarrow} : L^X \rightarrow L^Y$ is called an L -fuzzy mapping, if there exists a mapping $f : X \rightarrow Y$ such that $f_L^{\rightarrow}(A)(y) = \bigvee_{f(x)=y} A(x)$ for all $A \in L^X$ and all $y \in Y$. Also, $f_L^{\leftarrow} : L^Y \rightarrow L^X$ is defined as: $f_L^{\leftarrow}(B)(x) = B(f(x))$ for all $B \in L^Y$ and all $x \in X$ [27, 28, 29].

Definition 2.1. [29, 30, 32] A pair (X, \mathcal{C}) is called an (L, M) -fuzzy closure space, where $\mathcal{C} : L^X \rightarrow M$ is a mapping, if

$$(LMC1) \mathcal{C}(\underline{\perp}) = \mathcal{C}(\overline{\perp}) = \top;$$

$$(LMC2) \mathcal{C}(\bigwedge_{i \in \Omega} A_i) \geq \bigwedge_{i \in \Omega} \mathcal{C}(A_i) \text{ for all } \{A_i\}_{i \in \Omega} \subseteq L^X.$$

Further, (X, \mathcal{C}) is called an (L, M) -fuzzy convex space [32], if

$$(LMC3) \mathcal{C}(\bigvee_{i \in \Omega} A_i) \geq \bigwedge_{i \in \Omega} \mathcal{C}(A_i) \text{ for all totally ordered set } \{A_i\}_{i \in \Omega} \subseteq L^X.$$

Let \mathcal{C} and \mathcal{D} be (L, M) -fuzzy convex structures on X . \mathcal{C} is said to be finer than \mathcal{D} , denoted by $\mathcal{D} \leq \mathcal{C}$, if $\mathcal{D}(A) \leq \mathcal{C}(A)$ for all $A \in L^X$.

Theorem 2.2. [27, 29] The closure (resp. hull) operator $co_C : L^X \rightarrow M^{J(L^X)}$ (briefly, co) of an (L, M) -fuzzy closure (resp. convex) space (X, \mathcal{C}) is defined as:

$$\forall A \in L^X, \forall x_\lambda \in J(L^X), \quad co(A)(x_\lambda) = \bigwedge_{x_\lambda \not\leq^* B \geq A} (\mathcal{C}(B))'.$$

Then, for all $A, B \in L^X$ and $x_\lambda \in J(L^X)$, co satisfies the following conditions:

- (LMCO1) $co(\perp)(x_\lambda) = \perp$;
- (LMCO2) $x_\lambda \leq A$ implies that $co(A)(x_\lambda) = \top$;
- (LMCO3) $A \leq B$ implies that $co(A) \leq co(B)$;
- (LMCO4) $co(A)(x_\lambda) = \bigwedge_{x_\lambda \not\leq B \geq A} \bigvee_{y_\mu \not\leq^* B} co(B)(y_\mu)$.

Conversely, let $co : L^X \rightarrow M^{J(L^X)}$ be an operator satisfying (LMCO1)–(LMCO4) and a mapping $\mathcal{C}_{co} : L^X \rightarrow M$ be defined as:

$$\forall A \in L^X, \quad \mathcal{C}_{co}(A) = \bigwedge_{x_\lambda \not\leq^* A} (co(A)(x_\lambda))'.$$

Then \mathcal{C}_{co} is an (L, M) -fuzzy closure structure with $co_{\mathcal{C}_{co}} = co$.

Theorem 2.3. [28] If $co : L^X \rightarrow M^{J(L^X)}$ is an operator satisfying (LMCO1)–(LMCO4), then it satisfies (LMCO0).

(LMCO0) $co(A)(x_\lambda) = \bigwedge_{\mu \in \beta^*(\lambda)} co(A)(x_\mu)$ for all $A \in L^X$ and $x_\lambda \in J(L^X)$.

Remark 2.4. (1) An $(L, \mathbf{2})$ -fuzzy closure space (resp. $(L, \mathbf{2})$ -fuzzy convex space) (X, \mathcal{C}) is reduced to an L -closure space (resp. L -convex space [16]). That is, \mathcal{C} satisfies (LC1) and (LC2) (resp. (LC1)–(LC3)).

- (LC1) $\perp, \top \in \mathcal{C}$.
- (LC2) If $\{A_i\}_{i \in \Omega} \subseteq \mathcal{C}$, then $\bigwedge_{i \in \Omega} A_i \in \mathcal{C}$.
- (LC3) If $\{A_i\}_{i \in \Omega} \subseteq \mathcal{C}$ is totally ordered, then $\bigvee_{i \in \Omega} A_i \in \mathcal{C}$.

The L -closure operator $co : L^X \rightarrow L^X$ of an L -closure space (X, \mathcal{C}) is defined as: $co_C(A) = \bigwedge \{B \in \mathcal{C} : A \leq B\}$ for all $A \in L^X$, which is still denoted by co . Then

- (LCO1) $co(\perp) = \perp$;
- (LCO2) $A \leq co(A)$ for all $A \in L^X$;
- (LCO3) $co(A) \leq co(B)$ for all $A, B \in L^X$ with $A \leq B$;
- (LCO4) $co(co(A)) = co(A)$ for all $A \in L^X$.

Conversely, if an operator $co : L^X \rightarrow L^X$ satisfies (LCO1)–(LCO4) and $\mathcal{C}_{co} = \{A \in L^X : co(A) = A\}$, then (X, \mathcal{C}_{co}) is an L -closure space satisfying $co_{\mathcal{C}_{co}} = co$.

(2) (X, \mathcal{C}) is an (L, M) -fuzzy closure space iff $(X, \mathcal{C}_{[a]})$ is an L -closure space for all $a \in M \setminus \{\perp\}$, or $(X, \mathcal{C}^{[a]})$ is an L -closure space for all $a \in \alpha(\perp)$ [32].

(3) A $(\mathbf{2}, M)$ -fuzzy convex structure is reduced to an M -fuzzifying convex structure [31]. A $(\mathbf{2}, \mathbf{2})$ -fuzzy convex space is reduced to a convex space [34].

Definition 2.5. [30] An operator $\mathcal{H} : 2_{fin}^X \rightarrow M^X$ is called an M -fuzzifying restricted hull operator and the pair (X, \mathcal{H}) is called an M -fuzzifying restricted hull space, if

- (MRH1) $\mathcal{H}(\emptyset)(x) = \perp$ for all $x \in X$;
- (MRH2) $\mathcal{H}(U)(x) = \top$ for all $U \in 2_{fin}^X$ and $x \in U$;
- (MRH3) $\mathcal{H}(V)(x) \wedge \bigwedge_{y \in V} \mathcal{H}(U)(y) \leq \mathcal{H}(U)(x)$ for all $U, V \in 2_{fin}^X$ and $x \in X$.

Theorem 2.6. [30] If (X, \mathcal{C}) is an M -fuzzifying convex space, then the restriction $\mathcal{H}_C : 2_{fin}^X \rightarrow M^X$ of the hull operator co_C on 2_{fin}^X is an M -fuzzifying restricted hull operator.

Conversely, if $\mathcal{H} : 2_{fin}^X \rightarrow M^X$ is an M -fuzzifying restricted hull operator and $co_{\mathcal{H}} : 2^X \rightarrow M^X$ is defined as: $co_{\mathcal{H}}(U)(x) = \bigvee_{V \in 2_{fin}^U} \mathcal{H}(V)(x)$ for all $U \in 2^X$ and $x \in X$, then $co_{\mathcal{H}}$ is the hull operator of an M -fuzzifying convex space $(X, \mathcal{C}_{\mathcal{H}})$ satisfying $\mathcal{H}_{\mathcal{C}_{\mathcal{H}}} = \mathcal{H}$.

Let (X, \mathcal{H}_X) and (Y, \mathcal{H}_Y) be M -fuzzifying restricted hull spaces. $f : X \rightarrow Y$ is called an M -fuzzifying RHP mapping, if $\mathcal{H}_X(U)(x) \leq \mathcal{H}_Y(f(U))(f(x))$ for all $U \in 2_{fin}^X$ and $x \in X$.

Definition 2.7. (1) $B \in L^X$ is called a proper subset of $A \in L^X$, denoted by $B \not\leq A$, if $B \leq A$ and $B \neq A$. Clearly, $B \not\leq A$ iff $B \leq A$ and there exists $x_\lambda \prec A$ such that $x_\lambda \not\leq B$.

(2) $F \in L^X$ is called an L -fuzzy finite set relative to $A \in L^X$, if there exists a $\varphi \in 2_{fin}^{\beta^*(A)}$ such that $F = \bigvee \varphi$. We denote $\mathfrak{F}(A) = \{\bigvee \varphi : \varphi \in 2_{fin}^{\beta^*(A)}\}$. In particular, L -fuzzy finite sets relative to \perp are simply called L -fuzzy finite sets. We simply denote $\mathfrak{F}(\perp)$ by $\mathfrak{F}(X)$.

Proposition 2.8. *Let $A, B \in L^X$ and $F \in \mathfrak{F}(X)$. Then*

- (1) $\mathfrak{F}(A)$ is up-directed;
- (2) $B \leq A$ iff $\mathfrak{F}(B) \subseteq \mathfrak{F}(A)$;
- (3) $\beta^*(A) \subseteq \mathfrak{F}(A)$ and $\bigvee \mathfrak{F}(A) = A$;
- (4) $\mathfrak{F}(\bigvee_{i \in \Omega} A_i) = \bigcup_{i \in \Omega} \mathfrak{F}(A_i)$ for all up-directed set $\{A_i\}_{i \in \Omega} \subseteq L^X$.

Proof. (1)–(3) are clear. We only need to prove (4). Clearly, $\bigcup_{i \in \Omega} \mathfrak{F}(A_i) \subseteq \mathfrak{F}(\bigvee_{i \in \Omega} A_i)$. Conversely, if $F \in \mathfrak{F}(\bigvee_{i \in \Omega} A_i)$, then there exists $\varphi \in 2_{fin}^{\beta^*(\bigvee_{i \in \Omega} A_i)} = 2_{fin}^{\bigcup_{i \in \Omega} \beta^*(A_i)}$ such that $F = \bigvee \varphi$. Since the set $\{\beta^*(A_i)\}_{i \in \Omega}$ is also up-directed, there exists $i_0 \in \Omega$ such that $\varphi \in 2_{fin}^{\beta^*(A_{i_0})}$. Thus $F \in \mathfrak{F}(A_{i_0}) \subseteq \bigcup_{i \in \Omega} \mathfrak{F}(A_i)$. Therefore $\bigcup_{i \in \Omega} \mathfrak{F}(A_i) = \mathfrak{F}(\bigvee_{i \in \Omega} A_i)$. \square

The following proposition is similar to Lemma 3 in [31].

Proposition 2.9. *Let $p, q \in M$. The following statements are equivalent.*

- (1) $p \leq q$.
- (2) [31] $\forall a \in M, a \leq p$ implies $a \leq q$.
- (3) [31] $\forall a \in \alpha(\perp), a \not\leq \alpha(p)$ implies $a \not\leq \alpha(q)$.
- (4) [31] $\forall a \in J(M), a \leq p$ implies $a \leq q$.
- (5) $\forall a \in \beta(\top), a \prec p$ implies $a \leq q$.
- (6) $\forall a \in P(M), p \not\leq a$ implies $q \not\leq a$.
- (7) $\forall a \in \beta^*(\top), a \prec p$ implies $a \leq q$.

Definition 2.10. [1] *Let \mathbf{A}, \mathbf{B} be categories. A mapping $\mathbb{F} : \mathbf{A} \rightarrow \mathbf{B}$ is called a functor, if for all $A, B \in O(\mathbf{A})$, $f \in \text{hom}_{\mathbf{A}}(A, B)$ and $g \in \text{hom}_{\mathbf{A}}(B, C)$,*

- (1) $\mathbb{F}(A) \in O(\mathbf{B})$ and $\mathbb{F}(f) \in \text{hom}_{\mathbf{B}}(\mathbb{F}(A), \mathbb{F}(B))$;
- (2) $\mathbb{F}(g \circ f) = \mathbb{F}(g) \circ \mathbb{F}(f)$;
- (3) $\mathbb{F}(id_A) = id_{\mathbb{F}(A)}$.

In particular, the functor $\mathbb{I}_{\mathbf{A}} : \mathbf{A} \rightarrow \mathbf{A}$, defined as: $\mathbb{I}_{\mathbf{A}}(A) = A$ and $\mathbb{I}_{\mathbf{A}}(f) = f$ for all $A \in O(\mathbf{A})$ and $f \in \text{hom}_{\mathbf{A}}(A, B)$, is called the identity functor of \mathbf{A} .

Definition 2.11. [1] *Let \mathbf{A} and \mathbf{B} be categories. A functor $\mathbb{F} : \mathbf{A} \rightarrow \mathbf{B}$ is called an isomorphism provided that there is a functor $\mathbb{G} : \mathbf{B} \rightarrow \mathbf{A}$ such that $\mathbb{G} \circ \mathbb{F} = \mathbb{I}_{\mathbf{A}}$ and $\mathbb{F} \circ \mathbb{G} = \mathbb{I}_{\mathbf{B}}$. \mathbf{A}, \mathbf{B} are called isomorphic, denoted by $\mathbf{A} \cong \mathbf{B}$, if there exists an isomorphic functor $\mathbb{F} : \mathbf{A} \rightarrow \mathbf{B}$.*

Definition 2.12. [1] *A functor $\mathbb{F} : \mathbf{A} \rightarrow \mathbf{B}$ is said to be*

- (1) an embedding, if \mathbb{F} is injective on both objects and morphisms;
- (2) faithful, if $\mathbb{F} : \text{hom}_{\mathbf{A}}(A, B) \rightarrow \text{hom}_{\mathbf{B}}(\mathbb{F}(A), \mathbb{F}(B))$ is injective for all $A, B \in O(\mathbf{A})$;
- (3) full, if $\mathbb{F} : \text{hom}_{\mathbf{A}}(A, B) \rightarrow \text{hom}_{\mathbf{B}}(\mathbb{F}(A), \mathbb{F}(B))$ is surjective for all $A, B \in O(\mathbf{A})$.

