FUZZY SETS FROM A META-SYSTEM-THEORETIC POINT OF VIEW

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ABSTRACT. In this paper we present, for the first time, complete proofs of the facts that have already been announced in [A. Daneshgar, A. Hashemi, *A general model for I/O system theory*, Proc. AIMC31, 2000, 292–299]. Our approach here is to focus on the aspects related to fuzzy set theory and we leave other connections to mathematical disciplines such as quantum groups, categorical logic, homological algebra and measure theory to appear elsewhere. The main contribution is to introduce a general framework based on enriched category theory that covers the theory of translation invariant systems as a special case, as well as the construction of the Haar fraction on a locally compact group.

1. **The General Idea and Some Basic Concepts**

The general I/O system theory deals with modeling (i.e identifying) a system as a black box with given *input/output* characteristics. In this approach the space of inputs and the outputs is traditionally modeled as a function space with suitable topological and algebraic properties, and the system itself is usually modeled as a special type of an operator, mapping the elements of the input space to the elements of the output space. The functions in the input/output spaces are usually called *signals* and the maps are called *operators* (or *systems*), where the algebraic and topological structure of the function space as well as the invariance constraints are strong enough to guarantee a *reconstruction property.* Strictly speaking, if the outputs for a certain specified set of signals are known, this reconstruction means that one should be able to compute the output, for any input signal.
To be more specific, let us assume that a subspace Φ of real functions is chosen to be a model for the space of signals and (by abuse of language) assume that it is rich enough to be a Banach space containing distributions (for more details and generalizations see [16, 22, 36]). Also we assume that the class of systems under consideration is the space of linear and shift invariant (LSI) operators such as $T : \Phi \longrightarrow \Phi$ that satisfy

$$T(af + bg) = aT(f) + bT(g), \quad T(f(x - x_0)) = T(f)(x - x_0).$$

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Then it is well known that, under certain conditions, such an operator can be represented as a convolution integral [32, 36], i.e.

$$T(f)(x) = \int_{\mathbf{R}} f(x-y)h(y)\ dy \stackrel{\text{def}}{=} Conv(f,h)(x), \tag{1}$$

where $h \stackrel{\text{def}}{=} T(\delta)$ is the impulse response of the system and the input is the Dirac impulse δ (more general formulations can be traced in the theory of harmonic analysis on locally compact groups). This *reconstruction property* is strong enough to characterize any LSI system as an integral operator that is completely determined by a *fixed* function as a very special output of the system. Therefore, for any such system, one can *identify* the system just by knowing about one special output (i.e. the impulse response $h(x)$).

It is instructive to note that in the above-mentioned setup, since the structure of the input and output spaces are considered to be that of a topological vector space in which addition and scalar multiplication operations play a central role, the elements of these spaces are treated as functions. However, it is conceivable to think of weaker frameworks in which one uses only addition and possibly a weaker substitute for multiplication (say comparison). Needless to say, by this change of settings one loses linearity. But if one can still prove a reconstruction property, then one may be successful enough to identify a broader class of systems that also contains some nonlinear ones. On the other hand, it is quite natural to expect a weaker form of reconstruction since in the new setup one has weakened the operational properties of the signal space.

The next example considers such a framework in which we focus mainly on signals as objects that can be compared or added together, and this is a good justification for treating them as *generalized sets* rather than *functions*. In the sequel, it will become more clear that this sort of approach can also be expressed as an extension property from the Boolean algebra of sets to a Heyting algebra (justified by the Strong Reconstruction Theorem below).

Morphological mathematics and *translation invariant* (TI) systems have been used in signal processing [23, 24, 25], since Matheron studied morphological operators and random closed sets [26, 27, 29] and Tukey used median filters for noise cancelation [37] (also see [33, 34, 35]). Considering recent developments in ordered structures and category theory, the theory of TI systems has got the distinction of being one of the new areas of research where there is a nice coalition of mathematical structures such as ordered groups and semigroups, topologies and locals as well as fuzzy sets and categories [10, 14, 15, 17, 23, 24, 25, 38].

A general model for TI systems, based on residuated semigroups, is introduced in [11], and a reconstruction theorem is proved to show the consistency of the framework. Also, duality, thresholding and representation theorems are considered in this model [12, 13, 14] as important concepts in the theory.

We briefly review this reconstruction theorem [8, 9, 14, 25, 26]. Let G be an Abelian group with binary operation $+$ and let 0 be the identity element. We consider the function space $\Phi = \{A : G \longrightarrow \Omega \cup \{-\infty\}\}$, where Ω is a complete lattice ordered group, say $(\Omega, \leq, *, \div, 0)$. We define the extension of group operations to $\Omega \cup \{-\infty, +\infty\}$, for the virtual universal bounds $-\infty$ and $+\infty$ as follows:

- $\div(-\infty) = +\infty, \quad \div(+\infty) = -\infty,$
- $(-\infty) * (+\infty) = (+\infty) * (-\infty) = (+\infty) * (+\infty) = (+\infty),$
- $(-\infty) * (-\infty) = (-\infty),$
- $(+\infty) * p = p * (+\infty) = +\infty, \ (-\infty) * p = p * (-\infty) = -\infty \quad \forall\, p \in \Omega.$