Definition 2.13. [1] *A concrete category over a category \mathbf{X} is a pair (\mathbf{A}, \mathbb{U}) , where $\mathbb{U} : \mathbf{A} \rightarrow \mathbf{X}$ is a faithful functor. A concrete category over \mathbf{Set} is called a construct. A concrete functor from (\mathbf{A}, \mathbb{U}) to (\mathbf{B}, \mathbb{V}) is a functor $\mathbb{F} : \mathbf{A} \rightarrow \mathbf{B}$ such that $\mathbb{U} = \mathbb{V} \circ \mathbb{F}$.*

Definition 2.14. [1] *Let (\mathbf{A}, \mathbb{U}) and (\mathbf{B}, \mathbb{V}) be constructs and $X \in O(\mathbf{Set})$.*

- (1) *The fibre of X is a preordered class $(\mathfrak{F}_{\mathbf{A}}(X), \ll)$, where $\mathfrak{F}_{\mathbf{A}}(X) = \{A \in O(\mathbf{A}) : \mathbb{U}(A) = X\}$ and \ll is defined as: $A \ll B$ iff $id_X : \mathbb{U}(A) \rightarrow \mathbb{U}(B)$ is an \mathbf{A} -morphism.*
- (2) *If $\mathbb{F}, \mathbb{G} : \mathbf{A} \rightarrow \mathbf{B}$ are concrete functors, then \mathbb{F} is said to be finer than \mathbb{G} , denoted by $\mathbb{F} \ll \mathbb{G}$, provided that $\mathbb{F}(A) \ll \mathbb{G}(A)$ for each $A \in O(\mathbf{A})$.*
- (3) *If $\mathbb{F} : \mathbf{A} \rightarrow \mathbf{B}$ and $\mathbb{G} : \mathbf{B} \rightarrow \mathbf{A}$ are concrete functors, then the pair (\mathbb{F}, \mathbb{G}) is called a Galois correspondence provided that $\mathbb{I}_{\mathbf{A}} \ll \mathbb{G} \circ \mathbb{F}$ and $\mathbb{F} \circ \mathbb{G} \ll \mathbb{I}_{\mathbf{B}}$.*
- (4) *A functor $\mathbb{T} : (\mathbf{A}, \mathbb{U}) \rightarrow (\mathbf{B}, \mathbb{V})$ is called topological provided that every \mathbb{T} -spaced sink $(f_j : B \rightarrow \mathbb{T}(A_j))_{j \in J}$ has a unique \mathbb{T} -final lift $(\bar{f}_j : A \rightarrow A_j)_{j \in J}$.*
- (5) *(\mathbf{A}, \mathbb{U}) is called topological provided that $\mathbb{U} : \mathbf{A} \rightarrow \mathbf{Set}$ is topological.*

Theorem 2.15. [1] *Let (\mathbf{A}, \mathbb{U}) be a subcategory of a category (\mathbf{B}, \mathbb{V}) with the inclusion functor $\mathbb{E} : (\mathbf{A}, \mathbb{U}) \rightarrow (\mathbf{B}, \mathbb{V})$. Then (\mathbf{A}, \mathbb{U}) is coreflective in (\mathbf{B}, \mathbb{V}) iff there exists a functor $\mathbb{R} : (\mathbf{B}, \mathbb{V}) \rightarrow (\mathbf{A}, \mathbb{U})$ such that (\mathbb{E}, \mathbb{R}) is a Galois correspondence.*

Remark 2.16. (1) If $co : L^X \rightarrow M^{J(L^X)}$ satisfies (LMCO1)–(LMCO4), then (X, co) is called an (L, M) -fuzzy closure operator space. Let (X, co_X) and (Y, co_Y) be (L, M) -fuzzy closure operator spaces. A mapping $f : X \rightarrow Y$ is called an (L, M) -fuzzy closure operator preserving mapping, if $co_X(A)(x_\lambda) \leq co_Y(f_L^\rightarrow(A))(f_L^\rightarrow(x_\lambda))$ for $A \in L^X$ and $x_\lambda \in J(L^X)$.

(2) Let (X, \mathcal{C}_X) and (Y, \mathcal{C}_Y) be (L, M) -fuzzy convex spaces. A mapping $f : X \rightarrow Y$ is called an (L, M) -CP mapping, if $f : (X, co_{\mathcal{C}_X}) \rightarrow (Y, co_{\mathcal{C}_Y})$ is an (L, M) -fuzzy closure operator preserving mapping [32]. The category whose objects are (L, M) -fuzzy convex spaces and whose morphisms are (L, M) -CP mappings, is denoted by (L, M) -CS. The corresponding construct is denoted by $((L, M)$ -CS, \mathbb{C}).

(3) Let (X, \mathcal{C}_X) and (Y, \mathcal{C}_Y) be L -convex spaces. A mapping $f : X \rightarrow Y$ is called an L -CP mapping, if $f_L^\leftarrow(B) \in \mathcal{C}_X$ for all $B \in \mathcal{C}_Y$ [16]. $(L, \mathbf{2})$ -CS is denoted by L -CS [18].

(4) Let (X, \mathcal{C}_X) and (Y, \mathcal{C}_Y) be M -fuzzifying convex spaces. A mapping $f : X \rightarrow Y$ is called an M -fuzzifying CP mapping, if $co_{\mathcal{C}_X}(U)(x) \leq co_{\mathcal{C}_Y}(f(U))(f(x))$ for all $U \in 2^X$ and $x \in X$ [31]. $(\mathbf{2}, M)$ -CS is denoted by M -CS. In particular, $\mathbf{2}$ -CS is denoted by CS.

(5) The category, consisting of M -fuzzifying restricted hull spaces as objects and M -fuzzifying RHP mappings as morphisms, is denoted by M -RHS. The corresponding construct is denoted by $(M$ -RHS, \mathbb{H}). In particular, $\mathbf{2}$ -RHS is denoted by RHS.

3 Some Properties of L -convex Spaces

Theorem 3.1. Any L -convex space (X, \mathcal{C}) is stable for up-directed unions, i.e.,

$$(LC3^*) \bigvee_{i \in \Omega} A_i \in \mathcal{C} \text{ for each up-directed family } \{A_i\}_{i \in \Omega} \subseteq \mathcal{C}.$$

Proof. Let $\mathcal{D} = \{D_i\}_{i \in \Omega} \subseteq \mathcal{C}$ be up-directed and $D = \bigvee_{i \in \Omega} D_i$. Define a set

$$\mathcal{M} = \{A \in \mathcal{C} : \forall D_i \in \mathcal{D}, A \vee_{\mathcal{C}} D_i \leq D\},$$

where $A \vee_{\mathcal{C}} D_i = \bigwedge \{B \in \mathcal{C} : A \vee D_i \leq B\}$. Then we have the following results.

(i) $\mathcal{D} \subseteq \mathcal{M}$.

In fact, let $D_{i_0} \in \mathcal{D}$ and $D_i \in \mathcal{D}$. Since $\mathcal{D} \subseteq \mathcal{C}$ is up-directed, there exists $D_j \in \mathcal{D}$ such that $D_{i_0}, D_i \leq D_j$. Then $D_{i_0} \vee_{\mathcal{C}} D_i \leq D_j \leq D$, which shows $D_{i_0} \in \mathcal{M}$. Thus $\mathcal{D} \subseteq \mathcal{M}$.

(ii) $A \leq D$ for all $A \in \mathcal{M}$. It directly follows from definition of \mathcal{M} .

(iii) $A \vee_{\mathcal{C}} D_i \in \mathcal{M}$ for all $A \in \mathcal{M}$ and $D_i \in \mathcal{D}$.

In fact, by (LC2), $A \vee_{\mathcal{C}} D_i \in \mathcal{C}$. For each $D_j \in \mathcal{D}$, there exists $D_k \in \mathcal{D}$ such that $D_i, D_j \leq D_k$. Thus $D_j, A \vee_{\mathcal{C}} D_i \leq A \vee_{\mathcal{C}} D_k \in \mathcal{C}$. Hence $(A \vee_{\mathcal{C}} D_i) \vee_{\mathcal{C}} D_j \leq A \vee_{\mathcal{C}} D_k$, which shows $(A \vee_{\mathcal{C}} D_i) \vee_{\mathcal{C}} D_j \leq A \vee_{\mathcal{C}} D_k \leq D$. Therefore $(A \vee_{\mathcal{C}} D_i) \in \mathcal{M}$.

(iv) \mathcal{M} is inductive. i.e., each nonempty chain $\mathcal{K} = \{K_j\}_{j \in \Lambda} \subseteq \mathcal{M}$ is bounded in \mathcal{M} .

In fact, $K = \bigvee \mathcal{K} \in \mathcal{C}$ by (LC3). If $D_i \in \mathcal{D}$, the set $\{(K_j \vee_{\mathcal{C}} D_i)\}_{j \in \Lambda}$ is also a chain in \mathcal{C} . Thus $\bigvee_{j \in \Lambda} (K_j \vee_{\mathcal{C}} D_i) \in \mathcal{C}$ and $K \vee_{\mathcal{C}} D_i \leq \bigvee_{j \in \Lambda} (K_j \vee_{\mathcal{C}} D_i) \leq D$. Hence $K \in \mathcal{M}$.

By (iv) and Zorn's lemma, \mathcal{M} has a maximal element, namely, $M_{\mathcal{M}}$. Then $D \geq M_{\mathcal{M}}$ by (ii). Conversely, by (iii), we have $M_{\mathcal{M}} \vee_{\mathcal{C}} D_i \in \mathcal{M}$ for all $D_i \in \mathcal{D}$. By maximality of $M_{\mathcal{M}}$ and $M_{\mathcal{M}} \vee_{\mathcal{C}} D_i \geq M_{\mathcal{M}}$, we have $D_i \leq M_{\mathcal{M}} \vee_{\mathcal{C}} D_i = M_{\mathcal{M}}$. Therefore $D = M_{\mathcal{M}} \in \mathcal{C}$. \square

Theorem 3.2. For an L -closure space (X, \mathcal{C}) , the following results are equivalent.

(1) \mathcal{C} is an L -convex structure.

(2) \mathcal{C} is stable for up-directed unions.

(3) The closure operator co is domain finite, that is, co satisfies (LDF) as below.

(LDF) $co(A) = \bigvee_{F \in \mathfrak{F}(A)} co(F)$ for all $A \in L^X$.

(4) co is stable for up-directed union, that is, $co(\bigvee_{i \in \Omega} A_i) = \bigvee_{i \in \Omega} co(A_i)$ for all up-directed family $\{A_i\}_{i \in \Omega} \subseteq L^X$.

Proof. (1) \Rightarrow (2): It follows from Theorem 3.1.

(2) \Rightarrow (3): Clearly, $\bigvee_{F \in \mathfrak{F}(A)} co(F) \leq co(A)$. Conversely, since $\mathfrak{F}(A)$ is up-directed, the set $\{co(F) : F \in \mathfrak{F}(A)\}$ is also up-directed. Thus $\bigvee_{F \in \mathfrak{F}(A)} co(F) \in \mathcal{C}$. Hence $A = \bigvee_{F \in \mathfrak{F}(A)} F \leq \bigvee_{F \in \mathfrak{F}(A)} co(F)$ and $co(A) \leq \bigvee_{F \in \mathfrak{F}(A)} co(F)$. Therefore $co(A) = \bigvee_{F \in \mathfrak{F}(A)} co(F)$.

(3) \Rightarrow (4): Let $\{A_i\}_{i \in \Omega} \subseteq L^X$ be an up-directed family and $A = \bigvee_{i \in \Omega} A_i$. By (LDF), $co(A) = \bigvee_{F \in \mathfrak{F}(A)} co(F) = \bigvee_{F \in \bigcup_{i \in \Omega} \mathfrak{F}(A_i)} co(F) = \bigvee_{i \in \Omega} \bigvee_{F \in \mathfrak{F}(A_i)} co(F) = \bigvee_{i \in \Omega} co(A_i)$.

(4) \Rightarrow (1): If $\{A_i\}_{i \in \Omega} \subseteq \mathcal{C}$ is totally ordered, then it is up-directed. Thus $co(\bigvee_{i \in \Omega} A_i) = \bigvee_{i \in \Omega} co(A_i) = \bigvee_{i \in \Omega} A_i \in \mathcal{C}$. Therefore \mathcal{C} is an (L, M) -fuzzy convex structure. \square

Theorem 3.3. *Let (X, \mathcal{D}) be an L -closure space and*

$$\mathcal{C}_{\mathcal{D}} = \{A \in L^X : \exists \varphi \subseteq \mathcal{D} \text{ is up-directed, } A = \bigvee \varphi\}.$$

Then $\mathcal{C}_{\mathcal{D}}$ is an L -convex structure generated by \mathcal{D} .

Proof. (LC1): Clearly, $\perp, \top \in \mathcal{D} \subseteq \mathcal{C}_{\mathcal{D}}$.

(LC2): Let $\{A_i\}_{i \in \Omega} \subseteq \mathcal{C}_{\mathcal{D}}$ and $A = \bigwedge_{i \in \Omega} A_i$. By the definition of $\mathcal{C}_{\mathcal{D}}$, there exists an up-directed subfamily $\mathcal{D}_i \subseteq \mathcal{D}$ such that $A_i = \bigvee \mathcal{D}_i$ for each $i \in \Omega$.

Let \mathcal{S} be the set of all choice mappings $s : \Omega \rightarrow \bigcup_{i \in \Omega} \mathcal{D}_i$, where $s(i) \in \mathcal{D}_i$ for each $i \in \Omega$. Let $\mathcal{D}_0 = \{\bigwedge_{i \in \Omega} s(i) : s \in \mathcal{S}\}$. It has the following two properties.

(i) \mathcal{D}_0 is up-directed. In fact, let $B_1, B_2 \in \mathcal{D}_0$. Then there exist $s_1, s_2 \in \mathcal{S}$ such that $B_1 = \bigwedge_{i \in \Omega} s_1(i)$ and $B_2 = \bigwedge_{i \in \Omega} s_2(i)$. Thus, for each $i \in \Omega$, there exists $D_i \in \mathcal{D}_i$ such that $s_1(i), s_2(i) \leq D_i$. Define a mapping $s_3 : \Omega \rightarrow \bigcup_{i \in \Omega} \mathcal{D}_i$ as: $s_3(i) = D_i$ for all $i \in \Omega$. Then $s_3 \in \mathcal{S}$ and $B_1, B_2 \leq B_3 = \bigwedge_{i \in \Omega} s_3(i) \in \mathcal{D}_0$. This shows \mathcal{D}_0 is up-directed.

(ii) $A = \bigvee \mathcal{D}_0$. In fact, if $x_\lambda \prec A = \bigwedge_{i \in \Omega} A_i$, then $x_\lambda \prec A_i$ for each $i \in \Omega$. Thus, for each $i \in \Omega$, there exists $D_i \in \mathcal{D}_i$ such that $x_\lambda \leq D_i$. Let $s : \Omega \rightarrow \bigcup_{i \in \Omega} \mathcal{D}_i$ be defined as: $s(i) = D_i$ for each $i \in \Omega$. Then $s \in \mathcal{S}$ and $x_\lambda \leq \bigwedge_{i \in \Omega} s(i) = \bigwedge_{i \in \Omega} s(i) \in \mathcal{D}_0$. Hence $x_\lambda \leq \bigvee \mathcal{D}_0$. By Proposition 2.9(5), we have $A \leq \bigvee \mathcal{D}_0$. Conversely, if $y_\mu \prec \bigvee \mathcal{D}_0$, then there exists $s \in \mathcal{S}$ such that $y_\mu \leq \bigwedge_{i \in \Omega} s(i) \leq \bigwedge_{i \in \Omega} A_i = A$. Hence $\bigvee \mathcal{D}_0 \leq A$.

(LC3*): Let $\{A_i\}_{i \in \Omega} \subseteq \mathcal{C}_{\mathcal{D}}$ be up-directed and $A = \bigvee_{i \in \Omega} A_i$. Since $A_i \in \mathcal{C}_{\mathcal{D}}$ for each $i \in \Omega$, there exists an up-directed $\mathcal{D}_i \subseteq \mathcal{D}$ such that $A_i = \bigvee \mathcal{D}_i$. Let $\mathcal{D}_* = \bigcup_{i \in \Omega} \mathcal{D}_i$ and $\varphi_F = \{D : F \leq D \in \mathcal{D}_*\}$ for each $F \in \mathfrak{F}(A)$. We firstly check that φ_F is nonempty for all $F \in \mathfrak{F}(A)$. In fact, by Proposition 2.8(4), there exists $i_F \in \Omega$ such that $F \in \mathfrak{F}(A_{i_F}) = \mathfrak{F}(\bigvee \mathcal{D}_{i_F})$. Again, there exists $D_{i_F} \in \mathcal{D}_{i_F} \subseteq \mathcal{D}_*$ such that $F \in \mathfrak{F}(D_{i_F})$. Hence $D_{i_F} \in \varphi_F$.

Since $F \leq D_{i_F} \leq A_{i_F} \leq A$ for all $F \in \mathfrak{F}(A)$, we have $F \leq \bigwedge \varphi_F \leq A$. Thus $A = \bigvee_{F \in \mathfrak{F}(A)} F \leq \bigvee_{F \in \mathfrak{F}(A)} \bigwedge \varphi_F \leq A$ which implies that $A = \bigvee_{F \in \mathfrak{F}(A)} \bigwedge \varphi_F$. So we have $A \in \mathcal{C}_{\mathcal{D}}$ since $\{\bigwedge \varphi_F : F \in \mathfrak{F}(A)\} \subseteq \mathcal{D}$ is up-directed. Hence $\mathcal{C}_{\mathcal{D}}$ is an L -convex space. \square

Definition 3.4. *Let (X, \mathcal{C}) be an L -convex space and $\& \subseteq \mathcal{D}$. $\&$ is called a*

- (1) *subbase of \mathcal{C} , if \mathcal{C} is the coarsest L -convex structure on X containing $\&$;*
- (2) *base of \mathcal{C} , if elements of \mathcal{C} are supermums of up-directed subfamilies of $\&$.*

By Theorem 3.3, an L -closure space is a base of its generated L -convex space. Also, by Definition 3.4 and Theorem 3.1, a base of an L -convex space is a subbase.

Theorem 3.5. *Let (X, \mathcal{C}) be an L -convex space and $\& \subseteq \mathcal{C}$.*

- (1) *If $\{co(F) : F \in \mathfrak{F}(X) \setminus \{\perp\}\} \subseteq \&$, then $\&$ is a base of \mathcal{C} .*
- (2) *If $co(F) = \bigwedge \&_1$, where $\&_1 \subseteq \&$ for all $F \in \mathfrak{F}(X) \setminus \{\perp\}$, then $\&$ is a subbase of \mathcal{C} .*

Proof. We show $M_{\mathbb{I}}$ satisfies (1) and (2).

(1): Let $C \in \mathcal{C}$. By Proposition 2.8(3) and Theorem 3.2(4), we have $C = co(C) = co(\bigvee_{F \in \mathfrak{F}(C)} F) = \bigvee_{F \in \mathfrak{F}(C)} co(F) = \bigvee_{F \in \mathfrak{F}(C) \setminus \{\perp\}} co(F)$. Since the set $\{co(F) : F \in \mathfrak{F}(C) \setminus \{\perp\}\} \subseteq \&$ is up-directed, we know that $\&$ is a base of \mathcal{C} .

(2): Let $\mathcal{D} = \{\bigwedge \varphi \in L^X : \varphi \subseteq \&\}$. Since $\{co(F) : F \in \mathfrak{F}(X) \setminus \{\perp\}\} \subseteq \mathcal{D} \subseteq \mathcal{C}$, we know that \mathcal{D} is a base of \mathcal{C} by (1). Thus each element of \mathcal{C} is the supermum of an up-directed subset of \mathcal{D} . To show that $\&$ is a subbase of \mathcal{C} , let (X, \mathcal{F}) be an L -convex space with $\& \subseteq \mathcal{F}$. Then $\mathcal{D} \subseteq \mathcal{F}$ which shows $\mathcal{C} \subseteq \mathcal{F}$. Therefore $\&$ is a subbase of \mathcal{C} . \square

The inverse results of Theorem 3.5 are not true. We have the following example.

Example 3.6. *Let $X = \{x\}$ and $L = [0, 1]$. Then $\& = \{x_t : t \in [0, \frac{1}{3}] \cup [\frac{1}{2}, 1]\}$ is a base of the L -convex structure $\mathcal{C} = [0, \frac{1}{3}]^X \cup [\frac{1}{2}, 1]^X$. Thus $\&$ is also a subbase of \mathcal{C} . However, we have $x_{\frac{1}{3}} \in \mathfrak{F}(X)$, $co(x_{\frac{1}{3}}) = x_{\frac{1}{3}} \notin \&$ and $x_{\frac{1}{3}} \neq \bigwedge \varphi$ for any $\varphi \subseteq \&$.*

4 Characterizations of (L, M) -fuzzy Convex Spaces

Theorem 4.1. *The following are equivalent for an (L, M) -fuzzy closure space (X, \mathcal{C}) .*

- (1) *\mathcal{C} is an (L, M) -fuzzy convex structure.*
- (2) *co is domain finite, that is, co fulfills (LMDF) below.*
(LMDF) $\forall A \in L^X, \forall x_\lambda \in J(L^X), co(A)(x_\lambda) = \bigwedge_{\mu \in \beta^*(\lambda)} \bigvee_{F \in \mathfrak{F}(A)} co(F)(x_\mu)$.
- (3) *\mathcal{C} is stable for up-directed unions, that is, \mathcal{C} satisfies (LMC3*) below.*
(LMC3*) $\mathcal{C}(\bigvee_{i \in \Omega} A_i) \geq \bigwedge_{i \in \Omega} \mathcal{C}(A_i)$ for each up-directed subset $\{A_i\}_{i \in \Omega} \subseteq L^X$.

(4) co is stable for up-directed unions, that is,

$$co\left(\bigvee_{i \in \Omega} A_i\right)(x_\lambda) = \bigwedge_{\mu \in \beta^*(\lambda)} \bigvee_{i \in \Omega} co(A_i)(x_\mu)$$

for each up-directed subset $\{A_i\}_{i \in \Omega} \subseteq L^X$ and all $x_\lambda \in J(L^X)$.

Proof. (1) \Rightarrow (2): Let $A \in L^X$, $x_\lambda \in J(L^X)$ and $a \in \beta(\top)$ with $a \prec co(A)(x_\lambda)$. Then

$$\begin{aligned} co(A)(x_\lambda) &\Rightarrow \forall x_\lambda \not\leq B \geq A, \quad B \notin \mathcal{C}^{[a']} \\ &\Rightarrow x_\lambda \leq co_{\mathcal{C}^{[a']}}(A) = \bigvee_{F \in \mathfrak{F}(A)} co_{\mathcal{C}^{[a']}}(F) \\ &\Rightarrow \forall \mu \in \beta^*(\lambda), \quad \exists F_\mu \in \mathfrak{F}(A), \quad s.t. \quad x_\mu \leq co_{\mathcal{C}^{[a']}}(F_\mu) \\ &\Rightarrow \forall \mu \in \beta^*(\lambda), \quad \exists F_\mu \in \mathfrak{F}(A), \quad s.t. \quad \forall x_\mu \not\leq D \geq F_\mu, \quad D \notin \mathcal{C}^{[a']} \\ &\Rightarrow \forall \mu \in \beta^*(\lambda), \quad \exists F_\mu \in \mathfrak{F}(A), \quad s.t. \quad co(F_\mu)(x_\mu) \geq a \\ &\Rightarrow \bigwedge_{\mu \in \beta^*(\lambda)} \bigvee_{F \in \mathfrak{F}(A)} co(F)(x_\mu) \geq a. \end{aligned}$$

By Proposition 2.9(5), we have $co(A)(x_\lambda) \leq \bigwedge_{\mu \in \beta^*(\lambda)} \bigvee_{F \in \mathfrak{F}(A)} co(F)(x_\mu)$. Conversely, we have $co(A)(x_\lambda) = \bigwedge_{\mu \in \beta^*(\lambda)} co(A)(x_\mu) \geq \bigwedge_{\mu \in \beta^*(\lambda)} \bigvee_{F \in \mathfrak{F}(A)} co(F)(x_\mu)$.

(2) \Rightarrow (3): Let $\{A_i\}_{i \in \Omega} \subseteq L^X$ be up-directed and $A = \bigvee_{i \in \Omega} A_i$. If $a \in M$ with $\mathcal{C}(A) \not\geq a$, then there exists $b \prec a$ such that $\mathcal{C}(A) \not\geq b$. By Proposition 2.8(4), we have

$$\begin{aligned} \mathcal{C}(A) &= \bigwedge_{x_\lambda \not\leq^* A} (co(A)(x_\lambda))' \not\geq b \\ &\Rightarrow \exists x_\lambda \not\leq^* A, \quad \bigwedge_{\mu \in \beta^*(\lambda)} \bigvee_{F \in \mathfrak{F}(A)} co(F)(x_\mu) \not\geq b' \\ &\Rightarrow \exists x_\lambda \not\leq^* A, \quad s.t. \quad \forall \mu \in \beta^*(\lambda), \quad \exists F_\mu \in \mathfrak{F}(A), \quad co(F_\mu)(x_\mu) \not\geq b' \\ &\Rightarrow \exists x_\lambda \not\leq^* A, \quad s.t. \quad \forall \mu \in \beta^*(\lambda), \quad \exists F_\mu \in \mathfrak{F}(A_{i_\mu})(i_\mu \in \Omega), \quad (co(F_\mu)(x_\mu))' \not\geq b. \end{aligned}$$

Thus there exists $x_\lambda \not\leq^* A$ and $\bigvee_{\mu \prec \lambda} \bigwedge_{F \in \mathfrak{F}(A_{i_\mu})} (co(F)(x_\mu))' \not\geq a$. Hence we have

$$\bigwedge_{i \in \Omega} \mathcal{C}(A_i) = \bigwedge_{i \in \Omega} \bigwedge_{x_\lambda \not\leq A_i} (co_{\mathcal{C}}(A_i)(x_\lambda))' \leq \bigwedge_{x_\lambda \not\leq A} \bigvee_{\mu \in \beta^*(\lambda)} \bigwedge_{F \in \mathfrak{F}(A_{i_\mu})} (co_{\mathcal{C}}(F)(x_\mu))' \not\geq a.$$

This shows $\bigwedge_{i \in \Omega} \mathcal{C}(A_i) \not\geq a$. By Proposition 2.9(2), we have $\bigwedge_{i \in \Omega} \mathcal{C}(A_i) \leq \mathcal{C}(A)$.

(2) \Rightarrow (4): Let $\{A_i\}_{i \in \Omega} \subseteq L^X$ be up-directed, $A = \bigvee_{i \in \Omega} A_i$ and $x_\lambda \in J(L^X)$. Thus

$$\begin{aligned} co(A)(x_\lambda) &= \bigwedge_{\mu \in \beta^*(\lambda)} \bigvee_{i \in \Omega} \bigvee_{F \in \mathfrak{F}(A_i)} co(F)(x_\mu) = \bigwedge_{\mu \in \beta^*(\lambda)} \bigwedge_{\eta \in \beta^*(\mu)} \bigvee_{i \in \Omega} \bigvee_{F \in \mathfrak{F}(A_i)} co(F)(x_\eta) \\ &\geq \bigwedge_{\mu \in \beta^*(\lambda)} \bigvee_{i \in \Omega} \bigwedge_{\eta \in \beta^*(\mu)} \bigvee_{F \in \mathfrak{F}(A_i)} co(F)(x_\eta) = \bigwedge_{\mu \in \beta^*(\lambda)} \bigvee_{i \in \Omega} co(A_i)(x_\mu). \end{aligned}$$

Conversely, by Proposition 2.8(4), (LMCO3) and (LMDF), we have

$$co(A)(x_\lambda) = \bigwedge_{\mu \in \beta^*(\lambda)} \bigvee_{i \in \Omega} \bigvee_{F \in \mathfrak{F}(A_i)} co(F)(x_\mu) \leq \bigwedge_{\mu \in \beta^*(\lambda)} \bigvee_{i \in \Omega} co(A_i)(x_\mu).$$

(4) \Rightarrow (2): Let $A \in L^X$ and $x_\lambda \in J(L^X)$. By Proposition 2.8(1) and (3), we have

$$co(A)(x_\lambda) = co\left(\bigvee_{F \in \mathfrak{F}(A)} F\right)(x_\lambda) = \bigwedge_{\mu \in \beta^*(\lambda)} \bigvee_{F \in \mathfrak{F}(A)} co(F)(x_\mu).$$

(3) \Rightarrow (1): Clear. □

Definition 4.2. An operator $\mathcal{H} : \mathfrak{F}(X) \rightarrow M^{J(L^X)}$ is called an (L, M) -fuzzy restricted hull operator and the pair (X, \mathcal{H}) is called an (L, M) -fuzzy restricted hull space, if for all $x_\lambda \in J(L^X)$ and $F, G \in \mathfrak{F}(X)$, \mathcal{H} satisfies the following conditions.

- (LMRH0) $\mathcal{H}(F)(x_\lambda) = \bigwedge_{\mu \in \beta^*(\lambda)} \mathcal{H}(F)(x_\mu)$.
- (LMRH1) $\mathcal{H}(\perp)(x_\lambda) = \perp$.
- (LMRH2) If $x_\lambda \leq F$, then $\mathcal{H}(F)(x_\lambda) = \top$.
- (LMRH3) $\mathcal{H}(G)(x_\lambda) \wedge \bigwedge_{y_\mu \in \beta^*(G)} \mathcal{H}(F)(y_\mu) \leq \mathcal{H}(F)(x_\lambda)$.
- (LMRH4) $\mathcal{H}(F)(x_\lambda) = \bigwedge_{\mu \in \beta^*(\lambda)} \bigvee_{G \in \mathfrak{F}(F)} \mathcal{H}(G)(x_\mu)$.

Remark 4.3. (1) A $(2, M)$ -fuzzy restricted hull operator is an M -fuzzifying restricted hull operator (Definition 2.5). An $(L, 2)$ -fuzzy restricted hull operator is reduced to an L -restricted hull operator $\mathcal{H} : \mathfrak{F}(X) \rightarrow L^X$, which satisfies (LRH1)-(LRH4).

- (LRH1) $\mathcal{H}(\perp) = \perp$.
- (LRH2) $F \leq \mathcal{H}(F)$ for all $F \in \mathfrak{F}(X)$.
- (LRH3) $\mathcal{H}(G) \leq \mathcal{H}(F)$ for all $F, G \in \mathfrak{F}(X)$ with $G \leq \mathfrak{F}(\mathcal{H}(F))$.
- (LRH4) $\mathcal{H}(F) = \bigvee_{G \in \mathfrak{F}(F)} \mathcal{H}(G)$ for all $F \in \mathfrak{F}(X)$.

(2) Let (X, \mathcal{H}_X) and (Y, \mathcal{H}_Y) be (L, M) -fuzzy restricted hull spaces. A mapping $f : X \rightarrow Y$ is called an (L, M) -RHP mapping, if $\mathcal{H}_X(F)(x_\lambda) \leq \mathcal{H}_Y(f_L^\rightarrow(F))(f_L^\rightarrow(x_\lambda))$ for all $F \in \mathfrak{F}(X)$ and all $x_\lambda \in J(L^X)$. In particular, if $M = 2$, then $f : X \rightarrow Y$ is called an L -RHP mapping. That is, $f_L^\rightarrow(\mathcal{H}_X(F)) \leq \mathcal{H}_Y(f_L^\rightarrow(F))$ for all $F \in \mathfrak{F}(X)$.

Theorem 4.4. Let (X, \mathcal{C}) be an (L, M) -fuzzy convex space. Then the restriction of co on $\mathfrak{F}(X)$ is an (L, M) -fuzzy restricted hull operator, which is denoted by $\mathcal{H}_\mathcal{C}$.

Proof. Since $\mathcal{H}_\mathcal{C}(F) = co(F)$ for all $F \in \mathfrak{F}(X)$, we only need to check (LMRH3).

(LMRH3): Let $F, G \in \mathfrak{F}(X)$ and $x_\lambda \in J(L^X)$. If $x_\lambda \leq F$, then (LMRH3) is trivial. If $x_\lambda \leq G$, then $\mathcal{H}_\mathcal{C}(G)(x_\lambda) = \top$. This implies $\bigwedge_{y_\mu \in \beta^*(G)} \mathcal{H}_\mathcal{C}(F)(y_\mu) \leq \bigwedge_{\mu \in \beta^*(\lambda)} \mathcal{H}_\mathcal{C}(F)(x_\mu) = \bigwedge_{\mu \in \beta^*(\lambda)} co(F)(x_\mu) = \mathcal{H}_\mathcal{C}(F)(x_\lambda)$. Thus (LMRH3) holds.

Assume that $x_\lambda \not\leq F \vee G$. Then $\mathcal{H}_\mathcal{C}(F)(x_\lambda) = \bigwedge_{x_\lambda \not\leq A \geq F} \bigvee_{y_\mu \not\leq^* A} co(A)(y_\mu)$ by (LMCO4). Let $x_\lambda \not\leq A \geq F$. If $G \not\leq A$, then the set $\varphi = \{y_\mu \in \beta^*(G) : y_\mu \not\leq^* A\}$ is not empty and

$$\bigvee_{y_\mu \not\leq^* A} co(A)(y_\mu) \geq \bigvee_{y_\mu \in \varphi} co(A)(y_\mu) \geq \bigvee_{y_\mu \in \varphi} co(F)(y_\mu) \geq \bigwedge_{y_\mu \in \beta^*(G)} \mathcal{H}_\mathcal{C}(F)(y_\mu).$$

If $G \leq A$, then $\bigvee_{y_\mu \not\leq^* A} co(A)(y_\mu) \geq co(A)(x_\lambda) \geq co(G)(x_\lambda) = \mathcal{H}_\mathcal{C}(G)(x_\lambda)$.

Combining the above two inequalities, we conclude that (LMRH3) holds. \square

Theorem 4.5. Let (X, \mathcal{H}) be an (L, M) -fuzzy restricted hull space and $\mathcal{C}_\mathcal{H} : L^X \rightarrow M$ be defined as:

$$\forall A \in L^X, \quad \mathcal{C}_\mathcal{H}(A) = \bigwedge_{x_\lambda \not\leq^* A} \bigwedge_{F \in \mathfrak{F}(A)} (\mathcal{H}(F)(x_\lambda))'.$$

Then $\mathcal{C}_\mathcal{H}$ is an (L, M) -fuzzy convex structure with $\mathcal{H}_{\mathcal{C}_\mathcal{H}} = \mathcal{H}$.

Proof. (LMC1): $\mathcal{C}_\mathcal{H}(\perp) = \bigwedge_{x_\lambda \in J(L^X)} (\mathcal{H}(\perp)(x_\lambda))' = \top$ and $\mathcal{C}_\mathcal{H}(\top) = \bigwedge \emptyset = \top$.

(LMC2): Let $\{A_i\}_{i \in \Omega} \subseteq L^X$ and $A = \bigwedge_{i \in \Omega} A_i$. Then $\mathfrak{F}(A) \subseteq \bigcap_{i \in \Omega} \mathfrak{F}(A_i)$. If $x_\lambda \not\leq^* A$, then there exists $i_0 \in \Omega$ such that $x_\lambda \not\leq^* A_{i_0}$. Thus, if $F \in \mathfrak{F}(A)$, then

$$(\mathcal{H}(F)(x_\lambda))' \geq \bigwedge_{G \in \mathfrak{F}(A_{i_0})} (\mathcal{H}(G)(x_\lambda))' \geq \mathcal{C}_\mathcal{H}(A_{i_0}) \geq \bigwedge_{i \in \Omega} \mathcal{C}_\mathcal{H}(A_i).$$

Hence $\mathcal{C}_\mathcal{H}(A) = \bigwedge_{x_\lambda \not\leq^* A} \bigwedge_{F \in \mathfrak{F}(A)} (\mathcal{H}(F)(x_\lambda))' \geq \bigwedge_{i \in \Omega} \mathcal{C}_\mathcal{H}(A_i)$.

(LMC3*): Let $\{A_i\}_{i \in \Omega} \subseteq L^X$ be up-directed and $A = \bigvee_{i \in \Omega} A_i$. If $x_\lambda \not\leq^* A$, then $x_\lambda \not\leq^* A_i$ for each $i \in \Omega$. If $F \in \mathfrak{F}(A)$, then there exists $i_F \in \Omega$ such that $F \in \mathfrak{F}(A_{i_F})$. So

$$(\mathcal{H}(F)(x_\lambda))' \geq \bigwedge_{G \in \mathfrak{F}(A_{i_F})} (\mathcal{H}(G)(x_\lambda))' \geq \mathcal{C}_\mathcal{H}(A_{i_F}) \geq \bigwedge_{i \in \Omega} \mathcal{C}_\mathcal{H}(A_i).$$

Hence $\mathcal{C}_\mathcal{H}(A) = \bigwedge_{x_\lambda \not\leq^* A} \bigwedge_{F \in \mathfrak{F}(A)} (\mathcal{H}(F)(x_\lambda))' \geq \bigwedge_{i \in \Omega} \mathcal{C}_\mathcal{H}(A_i)$.

In order to prove that $\mathcal{H}_{\mathcal{C}_{\mathcal{H}}} = \mathcal{H}$, let $F \in \mathfrak{F}(X)$ and $x_\lambda \in J(L^X)$ with $x_\lambda \not\leq F$. Then

$$\mathcal{H}_{\mathcal{C}_{\mathcal{H}}}(F)(x_\lambda) = \text{coc}_{\mathcal{C}_{\mathcal{H}}}(F)(x_\lambda) = \bigwedge_{x_\lambda \not\leq B \geq F} (\mathcal{C}_{\mathcal{H}}(B))' = \bigwedge_{x_\lambda \not\leq B \geq F} \bigvee_{y_\mu \not\leq^* B} \bigvee_{G \in \mathfrak{F}(B)} \mathcal{H}(G)(y_\mu).$$

If $B \in L^X$ with $x_\lambda \not\leq B \geq F$, then there exists $\mu_B \in \beta^*(\lambda)$ such that $x_{\mu_B} \not\leq^* B$. Thus

$$\begin{aligned} \mathcal{H}_{\mathcal{C}_{\mathcal{H}}}(F)(x_\lambda) &= \text{coc}_{\mathcal{C}_{\mathcal{H}}}(F)(x_\lambda) = \bigwedge_{x_\lambda \not\leq B \geq F} (\mathcal{C}_{\mathcal{H}}(B))' = \bigwedge_{x_\lambda \not\leq B \geq F} \bigvee_{y_\mu \not\leq^* B} \bigvee_{G \in \mathfrak{F}(B)} \mathcal{H}(G)(y_\mu) \\ &\geq \bigwedge_{x_\lambda \not\leq B \geq F} \bigvee_{G \in \mathfrak{F}(B)} \mathcal{H}(G)(x_{\mu_B}) \geq \bigwedge_{\mu \in \beta^*(\lambda)} \bigvee_{G \in \mathfrak{F}(F)} \mathcal{H}(G)(x_\mu) = \mathcal{H}(F)(x_\lambda). \end{aligned}$$

Conversely, let $a \in \beta(\top)$ with $a \prec \mathcal{H}_{\mathcal{C}_{\mathcal{H}}}(F)(x_\lambda)$. We say $a \leq \mathcal{H}(F)(x_\lambda)$.

Assume that $x_\lambda \notin \mathcal{H}(F)_{[a]}$. If $x_\lambda \leq \bigvee \mathcal{H}(F)_{[a]} = B_0$, then $x_\mu \in \beta^*(B_0)$ for each $\mu \in \beta^*(\lambda)$. Thus $\mathcal{H}(F)(x_\mu) \geq a$ and $\mathcal{H}(F)(x_\lambda) = \bigwedge_{\mu \in \beta^*(\lambda)} \mathcal{H}(F)(x_\mu) \geq a$ by (LMRH0). Hence $x_\lambda \in \mathcal{H}(F)_{[a]}$ which is a contradiction. Thus $x_\lambda \not\leq B_0 \geq F$. By $a \prec \mathcal{H}_{\mathcal{C}_{\mathcal{H}}}(F)(x_\lambda)$, there exists $y_\mu \not\leq^* B_0$ such that $a \leq \bigvee_{G \in \mathfrak{F}(B_0)} \mathcal{H}(G)(y_\mu)$. Thus there exists $G_b \in \mathfrak{F}(B_0)$ such that $b \leq \mathcal{H}(G_b)(y_\mu)$ for each $b \prec a$. If $z_\zeta \prec G_b$, then there exists $z_\rho \in \beta^*(B_0)$ such that $z_\zeta \prec z_\rho$. Hence there further exists $z_\eta \in \mathcal{H}(F)_{[a]}$ such that $z_\rho \in \beta^*(z_\eta)$. So $\mathcal{H}(z_\zeta) \geq \mathcal{H}(z_\rho) \geq \mathcal{H}(z_\eta) \geq a \succ b$ by (LMRH0). Therefore $b \leq \mathcal{H}(G_b)(y_\mu) \wedge \bigwedge_{z_\zeta \in \beta^*(G_b)} \mathcal{H}(F)(z_\zeta) \leq \mathcal{H}(F)(y_\mu)$ by (LMRH3). However, by Proposition 2.9(5), $a \leq \mathcal{H}(F)(y_\mu)$ which contradicts $y_\mu \not\leq B_0$. Hence $a \leq \mathcal{H}(F)(x_\lambda)$. By Proposition 2.9(5) again, $\mathcal{H}_{\mathcal{C}_{\mathcal{H}}}(F)(x_\lambda) \leq \mathcal{H}(F)(x_\lambda)$. Therefore $\mathcal{H}_{\mathcal{C}_{\mathcal{H}}}(F)(x_\lambda) = \mathcal{H}(F)(x_\lambda)$ and so $\mathcal{H}_{\mathcal{C}_{\mathcal{H}}} = \mathcal{H}$. \square

Theorem 4.6. *If (X, \mathcal{C}) is an (L, M) -fuzzy convex space, then $\mathcal{C}_{\mathcal{H}_{\mathcal{C}}} = \mathcal{C}$.*

Proof. Let $A \in L^X$ and $x_\lambda \in J(L^X)$. In order to prove that $\mathcal{C}_{\mathcal{H}_{\mathcal{C}}}(A) \leq \mathcal{C}(A)$, we firstly prove that $\text{coc}_{\mathcal{C}_{\mathcal{H}_{\mathcal{C}}}}(A)(x_\lambda) \geq \text{coc}(A)(x_\lambda)$. Let $x_\lambda \not\leq^* A$. If $B \in L^X$ with $x_\lambda \not\leq B \geq A$, then there exists $\mu_B \in \beta^*(\lambda)$ such that $x_{\mu_B} \not\leq^* B$. By Theorem 4.5 and (LMDF), we have

$$\begin{aligned} \text{coc}_{\mathcal{C}_{\mathcal{H}_{\mathcal{C}}}}(A)(x_\lambda) &= \bigwedge_{x_\lambda \not\leq B \geq A} \bigvee_{y_\eta \not\leq^* B} \bigvee_{F \in \mathfrak{F}(B)} \mathcal{H}_{\mathcal{C}}(F)(y_\eta) \geq \bigwedge_{x_\lambda \not\leq B \geq A} \bigvee_{F \in \mathfrak{F}(A)} \text{coc}(F)(x_{\mu_B}) \\ &\geq \bigwedge_{\mu \in \beta^*(\lambda)} \bigvee_{F \in \mathfrak{F}(A)} \text{coc}(F)(x_\mu) = \text{coc}(A)(x_\lambda). \end{aligned}$$

Hence $\mathcal{C}_{\mathcal{H}_{\mathcal{C}}}(A) \leq \mathcal{C}(A)$. Conversely, we have

$$\mathcal{C}_{\mathcal{H}_{\mathcal{C}}}(A) = \bigwedge_{x_\lambda \not\leq^* A} \bigwedge_{F \in \mathfrak{F}(A)} (\text{coc}(F)(x_\lambda))' \geq \bigwedge_{x_\lambda \not\leq^* A} (\text{coc}(A))' = \mathcal{C}(A).$$

Thus $\mathcal{C}_{\mathcal{H}_{\mathcal{C}}}(A) = \mathcal{C}(A)$. Therefore $\mathcal{C}_{\mathcal{H}_{\mathcal{C}}} = \mathcal{C}$. \square

Theorem 4.7. *Let (X, \mathcal{C}_X) and (Y, \mathcal{C}_Y) be (L, M) -fuzzy convex spaces. If a mapping $f : (X, \mathcal{C}_X) \rightarrow (Y, \mathcal{C}_Y)$ is an (L, M) -CP mapping, then $f : (X, \mathcal{H}_{\mathcal{C}_X}) \rightarrow (Y, \mathcal{H}_{\mathcal{C}_Y})$ is an (L, M) -RHP mapping.*

Proof. Let $F \in \mathfrak{F}(X)$ and $x_\lambda \in J(L^X)$. Then $f_L^\rightarrow(F) \in \mathfrak{F}(Y)$, $f_L^\rightarrow(x_\lambda) \in J(Y)$ and

$$\mathcal{H}_{\mathcal{C}_X}(F)(x_\lambda) = \text{coc}_X(F)(x_\lambda) \leq \text{coc}_Y(f_L^\rightarrow(F))(f_L^\rightarrow(x_\lambda)) = \mathcal{H}_{\mathcal{C}_Y}(f_L^\rightarrow(F))(f_L^\rightarrow(x_\lambda)).$$

Thus f is an (L, M) -RHP mapping. \square

Theorem 4.8. *Let (X, \mathcal{H}_X) and (Y, \mathcal{H}_Y) be (L, M) -fuzzy restricted hull spaces. If $f : (X, \mathcal{H}_X) \rightarrow (Y, \mathcal{H}_Y)$ be an (L, M) -RHP mapping, then $f : (X, \mathcal{C}_{\mathcal{H}_X}) \rightarrow (Y, \mathcal{C}_{\mathcal{H}_Y})$ is an (L, M) -CP mapping.*

Proof. Let $A \in L^X$, $F \in \mathfrak{F}(A)$ and $x_\lambda \in J(L^X)$. Then $f_L^\rightarrow(F) \in \mathfrak{F}(f_L^\rightarrow(A))$ and $f_L^\rightarrow(x_\lambda) \in \mathfrak{F}(Y)$. Thus, by Theorem 4.5, we have

$$\begin{aligned} \text{coc}_{\mathcal{H}_X}(A)(x_\lambda) &= \bigwedge_{\mu \in \beta^*(\lambda)} \bigvee_{F \in \mathfrak{F}(A)} \text{coc}_{\mathcal{H}_X}(F)(x_\mu) = \bigwedge_{\mu \in \beta^*(\lambda)} \bigvee_{F \in \mathfrak{F}(A)} \mathcal{H}_{\mathcal{H}_X}(F)(x_\mu) \\ &= \bigwedge_{\mu \in \beta^*(\lambda)} \bigvee_{F \in \mathfrak{F}(A)} \mathcal{H}_X(F)(x_\mu) \leq \bigwedge_{\mu \in \beta^*(\lambda)} \bigvee_{F \in \mathfrak{F}(A)} \mathcal{H}_Y(f_L^\rightarrow(F))(f_L^\rightarrow(x_\mu)) \\ &\leq \bigwedge_{\mu \in \beta^*(\lambda)} \bigvee_{G \in \mathfrak{F}(f_L^\rightarrow(A))} \mathcal{H}_Y(G)(f_L^\rightarrow(x_\mu)) = \text{coc}_{\mathcal{H}_Y}(f_L^\rightarrow(A))(f_L^\rightarrow(x_\lambda)). \end{aligned}$$

Hence $f : (X, \mathcal{C}_{\mathcal{H}_X}) \rightarrow (Y, \mathcal{C}_{\mathcal{H}_Y})$ is an (L, M) -CP mapping. \square

Theorem 4.9. Let (X, \mathcal{D}) be an (L, M) -fuzzy closure space, and $\mathcal{C}_{\mathcal{D}} : L^X \rightarrow M$ be defined as:

$$\forall A \in L^X, \quad \mathcal{C}_{\mathcal{D}}(A) = \bigvee_{\substack{\text{dir} \\ j \in J}} \bigwedge_{B_j = A} \mathcal{D}(B_j).$$

Then $\mathcal{C}_{\mathcal{D}}$ is an (L, M) -fuzzy convex structure.

Proof. (LMC1) directly follows from the fact that $\mathcal{C}_{\mathcal{D}} \geq \mathcal{D}$.

(LMC2): Let $\{A_i\}_{i \in \Omega} \subseteq L^X$ and $a \in \beta(\top)$ with $a \prec \bigwedge_{i \in \Omega} \mathcal{C}_{\mathcal{D}}(A_i)$. Then $a \prec \mathcal{C}_{\mathcal{D}}(A_i)$ for each $i \in \Omega$. Thus, for each $i \in \Omega$, there exists an up-directed set $\{B_{i,j} : j \in J_i\} \subseteq L^X$ such that $A_i = \bigvee_{j \in J_i}^{\text{dir}} B_{i,j}$ and $\mathcal{D}(B_{i,j}) \geq a$ for all $j \in J_i$. Hence

$$\bigwedge_{i \in \Omega} A_i = \bigwedge_{i \in \Omega} \bigvee_{j \in J_i} B_{i,j} = \bigvee_{f \in \Pi_{i \in \Omega} J_i} \bigwedge_{i \in \Omega} B_{i,f(i)}.$$

Clearly, $\{B_{i,f(i)} : f \in \Pi_{i \in \Omega} J_i\}$ is up-directed and $\mathcal{D}(B_{i,f(i)}) \geq a$. Hence

$$\mathcal{C}_{\mathcal{D}}\left(\bigwedge_{i \in \Omega} A_i\right) = \mathcal{C}_{\mathcal{D}}\left(\bigvee_{f \in \Pi_{i \in \Omega} J_i} \bigwedge_{i \in \Omega} B_{i,f(i)}\right) \geq \bigwedge_{f \in \Pi_{i \in \Omega} J_i} \bigwedge_{i \in \Omega} \mathcal{D}(B_{i,f(i)}) \geq a.$$

Therefore, $\mathcal{C}_{\mathcal{D}}(\bigwedge_{i \in \Omega} A_i) \geq \bigwedge_{i \in \Omega} \mathcal{C}_{\mathcal{D}}(A_i)$ by Proposition 2.9(5).

(LMC3*): Let $\{A_i\}_{i \in \Omega} \subseteq L^X$ be up-directed, $A = \bigvee_{i \in \Omega} A_i$ and $a \in \beta(\top)$ with $a \prec \bigwedge_{i \in \Omega} \mathcal{C}_{\mathcal{D}}(A_i)$. Then $a \prec \mathcal{C}_{\mathcal{D}}(A_i)$ for each $i \in \Omega$. Thus there exists an up-directed set $\{B_{i,j} : j \in J_i\} \subseteq L^X$ such that $\bigvee_{j \in J_i}^{\text{dir}} B_{i,j} = A_i$ and $\mathcal{D}(B_{i,j}) \geq a$ for each $j \in J_i$.

Let $\mathcal{B}_* = \{B_{i,j} : i \in \Omega, j \in J_i\}$ and $\varphi_F = \{B \in \mathcal{B}_* : F \leq B\}$ for all $F \in \mathfrak{F}(A)$. Next, we check that φ_F is nonempty for all $F \in \mathfrak{F}(A)$. In fact, by Proposition 2.8(4), there exists $i_F \in \Omega$ such that $F \in \mathfrak{F}(A_{i_F}) = \mathfrak{F}(\bigvee_{j \in J_{i_F}}^{\text{dir}} B_{i_F,j})$. Thus there exists $B_{i_F,j_0} \in \mathcal{B}_*$ such that $F \in \mathfrak{F}(B_{i_F,j_0})$. So $B_{i_F,j_0} \in \varphi_F$. Now, since $F \leq B_{i_F,j_0} \leq A_{i_F} \leq A$ for all $F \in \mathfrak{F}(A)$, we have $F \leq \bigwedge \varphi_F \leq A$. Thus $A = \bigvee_{F \in \mathfrak{F}(A)} F \leq \bigvee_{F \in \mathfrak{F}(A)} \bigwedge \varphi_F \leq A$ and $A = \bigvee_{F \in \mathfrak{F}(A)} \bigwedge \varphi_F$.

If $F, G \in \mathfrak{F}(A)$, then $F \vee G \in \mathfrak{F}(A)$. Thus $\varphi_{F \vee G} \subseteq \varphi_F \cap \varphi_G$ and $\bigwedge \varphi_F, \bigwedge \varphi_G \leq \bigwedge \varphi_{F \vee G}$. Hence the set $\{\bigwedge \varphi_F : F \in \mathfrak{F}(A)\} \subseteq L^X$ is an up-directed set. It follows that

$$\mathcal{C}_{\mathcal{D}}(A) \geq \bigwedge_{F \in \mathfrak{F}(A)} \mathcal{D}\left(\bigwedge \varphi_F\right) \geq \bigwedge_{F \in \mathfrak{F}(A)} \bigwedge_{B \in \varphi_F} \mathcal{D}(B) \geq a.$$

Thus $\mathcal{C}_{\mathcal{D}}(A) \geq \bigwedge_{i \in \Omega} \mathcal{C}_{\mathcal{D}}(A_i)$ by Proposition 2.9(5). □

Corollary 4.10. An (L, M) -fuzzy closure space (X, \mathcal{D}) is an (L, M) -fuzzy convex space iff $\mathcal{C}_{\mathcal{D}} = \mathcal{D}$.

Remark 4.11. (1) The category, consisting of (L, M) -fuzzy restricted hull spaces as objects and (L, M) -RHP mappings as morphisms, is denoted by (L, M) -RHS. Its construct is denoted by $((L, M)$ -RHS, \mathbb{H}). In particular, when $M = \mathbf{2}$, we obtain a category L -RHS.

(2) The category, consisting of (L, M) -fuzzy domain finite closure operator spaces (refer to Remark 2.16 and Theorem 4.1(2)) as objects and (L, M) -closure operator preserving mappings as morphisms, is denoted by (L, M) -DFCOS. When $M = \mathbf{2}$, we obtain a category L -DFCOS (refer to Theorem 3.2(3) and Remark 2.14(1)). When $L = \mathbf{2}$, we obtain a category M -DFCOS.

Theorem 4.12. (L, M) -CS $\cong (L, M)$ -RHS $\cong (L, M)$ -DFCOS.

Proof. Let $\mathbb{F}_h : (L, M)$ -RHS $\rightarrow (L, M)$ -CS be defined as: $\mathbb{F}_h((X, \mathcal{H})) = (X, \mathcal{C}_{\mathcal{H}})$ and $\mathbb{F}_h(f) = f$ for all $(X, \mathcal{H}) \in O((L, M)$ -RHS) and $f \in \text{hom}_{LM\text{RHS}}((X, \mathcal{H}_X), (Y, \mathcal{H}_Y))$.

Conversely, let $\mathbb{G}_h : (L, M)$ -CS $\rightarrow (L, M)$ -RHS be defined as: $\mathbb{G}_h((X, \mathcal{C})) = (X, \mathcal{H}_{\mathcal{C}})$ and $\mathbb{G}_h(f) = f$ for all $(X, \mathcal{C}) \in O((L, M)$ -CS) and $f \in \text{hom}_{LM\text{CS}}((X, \mathcal{C}_X), (Y, \mathcal{C}_Y))$.

By Theorems 4.4–4.8, $\mathbb{G}_h \circ \mathbb{F}_h = \mathbb{I}_{LM\text{RHS}}$ and $\mathbb{F}_h \circ \mathbb{G}_h = \mathbb{I}_{LM\text{CS}}$. Thus (L, M) -RHS $\cong (L, M)$ -CS. Also, (L, M) -DFCOS $\cong (L, M)$ -CS by Theorems 2.2 and 4.1(2). □

Corollary 4.13. (1) L -CS $\cong L$ -RHS $\cong L$ -DFCOS.

(2) M -CS $\cong M$ -RHS $\cong M$ -DFCOS. In particular, RHS \cong CS.

Next, we discuss relations between L -CS and (L, M) -CS.

Let $(X, \mathcal{C}) \in \mathfrak{F}_{LMCS}(X)$ and $\&_{\mathcal{C}} = \{\bigwedge \varphi : \varphi \subseteq \bigcup_{a \in J(L)} \mathcal{C}_{[a]}\}$.

Clearly, we have $\perp, \top \in \&_{\mathcal{C}}$. If $\{A_i\}_{i \in \Omega} \subseteq \&_{\mathcal{C}}$, then there exists $\varphi_i \subseteq \bigcup_{a \in J(L)} \mathcal{C}_{[a]}$ such that $A_i = \bigwedge \varphi_i$ for each $i \in \Omega$. Thus $\varphi = \bigcup_{i \in \Omega} \varphi_i \subseteq \bigcup_{a \in J(L)} \mathcal{C}_{[a]}$ and $\bigwedge_{i \in \Omega} A_i = \bigwedge_{i \in \Omega} \bigwedge \varphi_i = \bigwedge \varphi \in \&_{\mathcal{C}}$. Hence $(X, \&_{\mathcal{C}})$ is an L -closure structure. By Theorem 3.3, $(X, \&_{\mathcal{C}})$ is a base of an L -convex space which is denoted by $(X, \&_{\mathcal{C}}^*)$.

An (L, M) -fuzzy convex structure \mathcal{C} on X is called an (L, M) -fuzzy crisp convex structure and (X, \mathcal{C}) is called an (L, M) -fuzzy crisp convex space, if there exists an L -convex structure \mathcal{D} such that $\mathcal{C} = \chi_{\mathcal{D}}$, where $\chi_{\mathcal{D}}$ is the characterization function of \mathcal{D} .

Theorem 4.14. $\text{hom}_{LMCS}((X, \mathcal{C}), (Y, \mathcal{D})) \subseteq \text{hom}_{LCS}((X, \&_{\mathcal{C}}^*), (Y, \&_{\mathcal{D}}^*))$.

Proof. Let $f \in \text{hom}_{LMCS}((X, \mathcal{C}), (Y, \mathcal{D}))$. If $B \in \&_{\mathcal{D}}^*$, then there exists an up-directed family $\{C_i\}_{i \in \Omega}^{dir} \subseteq \&_{\mathcal{D}}$ such that $B = \bigvee_{i \in \Omega}^{dir} C_i$. Further, there exists $\{D_{ij}\}_{j \in J_i} \subseteq \bigcup_{a \in J(L)} \mathcal{D}_{[a]}$ such that $C_i = \bigwedge_{j \in J_i} D_{ij}$ for each $i \in \Omega$. Thus $B = \bigvee_{i \in \Omega}^{dir} \bigwedge_{j \in J_i} D_{ij}$. Hence $f_L^{\leftarrow}(B) = f_L^{\leftarrow}(\bigvee_{i \in \Omega}^{dir} \bigwedge_{j \in J_i} D_{ij}) = \bigvee_{i \in \Omega}^{dir} \bigwedge_{j \in J_i} f_L^{\leftarrow}(D_{ij})$.

Since $D_{ij} \in \bigcup_{a \in J(L)} \mathcal{D}_{[a]}$ for each $i \in \Omega$ and each $j \in J_i$, there exists $a \in J(L)$ such that $\mathcal{D}(D_{ij}) \geq a$. Thus $\mathcal{C}(f_L^{\leftarrow}(D_{ij})) \geq \mathcal{D}(D_{ij}) \geq a$. Hence $f_L^{\leftarrow}(D_{ij}) \in \mathcal{C}_{[a]}$ and $f_L^{\leftarrow}(B) = \bigvee_{i \in \Omega}^{dir} \bigwedge_{j \in J_i} f_L^{\leftarrow}(D_{ij}) \in \&_{\mathcal{C}}^*$. Therefore $f : (X, \&_{\mathcal{C}}^*) \rightarrow (Y, \&_{\mathcal{D}}^*)$ is an L -CP mapping. \square

Theorem 4.15. $\text{hom}_{LCS}((X, \mathcal{C}), (Y, \mathcal{D})) = \text{hom}_{LMCS}((X, \chi_{\mathcal{C}}), (Y, \chi_{\mathcal{D}}))$ for all $(X, \mathcal{C}), (X, \mathcal{D}) \in \mathfrak{F}_{LCS}(X)$.

Define $\mathbb{E}_c : L\text{-CS} \rightarrow (L, M)\text{-CS}$ as: $\mathbb{E}_c((X, \mathcal{C})) = (X, \chi_{\mathcal{C}})$ and $\mathbb{E}(f) = f$ for all $(X, \mathcal{C}) \in O(L\text{-CS})$ and all $f \in \text{hom}_{LCS}((X, \mathcal{C}), (Y, \mathcal{D}))$.

Conversely, define $\mathbb{T} : (L, M)\text{-CS} \rightarrow L\text{-CS}$ as: $\mathbb{T}((X, \mathcal{C})) = (X, \&_{\mathcal{C}}^*)$ and $\mathbb{T}(f) = f$ for all $(X, \mathcal{C}) \in O((L, M)\text{-CS})$ and all $f \in \text{hom}_{LMCS}((X, \mathcal{C}), (Y, \mathcal{D}))$.

Theorem 4.16. $(\mathbb{E}_c, \mathbb{T})$ is a Galois's correspondence and \mathbb{T} is a left inverse of \mathbb{E}_c .

Proof. By Theorems 4.14 and 4.15, $\mathbb{E}_c \circ \mathbb{T} \ll \mathbb{I}_{LMCS}$ and $\mathbb{T} \circ \mathbb{E}_c = \mathbb{I}_{LCS}$. Therefore $(\mathbb{E}_c, \mathbb{T})$ is a Galois's correspondence and \mathbb{T} is a left inverse of \mathbb{E}_c . \square

Corollary 4.17. $L\text{-CS}$ is a coreflective subcategory of $(L, M)\text{-CS}$. In particular, CS is a coreflective subcategory of $M\text{-CS}$.

Now, we introduce and characterize (L, M) -fuzzy concave spaces.

Definition 4.18. A mapping $\mathcal{T} : L^X \rightarrow M$ is an (L, M) -fuzzy concave structure and the pair (X, \mathcal{T}) is called an (L, M) -fuzzy concave space, if

(LMCA1) $\mathcal{T}(\perp) = \mathcal{T}(\top) = \top$;

(LMCA2) $\mathcal{T}(\bigvee_{i \in \Omega} A_i) \geq \bigwedge_{i \in \Omega} \mathcal{T}(A_i)$ for each subset $\{A_i\}_{i \in \Omega} \subseteq L^X$;

(LMCA3) $\mathcal{T}(\bigwedge_{i \in \Omega} A_i) \geq \bigwedge_{i \in \Omega} \mathcal{T}(A_i)$ for each totally ordered subset $\{A_i\}_{i \in \Omega} \subseteq L^X$.

Let (X, \mathcal{T}_X) and (Y, \mathcal{T}_Y) be (L, M) -fuzzy concave structures. A mapping $f : X \rightarrow Y$ is called an (L, M) -fuzzy concave preserving mapping, if $\mathcal{T}_X(f_L^{\leftarrow}(A)) \geq \mathcal{T}_Y(A)$ for all $A \in L^Y$. The category, consisting of (L, M) -fuzzy concave spaces as objects and (L, M) -fuzzy concave preserving mappings as morphisms, is denoted by $(L, M)\text{-CAS}$. In particular, when $M = \mathbf{2}$, then $(L, M)\text{-CAS}$ is reduced to the category $L\text{-CAS}$, consisting of L -concave spaces as objects and L -concave preserving mappings as morphisms [19].

Theorem 4.19. $(L, M)\text{-CS} \cong (L, M)\text{-CAS}$.

5 The Category of (L, M) -fuzzy Stratified Convex Spaces

Definition 5.1. An (L, M) -fuzzy convex structure \mathcal{C} on X is called an (L, M) -fuzzy stratified convex structure and (X, \mathcal{C}) is called an (L, M) -fuzzy stratified convex space, if

(LMSC) $\mathcal{C}(\lambda) = \top$ for all $\lambda \in L$.

The finest (L, M) -convex structures is an (L, M) -fuzzy stratified convex structure. Also, a mapping $\mathcal{C} : L^X \rightarrow M$, defined as: $\mathcal{C}(\lambda) = \top$ for all $\lambda \in L$ and $\mathcal{C}(A) = \perp$ otherwise, is an (L, M) -fuzzy stratified convexity. The full subcategory of $(L, M)\text{-CS}$, consisting of (L, M) -fuzzy stratified convex spaces as objects and (L, M) -CP mappings as morphisms, is denoted by $(L, M)\text{-SCS}$. Its construct is denoted by $((L, M)\text{-SCS}, \mathcal{C}_s)$.

Theorem 5.2. Both $\mathfrak{F}_{LM\text{SCS}}(X)$ and $\mathfrak{F}_{LM\text{CS}}(X)$ are complete lattices.

Proof. Let $\{(X, \mathcal{C}_\lambda)\}_{\lambda \in \Lambda} \subseteq \mathfrak{F}_{LM\text{SCS}}(X)$. Define $\mathcal{C}_{\wedge_s}, \mathcal{C}_{\vee_s} : L^X \rightarrow M$ as:

$$\begin{aligned} \forall A \in L^X, \quad \mathcal{C}_{\wedge_s}(A) &= \bigwedge \{\mathcal{C}(A) : (X, \mathcal{C}) \in \mathfrak{F}_{LM\text{SCS}}(X), \forall \lambda \in \Lambda, \mathcal{C} \ll \mathcal{C}_\lambda\}; \\ \mathcal{C}_{\vee_s}(A) &= \bigwedge_{\lambda \in \Lambda} \mathcal{C}_\lambda(A). \end{aligned}$$

We have $(X, \mathcal{C}_{\wedge_s}), (X, \mathcal{C}_{\vee_s}) \in \mathfrak{F}_{LM\text{SCS}}(X)$ with $\mathcal{C}_{\wedge_s} \ll \mathcal{C}_\lambda \ll \mathcal{C}_{\vee_s}$ for all $\lambda \in \Lambda$. If $(X, \mathcal{D}_1), (X, \mathcal{D}_2) \in \mathfrak{F}_{LM\text{SCS}}(X)$ satisfies $\mathcal{D}_1 \ll \mathcal{C}_\lambda \ll \mathcal{D}_2$ for all $\lambda \in \Lambda$, then $\mathcal{D}_1 \ll \mathcal{C}_{\wedge_s}$ and $\mathcal{C}_{\vee_s} \ll \mathcal{D}_2$. Thus $\mathfrak{F}_{LM\text{SCS}}(X)$ is a complete lattice. Similarly, $\mathfrak{F}_{LM\text{CS}}(X)$ is a complete lattice. \square

Theorem 5.3. Both (L, M) -SCS and (L, M) -CS are topological categories.

Proof. Let $(X, (f_k, (X_k, \mathcal{C}_{X_k}))_{k \in K})$ be a \mathbb{C}_s -structured sink (i.e., $\{(X_k, \mathcal{C}_{X_k})\}_{k \in K}$ is a family of (L, M) -fuzzy stratified convex spaces and $f_k : X_k \rightarrow X$ is a mapping for each $k \in K$). We need to prove that there exists an (L, M) -fuzzy stratified convex space (X, \mathcal{C}_s) such that for each (L, M) -fuzzy stratified convex space (Y, \mathcal{C}_Y) and each mapping $g : X \rightarrow Y, g \in \text{hom}_{LM\text{SCS}}((X, \mathcal{C}_s), (Y, \mathcal{C}_Y))$ iff $g \circ f_k \in \text{hom}_{LM\text{SCS}}((X_k, \mathcal{C}_{X_k}), (Y, \mathcal{C}_Y))$ for all $k \in K$. To this end, define $\mathcal{C}_s : L^X \rightarrow M$ as:

$$\forall A \in L^X, \quad \mathcal{C}_s(A) = \bigwedge_{k \in K} \mathcal{C}_k(f_k^{\leftarrow}(A)).$$

We firstly check that \mathcal{C}_s is an (L, M) -fuzzy stratified convex space.

(LMC1). $\mathcal{C}_s(\perp_X) = \bigwedge_{k \in K} \mathcal{C}_k(\perp_{X_k}) = \top$ and $\mathcal{C}_s(\top_X) = \bigwedge_{k \in K} \mathcal{C}_k(\top_{X_k}) = \top$.

(LMC2). Let $\{A_i\}_{i \in \Omega} \subseteq L^X$ and $A = \bigwedge_{i \in \Omega} A_i$. Then we have

$$\mathcal{C}_s(A) = \bigwedge_{k \in K} \mathcal{C}_k\left(\bigwedge_{i \in \Omega} (f_k)_L^{\leftarrow}(A_i)\right) \geq \bigwedge_{k \in K} \bigwedge_{i \in \Omega} \mathcal{C}_k((f_k)_L^{\leftarrow}(A_i)) = \bigwedge_{i \in \Omega} \mathcal{C}_s(A_i).$$

(LMC3). If $\{A_i\}_{i \in \Omega} \subseteq L^X$ is totally ordered and $A = \bigvee_{i \in \Omega} A_i$, then

$$\mathcal{C}_s(A) = \bigwedge_{k \in K} \mathcal{C}_k\left(\bigvee_{i \in \Omega} (f_k)_L^{\leftarrow}(A_i)\right) \geq \bigwedge_{k \in K} \bigwedge_{i \in \Omega} \mathcal{C}_k((f_k)_L^{\leftarrow}(A_i)) = \bigwedge_{i \in \Omega} \mathcal{C}_s(A_i).$$

(LMSC). If $\lambda \in L$, then $(f_k)_L^{\leftarrow}(\lambda_X) = \lambda_{X_k}$ for all $k \in K$. So $\mathcal{C}_s(\lambda_X) = \bigwedge_{k \in K} \mathcal{C}_k((f_k)_L^{\leftarrow}(\lambda_X)) = \bigwedge_{k \in K} \mathcal{C}_k(\lambda_{X_k}) = \top$. Thus \mathcal{C}_s is stratified.

If $g \in \text{hom}_{LM\text{SCS}}((X, \mathcal{C}_s), (Y, \mathcal{C}_Y))$ and $B \in L^Y$, then

$$\mathcal{C}_Y(B) \leq \mathcal{C}_s(g_L^{\leftarrow}(B)) = \bigwedge_{k \in K} \mathcal{C}_k(((f_k)_L^{\leftarrow} \circ g_L^{\leftarrow})(B)) \leq \mathcal{C}_k(((f_k)_L^{\leftarrow} \circ g_L^{\leftarrow})(B)).$$

Conversely, if $g \circ f_k \in \text{hom}_{LM\text{SCS}}((X_k, \mathcal{C}_{X_k}), (Y, \mathcal{C}_Y))$ for each $k \in K$, then for each $B \in L^Y$,

$$\mathcal{C}_Y(B) \leq \bigwedge_{k \in K} \mathcal{C}_k(((f_k)_L^{\leftarrow} \circ g_L^{\leftarrow})(B)) = \bigwedge_{k \in K} \mathcal{C}_k((f_k)_L^{\leftarrow}(g_L^{\leftarrow}(B))) = \mathcal{C}_s(g_L^{\leftarrow}(B)).$$

Therefore (L, M) -SCS is topological. Similarly, (L, M) -CS is topological. \square

Corollary 5.4. Let $(X, \mathcal{C}_X) \in \mathfrak{F}_{LM\text{SCS}}(X)$ and $f : X \rightarrow Y$ be a mapping. Define $\mathcal{C}_X^* : L^Y \rightarrow M$ as: $\mathcal{C}_X^*(B) = \mathcal{C}_X(f_L^{\leftarrow}(B))$ for all $B \in L^Y$. Then $(X, \mathcal{C}_X^*) \in \mathfrak{F}_{LM\text{SCS}}(Y)$.

Theorem 5.5. Let $(X, \mathcal{C}) \in \mathfrak{F}_{LM\text{CS}}(X)$ and $\bar{\mathcal{C}} : L^X \rightarrow M$ be defined as:

$$\forall A \in L^X, \quad \bar{\mathcal{C}}(A) = \bigwedge \{\mathcal{D}(A) : (X, \mathcal{D}) \in \mathfrak{F}_{LM\text{SCS}}(X), \mathcal{D} \ll \mathcal{C}\}.$$

Then $(X, \bar{\mathcal{C}}) \in \mathfrak{F}_{LM\text{SCS}}(X)$ satisfying $\bar{\mathcal{C}} \ll \mathcal{C}$.

Corollary 5.6. An (L, M) -fuzzy convex space (X, \mathcal{C}) is stratified iff $\bar{\mathcal{C}} = \mathcal{C}$.

Theorem 5.7. $\text{hom}_{LM\text{CS}}((X, \mathcal{C}_X), (Y, \mathcal{C}_Y)) \subseteq \text{hom}_{LM\text{SCS}}((X, \bar{\mathcal{C}}_X), (Y, \bar{\mathcal{C}}_Y))$.

Proof. Let $f \in \text{hom}_{LMCS}((X, \mathcal{C}_X), (Y, \mathcal{C}_Y))$. By Corollary 5.4, we have $\overline{\mathcal{C}_X}^*(B) = \overline{\mathcal{C}_X}(f_L^-(B)) \geq \mathcal{C}_X(f_L^-(B)) \geq \mathcal{C}_Y(B)$ for all $B \in L^Y$. Thus $\mathcal{C}_Y \leq \overline{\mathcal{C}_X}^*$ and $(Y, \overline{\mathcal{C}_X}^*) \in \mathfrak{F}_{LMSCS}(Y)$. Hence $\overline{\mathcal{C}_Y}(B) \leq \overline{\mathcal{C}_X}^*(B) = \overline{\mathcal{C}_X}(f_L^-(B))$. Therefore $f \in \text{hom}_{LMSCS}((X, \overline{\mathcal{C}_X}), (Y, \overline{\mathcal{C}_Y}))$. \square

Remark 5.8. (1) Let $\mathbb{E}_s : (L, M)\text{-SCS} \rightarrow (L, M)\text{-CS}$ be defined as: $\mathbb{E}_s((X, \mathcal{C})) = (X, \mathcal{C})$ and $\mathbb{E}_s(f) = f$ for all $(X, \mathcal{C}) \in O((L, M)\text{-SCS})$ and all $f \in \text{hom}_{LMSCS}(\mathcal{C}_X, \mathcal{C}_Y)$. Since $(L, M)\text{-SCS}$ is a full subcategory of $(L, M)\text{-CS}$, \mathbb{E}_s is a full embedding satisfying $\mathbb{C}_s = \mathbb{C} \circ \mathbb{E}_s : (L, M)\text{-SCS} \rightarrow \text{Set}$. So $(L, M)\text{-SCS}$ is a concrete subcategory of $(L, M)\text{-CS}$.

(2) Let $\mathbb{S}_1 : (L, M)\text{-CS} \rightarrow (L, M)\text{-SCS}$ be defined as: $\mathbb{S}_1((X, \mathcal{C})) = (X, \overline{\mathcal{C}})$ and $\mathbb{S}_1(f) = f$ for all $(X, \mathcal{C}) \in O((L, M)\text{-CS})$ and all $f \in \text{hom}_{LMCS}(\mathcal{C}_X, \mathcal{C}_Y)$. By Theorems 5.5 and 5.7, $\mathbb{S}_1 : ((L, M)\text{-SCS}, \mathbb{C}_s) \rightarrow ((L, M)\text{-CS}, \mathbb{C})$ is a concrete functor with $\mathbb{C}_s = \mathbb{C} \circ \mathbb{S}_1$.

Theorem 5.9. $(\mathbb{E}_s, \mathbb{S}_1)$ is a Galois correspondence and \mathbb{S}_1 is a left inverse of \mathbb{E}_s .

Proof. $\mathbb{E}_s \circ \mathbb{S}_1 \ll \mathbb{I}_{LMCS}$ by Theorem 5.5, and $\mathbb{S}_1 \circ \mathbb{E}_s = \mathbb{I}_{LMSCS}$ by Corollary 5.6. \square

Corollary 5.10. $(L, M)\text{-SCS}$ is a coreflective subcategory of $(L, M)\text{-CS}$.

6 The Category of (L, M) -fuzzy Weakly Induced Convex Spaces

Definition 6.1. An (L, M) -fuzzy convex structure \mathcal{C} on X is called an (L, M) -fuzzy weakly induced convex structure and the pair (X, \mathcal{C}) is called an (L, M) -fuzzy weakly induced convex space, if

$$(LMWIC) \mathcal{C}(A) \leq \bigwedge_{a \in L} \mathcal{C}(\chi_{A|a}) \text{ for all } A \in L^X.$$

The finest (L, M) -fuzzy convex structure is an (L, M) -fuzzy weakly induced convex structure. Also, if $(X, \mathcal{C}) \in \mathfrak{F}_{MCS}$, then a mapping $\mathcal{C} : L^X \rightarrow M$, defined as: $\mathcal{C}(A) = \mathcal{C}(U)$ if $A = \chi_U$ for some $U \in 2^X$ and $\mathcal{C}(A) = \perp$ otherwise, is an (L, M) -fuzzy weakly induced convex structure.

The full subcategory of $(L, M)\text{-CS}$, consisting of (L, M) -fuzzy weakly induced convex spaces as objects and (L, M) -CP mappings as morphisms, is denoted by $(L, M)\text{-WICS}$. Its construct is denoted by $((L, M)\text{-WICS}, \mathbb{C}_w)$.

Theorem 6.2. $\mathfrak{F}_{LMWICS}(X)$ is a complete lattice.

Proof. Let $\{(X, \mathcal{C}_\lambda)\}_{\lambda \in \Lambda} \subseteq \mathfrak{F}_{LMWICS}(X)$. Define $\mathcal{C}_{\wedge_w}, \mathcal{C}_{\vee_w} : L^X \rightarrow M$ as:

$$\begin{aligned} \forall A \in L^X, \quad \mathcal{C}_{\wedge_w}(A) &= \bigwedge \{ \mathcal{C}(A) : (X, \mathcal{C}) \in \mathfrak{F}_{LMWICS}(X), \forall \lambda \in \Lambda, \mathcal{C} \ll \mathcal{C}_\lambda \} \\ \mathcal{C}_{\vee_w}(A) &= \bigwedge_{\lambda \in \Lambda} \mathcal{C}_\lambda(A). \end{aligned}$$

Then $(X, \mathcal{C}_{\wedge_w}), (X, \mathcal{C}_{\vee_w}) \in \mathfrak{F}_{LMWICS}(X)$ with $\mathcal{C}_{\wedge_w} \ll \mathcal{C}_\lambda \ll \mathcal{C}_{\vee_w}$ for each $\lambda \in \Lambda$. Further, let $(X, \mathcal{D}_1), (X, \mathcal{D}_2) \in \mathfrak{F}_{LMWICS}(X)$ with $\mathcal{D}_1 \ll \mathcal{C}_\lambda \ll \mathcal{D}_2$ for each $\lambda \in \Lambda$. Then $\mathcal{D}_1 \ll \mathcal{C}_{\wedge_w}$ and $\mathcal{C}_{\vee_w} \ll \mathcal{D}_2$. Therefore $\mathfrak{F}_{LMWICS}(X)$ is a complete lattice. \square

Theorem 6.3. $(L, M)\text{-WICS}$ is a topological category.

Proof. Let $(X, (f_k, (X_k, \mathcal{C}_k))_{k \in K})$ be a \mathbb{C}_w -structured sink, (that is, $\{(X_k, \mathcal{C}_k)\}_{k \in K}$ is a family of (L, M) -fuzzy weakly induced convex spaces and $f_k : X_k \rightarrow X$ is a mapping). We shall prove that there exists an (L, M) -fuzzy weakly induced convex space (X, \mathcal{C}_w) such that for each (L, M) -fuzzy weakly induced convex space (Y, \mathcal{C}_Y) and each mapping $g : X \rightarrow Y$, $g \in \text{hom}_{LMWICS}((X, \mathcal{C}_w), (Y, \mathcal{C}_Y))$ iff $g \circ f_k \in \text{hom}_{LMWICS}((X_k, \mathcal{C}_k), (Y, \mathcal{C}_Y))$ for all $k \in K$. To this end, define $\mathcal{C}_w : L^X \rightarrow M$ as:

$$\forall A \in L^X, \quad \mathcal{C}_w(A) = \bigwedge_{k \in K} \mathcal{C}_k((f_k)_L^-(A)).$$

It is easy to check that \mathcal{C}_w is an (L, M) -fuzzy convex structure. In addition, for each $A \in L^X$, $\bigwedge_{a \in L} \mathcal{C}_w(\chi_{A|a}) = \bigwedge_{a \in L} \bigwedge_{k \in K} \mathcal{C}_k(\chi_{(f_k)_L^-(A)|a}) \geq \bigwedge_{k \in K} \mathcal{C}_k((f_k)_L^-(A)) = \mathcal{C}_w(A)$. Thus \mathcal{C}_w is weakly induced. In addition, similar to Theorem 5.3, $g \in \text{hom}_{LMWICS}((X, \mathcal{C}_w), (Y, \mathcal{C}_Y))$ iff $g \circ f_k \in \text{hom}_{LMWICS}((X_k, \mathcal{C}_k), (Y, \mathcal{C}_Y))$ for all $k \in K$. \square

Corollary 6.4. If $(X, \mathcal{C}_X) \in \mathfrak{F}_{LMWICS}(X)$ and $f : X \rightarrow Y$ is a mapping, then $(Y, \mathcal{C}_Y^*) \in \mathfrak{F}_{LMWICS}(Y)$, where $\mathcal{C}_Y^* : L^Y \rightarrow M : \mathcal{C}_Y^*(B) = \mathcal{C}_X(f_L^-(B))$ for all $B \in L^Y$.

Theorem 6.5. Let $(X, \mathcal{C}) \in \mathfrak{F}_{LMCS}(X)$. Define $\underline{\mathcal{C}} : L^X \rightarrow M$ as:

$$\forall A \in L^X, \quad \underline{\mathcal{C}}(A) = \bigwedge \{ \mathcal{D}(A) : (X, \mathcal{D}) \in \mathfrak{F}_{LMWIC}(X), \mathcal{D} \ll \mathcal{C} \}.$$

Then $\underline{\mathcal{C}}$ is an (L, M) -fuzzy weakly induced convex structure satisfying $\underline{\mathcal{C}} \ll \mathcal{C}$.

Corollary 6.6. An (L, M) -fuzzy convex space (X, \mathcal{C}) is weakly induced iff $\underline{\mathcal{C}} = \mathcal{C}$.

Theorem 6.7. $\text{hom}_{LMCS}((X, \mathcal{C}_X), (Y, \mathcal{C}_Y)) \subseteq \text{hom}_{LMWICS}((X, \underline{\mathcal{C}}_X), (Y, \underline{\mathcal{C}}_Y))$.

Proof. We have $\mathcal{C}_X \leq \underline{\mathcal{C}}_X$ by Theorem 6.5. If $f \in \text{hom}_{LMCS}((X, \mathcal{C}_X), (Y, \mathcal{C}_Y))$, then $\underline{\mathcal{C}}_X(f_L^+(B)) \geq \mathcal{C}_X(f_L^+(B)) \geq \mathcal{C}_Y(B)$ for all $B \in L^Y$. Thus $\underline{\mathcal{C}}_Y(B) \leq \mathcal{C}_Y^*(B) = \mathcal{C}_X(f_L^+(B)) \leq \underline{\mathcal{C}}_X(f_L^+(B))$ by Corollary 6.4. So $f \in \text{hom}_{LMWICS}((X, \underline{\mathcal{C}}_X), (Y, \underline{\mathcal{C}}_Y))$. \square

Remark 6.8. (1) Let $\mathbb{E}_w : (L, M)\text{-WICS} \rightarrow (L, M)\text{-CS}$ be defined as: $\mathbb{E}_w((X, \mathcal{C})) = (X, \mathcal{C})$ and $\mathbb{E}_w(f) = f$ for all $(X, \mathcal{C}) \in O((L, M)\text{-WICS})$ and all $f \in \text{hom}_{LMWICS}((X, \mathcal{C}_X), (Y, \mathcal{C}_Y))$. Since $(L, M)\text{-WICS}$ is a full subcategory of $(L, M)\text{-CS}$, \mathbb{E}_w is a full embedding and $\mathbb{C}_w = \mathbb{C} \circ \mathbb{E}_w$ is the forgetful functor of $(L, M)\text{-WICS}$. In this sense, $(L, M)\text{-WICS}$ can be naturally regarded as a concrete subcategory of $(L, M)\text{-CS}$.

(2) Let $\mathbb{W}_1 : (L, M)\text{-CS} \rightarrow (L, M)\text{-WICS}$ be defined as: $\mathbb{W}_1((X, \mathcal{C})) = (X, \underline{\mathcal{C}})$ and $\mathbb{W}_1(f) = f$ for all $(X, \mathcal{C}) \in O((L, M)\text{-CS})$ and $f \in \text{hom}_{CS}((X, \mathcal{C}_X), (Y, \mathcal{C}_Y))$. By Theorems 6.5 and 6.7, $\mathbb{W}_1 : ((L, M)\text{-WICS}, \mathbb{C}_w) \rightarrow ((L, M)\text{-CS}, \mathbb{C})$ is a concrete functor with $\mathbb{C}_w = \mathbb{C} \circ \mathbb{W}_1$.

Theorem 6.9. $(\mathbb{E}_w, \mathbb{W}_1)$ is a Galois correspondence, where \mathbb{W}_1 is a left inverse of \mathbb{E}_w .

Proof. We have $\mathbb{E}_w \circ \mathbb{W}_1 \ll \mathbb{I}_{LMCS}$ by Theorem 6.5 and $\mathbb{W}_1 \circ \mathbb{E}_w = \mathbb{I}_{LMWICS}$ by Corollary 6.6. Thus $(\mathbb{E}_w, \mathbb{W}_1)$ is a Galois correspondence and \mathbb{W}_1 is a left inverse of \mathbb{E}_w . \square

Corollary 6.10. $(L, M)\text{-WICS}$ is concretely coreflective in $(L, M)\text{-CS}$.

Theorem 6.11. (1) $L\text{-SCS}$ is coreflective in $(L, M)\text{-SCS}$.

(2) $L\text{-WICS}$ is coreflective in $(L, M)\text{-WICS}$.

7 The Category of (L, M) -fuzzy Induced Convex Spaces

In this section, L is a β -lattice, that is, $\beta(a \wedge b) = \beta(a) \cap \beta(b)$ for all $a, b \in L$.

Definition 7.1. An (L, M) -fuzzy convex structure \mathcal{C} on X is called an (L, M) -fuzzy induced convex structure and the pair (X, \mathcal{C}) is called an (L, M) -fuzzy induced convex space, if $\mathcal{C}(A) = \bigwedge_{a \in L} \mathcal{C}(\chi_{A[a]})$ for all $A \in L^X$.

Clearly, an (L, M) -fuzzy induced convex space is both weakly induced and stratified.

Let $(X, \mathcal{C}) \in \mathfrak{F}_{LMCS}(X)$. Define $\varphi_{\mathcal{C}} : 2^X \rightarrow M$ as: $\varphi_{\mathcal{C}}(U) = \bigvee_{a \in L} \bigvee_{A[a]=U} \mathcal{C}(A)$ for all $U \subseteq 2^X$. The M -fuzzifying convex structure with $\varphi_{\mathcal{C}}$ as its subbase is denoted by $\iota(\mathcal{C})$.

Theorem 7.2. Let $(X, \mathcal{C}) \in \mathfrak{F}_{LMCS}(X)$. Define $[\mathcal{C}] : 2^X \rightarrow M$ as: $[\mathcal{C}](U) = \mathcal{C}(\chi_U)$ for all $U \in 2^X$. Then $[\mathcal{C}]$ is an M -fuzzifying convex structure on X .

Theorem 7.3. [32] Let $(X, \mathcal{C}) \in \mathfrak{F}_{MCS}(X)$, and $\mathcal{C}_{\mathcal{C}} : L^X \rightarrow M$ be defined as:

$$\forall A \in L^X, \quad \mathcal{C}_{\mathcal{C}}(A) = \bigwedge_{a \in L} \mathcal{C}(A[a]).$$

Then $(X, \mathcal{C}_{\mathcal{C}})$ is an (L, M) -fuzzy convex space induced by \mathcal{C} .

Theorem 7.4. Let $(X, \mathcal{C}) \in \mathfrak{F}_{LMCS}(X)$ and $(X, \mathcal{C}) \in \mathfrak{F}_{MCS}(X)$. Then

(1) $[\mathcal{C}] \leq \varphi_{\mathcal{C}} \leq \iota(\mathcal{C})$;

(2) \mathcal{C} is weakly induced iff $[\mathcal{C}] = \iota(\mathcal{C})$;

(3) $\iota(\mathcal{C}_{\mathcal{C}}) = [\mathcal{C}] = \mathcal{C}$ and $\mathcal{C} \leq \mathcal{C}_{\iota(\mathcal{C})}$;

(4) \mathcal{C} is induced by an M -fuzzifying convex structure iff $\mathcal{C}_{\iota(\mathcal{C})} = \mathcal{C}$ iff $[\mathcal{C}] = \mathcal{C}$ iff \mathcal{C} is an (L, M) -fuzzy induced convex structure.

Proof. (1) is direct. Also, (4) directly follows from (3).

(2): If \mathcal{C} is weakly induced, then $\varphi_{\mathcal{C}}(U) \leq \bigvee_{a \in L} \bigvee_{A_{[a]}=U} \bigwedge_{b \in L} \mathcal{C}(\chi_{A_{[b]}}) \leq [\mathcal{C}](U)$ for all $U \in 2^X$. So $\varphi_{\mathcal{C}}(U) = [\mathcal{C}](U)$ by (1), and $\iota(\mathcal{C}) = [\mathcal{C}]$. Conversely, if $\iota(\mathcal{C}) = [\mathcal{C}]$, then $\bigwedge_{a \in L} \mathcal{C}(\chi_{A_{[a]}}) = \bigwedge_{a \in L} [\mathcal{C}](A_{[a]}) = \bigwedge_{a \in L} \iota(\mathcal{C})(A_{[a]}) \geq \mathcal{C}(A)$. Thus \mathcal{C} is weakly induced.

(3): $\varphi_{\mathcal{C}_c}(U) = \bigvee_{a \in L} \bigvee_{A_{[a]}=U} \bigwedge_{b \in L} \mathcal{C}(A_{[b]}) \leq \bigvee_{a \in L} \bigvee_{A_{[a]}=U} \mathcal{C}(U) = \mathcal{C}(U)$. Conversely, $\varphi_{\mathcal{C}_c}(U) = \bigvee_{a \in L} \bigvee_{A_{[a]}=U} \mathcal{C}_c(A) \geq \mathcal{C}_c(\chi_U) = \mathcal{C}(U)$. Thus $\varphi_{\mathcal{C}_c}(U) = \mathcal{C}(U)$, which shows $\varphi_{\mathcal{C}_c} = \mathcal{C}$. Hence $\iota(\mathcal{C}_c) = \mathcal{C}$. Also, $[\mathcal{C}_c](U) = \bigwedge_{a \in L} \mathcal{C}((\chi_U)_{[a]}) = \mathcal{C}(U)$. Thus $[\mathcal{C}_c] = \mathcal{C}$. Finally, $\mathcal{C}_{\iota(\mathcal{C})}(A) = \bigwedge_{a \in L} \iota(\mathcal{C})(A_{[a]}) \geq \bigwedge_{a \in L} \varphi_{\mathcal{C}}(A_{[a]}) \geq \mathcal{C}(A)$. \square

Theorem 7.5. (1) $\text{hom}_{M\text{CS}}((X, \mathcal{C}_X), (Y, \mathcal{C}_Y)) \subseteq \text{hom}_{LM\text{ICS}}((X, \mathcal{C}_X), (Y, \mathcal{C}_Y))$.

(2) $\text{hom}_{LM\text{CS}}((X, \mathcal{C}_X), (Y, \mathcal{C}_Y)) \subseteq \text{hom}_{M\text{CS}}((X, \iota(\mathcal{C}_X)), (Y, \iota(\mathcal{C}_Y)))$.

Remark 7.6. (1) The full subcategory of (L, M) -**CS**, consisting of (L, M) -fuzzy induced convex spaces as objects and (L, M) -CP mappings as morphisms, is denoted by (L, M) -**ICS**. Its construct is denoted by $((L, M)$ -**ICS**, \mathbb{C}_i); the full subcategory of (L, M) -**CS**, consisting of (L, M) -fuzzy convex spaces induced by M -fuzzifying convex spaces as objects and (L, M) -CP mappings as morphisms, is denoted by (L, M) -**CGCS**.

(2) Define $\mathbb{E}_i^1 : (L, M)$ -**ICS** \rightarrow (L, M) -**SCS** as: $\mathbb{E}_i^1((X, \mathcal{C})) = (X, \mathcal{C})$ and $\mathbb{E}_i^1(f) = f$ for all $(X, \mathcal{C}) \in O((L, M)$ -**ICS**) and all $f \in \text{hom}_{LM\text{ICS}}((X, \mathcal{C}_X), (Y, \mathcal{C}_Y))$. Since (L, M) -**ICS** is a full subcategory of (L, M) -**SCS**, \mathbb{E}_i^1 is a full embedding with $\mathbb{C}_i = \mathbb{C}_s \circ \mathbb{E}_i^1$. Thus (L, M) -**ICS** is a concrete subcategory of (L, M) -**SCS**.

(3) Define $\mathbb{E}_i^2 : (L, M)$ -**ICS** \rightarrow (L, M) -**WICS** as: $\mathbb{E}_i^2((X, \mathcal{C})) = (X, \mathcal{C})$ and $\mathbb{E}_i^2(f) = f$ for $(X, \mathcal{C}) \in O((L, M)$ -**ICS**) and all $f \in \text{hom}_{LM\text{ICS}}((X, \mathcal{C}_X), (Y, \mathcal{C}_Y))$. Since \mathbb{E}_i^2 is a full embedding with $\mathbb{C}_i = \mathbb{C}_w \circ \mathbb{E}_i^2$, (L, M) -**ICS** is a concrete subcategory of (L, M) -**WICS**.

(4) Define $\mathbb{S}_2 : (L, M)$ -**SCS** \rightarrow (L, M) -**ICS** as: $\mathbb{S}_2((X, \mathcal{C})) = (X, \mathcal{C}_{\iota(\mathcal{C})})$ and $\mathbb{S}_2(f) = f$ for all $(X, \mathcal{C}) \in O((L, M)$ -**SCS**) and all $f \in \text{hom}_{LM\text{SCS}}((X, \mathcal{C}_X), (Y, \mathcal{C}_Y))$.

(5) Define $\mathbb{W}_2 : (L, M)$ -**WICS** \rightarrow (L, M) -**ICS** as: $\mathbb{W}_2((X, \mathcal{C})) = (X, \mathcal{C}_{[\mathcal{C}]})$ and $\mathbb{W}_2(f) = f$ for $(X, \mathcal{C}) \in O((L, M)$ -**WICS**) and $f \in \text{hom}_{LM\text{WICS}}((X, \mathcal{C}_X), (Y, \mathcal{C}_Y))$.

Theorem 7.7. M -**CS** $\cong (L, M)$ -**ICS** $\cong (L, M)$ -**CGCS**.

Proof. Define $\mathbb{F}_3 : M$ -**CS** $\rightarrow (L, M)$ -**ICS** as: $\mathbb{F}_3((X, \mathcal{C})) = (X, \mathcal{C}_c)$ and $\mathbb{F}_3(f) = f$ for all $(X, \mathcal{C}) \in O(M$ -**CS**) and all $f \in \text{hom}_{M\text{CS}}((X, \mathcal{C}_X), (Y, \mathcal{C}_Y))$.

Conversely, define $\mathbb{G}_3 : (L, M)$ -**ICS** $\rightarrow M$ -**CS** as: $\mathbb{G}_3((X, \mathcal{C})) = (X, \iota(\mathcal{C}))$ and $\mathbb{G}_3(f) = f$ for all $(X, \mathcal{C}) \in O((L, M)$ -**ICS**) and all $f \in \text{hom}_{LM\text{ICS}}((X, \mathcal{C}_X), (Y, \mathcal{C}_Y))$. By Theorems 7.3-7.5, $\mathbb{F}_3, \mathbb{G}_3$ are concrete functors satisfying $\mathbb{F}_3 \circ \mathbb{G}_3 = \mathbb{I}_{LM\text{ICS}}$ and $\mathbb{G}_3 \circ \mathbb{F}_3 = \mathbb{I}_{M\text{CS}}$. Thus (L, M) -**ICS** and M -**CS** are isomorphic.

Define $\mathbb{F}_4 : (L, M)$ -**CGCS** $\rightarrow (L, M)$ -**ICS** as: $\mathbb{F}_4((X, \mathcal{C})) = (X, \mathcal{C})$ and $\mathbb{F}_4(f) = f$ for all $(X, \mathcal{C}) \in O((L, M)$ -**CGCS**) and all $f \in \text{hom}_{LM\text{CGCS}}((X, \mathcal{C}_X), (Y, \mathcal{C}_Y))$.

Conversely, define $\mathbb{G}_4 : (L, M)$ -**ICS** $\rightarrow (L, M)$ -**CGCS** as: $\mathbb{G}_4((X, \mathcal{C})) = (X, \mathcal{C}_{[\mathcal{C}]})$ and $\mathbb{G}_4(f) = f$ for all $(X, \mathcal{C}) \in O((L, M)$ -**ICS**) and $f \in \text{hom}_{LM\text{ICS}}((X, \mathcal{C}_X), (Y, \mathcal{C}_Y))$. By Theorem 7.4(4), $\mathbb{F}_4 \circ \mathbb{G}_4 = \mathbb{I}_{LM\text{ICS}}$ and $\mathbb{G}_4 \circ \mathbb{F}_4 = \mathbb{I}_{LM\text{CGCS}}$. Hence \mathbb{F}_4 is isomorphic. \square

Theorem 7.8. (1) $(\mathbb{E}_i^1, \mathbb{S}_2)$ is a Galois correspondence and \mathbb{S}_2 is a left inverse of \mathbb{E}_i^1 .

(2) $(\mathbb{E}_i^2, \mathbb{W}_2)$ is a Galois correspondence and \mathbb{W}_2 is a left inverse of \mathbb{E}_i^2 .

Corollary 7.9. (1) $(\mathbb{E}_i^1 \circ \mathbb{F}_3, \mathbb{G}_3 \circ \mathbb{S}_2)$ is a Galois correspondence and $\mathbb{G}_3 \circ \mathbb{S}_2$ is a left inverse of $\mathbb{E}_i^1 \circ \mathbb{F}_3$. So M -**CS** can be embedded in (L, M) -**SCS** as a coreflective subcategory.

(2) $(\mathbb{E}_i^2 \circ \mathbb{F}_3, \mathbb{G}_3 \circ \mathbb{W}_2)$ is a Galois correspondence and $\mathbb{G}_3 \circ \mathbb{W}_2$ is a left inverse of $\mathbb{E}_i^2 \circ \mathbb{F}_3$. So M -**CS** can be embedded in (L, M) -**WICS** as a coreflective subcategory.

8 Conclusion

Fuzzy convex spaces has been studied in various aspects [16, 20, 21, 24, 25, 30, 31, 32, 36, 37, 38, 39, 41, 42, 40]. Apparently, studies on M -fuzzifying convex spaces are in a deeper and broader level than that on (L, M) -fuzzy convex spaces. This because that domain finiteness of convex spaces has been introduced successfully in M -fuzzifying convex spaces [31]. Yet, it hasn't been studied in either L -convex convex spaces or (L, M) -fuzzy convex spaces. We define and characterize domain finiteness of both L -convex spaces and (L, M) -fuzzy convex spaces. We find that domain finiteness of (L, M) -fuzzy convex spaces perfectly matches that of L -convex spaces (resp. M -fuzzifying convex spaces) when $M = \mathbf{2}$ (resp. $L = \mathbf{2}$).

Research on fuzzy convex spaces in a view point of category is also meaningful. Pang and Shi have studied relations among the category of L -convex spaces and its subcategories [18]. Since the notion of (L, M) -fuzzy convex spaces has

been introduced [32], many new categories related to (L, M) -fuzzy convex spaces emerged. Thus relations among the category and subcategories of (L, M) -fuzzy convex spaces need to be discussed. In this paper, we introduce the category and several subcategories of (L, M) -fuzzy convex spaces. In a view of category aspect, we study their relations which can be showed by the diagram and tables in the following page where $L(\beta)$ stands for L if L is a β -lattice. Clearly, it contains the diagram described in [18]. Also, we conclude from Theorems 5.2, 5.3, 6.2 and 6.3 that all categories in the diagram are topological constructs whose fibres are complete lattices.

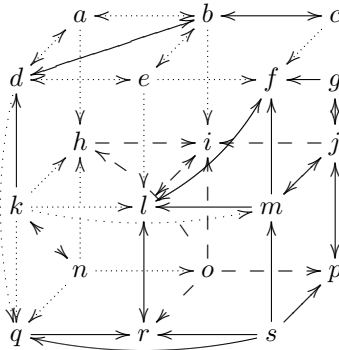


Figure 1: Relations

Symbol	Meaning
\longrightarrow	<i>coreflective</i> \longrightarrow
$\cdots\longrightarrow$	<i>coreflective</i> $\xrightarrow{L(\beta)}$
\longleftrightarrow	\cong \longleftrightarrow
$- -$	$- -$

Table 1: Symbols

No.	Category	No.	Category
<i>a</i>	(L, M) -CGCS	<i>j</i>	L -DFCOS
<i>b</i>	M -RHS	<i>k</i>	CS
<i>c</i>	M -DFCOS	<i>l</i>	(L, M) -CS
<i>d</i>	M -CS	<i>m</i>	L -CS
<i>e</i>	(L, M) -ICS	<i>n</i>	RHS
<i>f</i>	(L, M) -CAS	<i>o</i>	L -SCS
<i>g</i>	L -CAS	<i>p</i>	L -RHS
<i>h</i>	(L, M) -SCS	<i>q</i>	(L, M) -WICS
<i>i</i>	(L, M) -RHS	<i>r</i>	(L, M) -DFCOS
<i>s</i>	L -WICS		

Table 2: Categories

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