Hereafter, Φ will be called the space of *LG-fuzzy* sets, and the subspace

$$\{\ A : G \longrightarrow \{0, -\infty\}\}$$

the space of *crisp* sets. The complement of an ordered pair $(g,p)'$ is defined to be $(-g, \div p)$. The translated version of an LG-fuzzy set $A = \{(x, A(x)) \mid x \ \in \ G\ \}$ by an ordered pair (g,p) is defined to be the LG-fuzzy set

$$A[g,p] = \{(x+g, A(x) * p) \mid x \in G\}.$$

The *reflection* of an LG-fuzzy set A is defined as $A^s = \{(-x, A(x)) \mid x \in G\}$, while its *complement* is defined to be the set $A^c = \{(x, \div A(x)) \mid x \in G\}$. Accordingly, *Minkowski addition* and *subtraction* of LG-fuzzy sets A and B are defined as follows,

$$A \oplus B = \sup_g\ A[g, B(g)], \quad A \ominus B = \inf_g\ A[g, \div B(g)].$$

Also, the most important morphological operators *Dilation* and *Erosion* (morphological convolution) of A and B are defined to be

$$Di(A,B) = A \oplus B^s, \quad Er(A,B) = A \ominus B^s.$$

Now, consider the crisp set

$$P_0 = \begin{cases} 0 & x = 0 \\ -\infty & \text{otherwise.} \end{cases}$$

Then one can easily check that $(\Phi, \oplus)$ is a residuated semigroup with identity P_0, involution A^s and residuation Er [3]. Moreover, the Galois connection $(\oplus, Er)$ gives rise to a closure operator $C(A,B) = Er(A \oplus B, B)$ and an opening operator $O(A,B) = Er(A,B) \oplus B$ as its dual.

An *isotone* operator is an order preserving map on Φ. In other words $T : \Phi \longrightarrow \Phi$ is isotone if

$$\forall\ A, B \in \Phi \quad A \leq B \ \Rightarrow\ T(A) \leq T(B).$$

Moreover, an operator $T : \Phi \longrightarrow \Phi$ is *translation invariant* (TI) if

$$\forall\ g \in G,\ \forall\ p \in \Omega \cup \{-\infty\} \quad T(A[g,p]) = T(A)[g,p].$$

The *Kernel* of an operator T is defined to be the set

$$Ker(T) \stackrel{\text{def}}{=} \{A \in \Phi \mid 0 \leq T(A)(0)\}.$$

In [11] it is proved that, under certain conditions, one has the Strong Reconstruction Theorem for any isotone TI-operator T, i.e.

$$T(A) = \sup_{D \in Ker(T)} Er(A,D) = \sup_{D \in B(T)} Er(A,D), \tag{2}$$

where $B(T)$, the set of minimal elements of $Ker(T)$, is called the *base* of the system and usually contains more than one element. This is quite natural since, intuitively, linearity in LSI systems is weakened to "DC gain one" in TI systems (i.e. translation invariance on the valuation domain).
In what follows we discuss two simple examples in the discrete case to illustrate some basic facts.

Example 1.1. In this example we consider the discrete system

$$T(f)(n) \stackrel{\text{def}}{=} \frac{1}{3}(f(n) + f(n-1) + f(n-2))$$

as a smoother (low-pass filter) on discrete signals $f : \mathbf{Z} \longrightarrow \mathbf{R}$. Note that the design of this filter is strongly dependent on the property of being able to *divide by* 3.
The first important observation is that T is an LSI system and its impulse response h is the following:

$$h(n) \stackrel{\text{def}}{=} \begin{cases} \frac{1}{3} & n = 0, 1, 2 \\ 0 & n \neq 0, 1, 2. \end{cases}$$

Therefore, T can be expressed as the convolution

$$T(f)(n) = \sum_{m=0}^{\infty} f(n-m)h(m) = \frac{1}{3}(f(n) + f(n-1) + f(n-2)).$$

An important observation is that in this setting, 0 as the identity of addition, plays the very important role of being the benign element.
On the other hand, one may note that T is also an isotone TI system with the basis consisting of all maps $h_{r,s}$, $(r, s \in \mathbf{R})$ defined as follows:

$$h_{r,s}(n) \stackrel{\text{def}}{=} \begin{cases} -r-s & n = -2 \\ s & n = -1 \\ r & n = 0 \\ -\infty & otherwise. \end{cases}$$

Hence, by the Strong Reconstruction Theorem (Equation 2) we have:

$$\begin{aligned} T(f)(n) &= \sup_{r,s \in \mathbf{R}} Er(f, h_{r,s})(n) \\ &= \sup_{r,s \in \mathbf{R}} \inf_{m \in \mathbf{Z}} (f(n+m) - h_{r,s}(m)) \\ &= \sup_{r,s \in \mathbf{R}} \inf \, (f(n) - r, f(n-1) - s, f(n-2) + r + s). \end{aligned}$$

It is interesting to note that, in this case, since we are mainly using the comparison as the basic operator, $-\infty$ is used as the benign value (also see the definition of *crisp* sets above). Moreover, it is interesting to see how division is simulated by an infinite number of comparisons (i.e. sup as an ordered limit).

The above example shows that an LSI system may have a simple representation when multiplication and division are allowed, while it may not even have a finite representation as a TI system. On the other hand, it may still be feasible to look for similar nonlinear systems that have simple representations as TI systems. The next example illustrates such a case and may be compared with Example 1.1.

Example 1.2. In this example we consider the three point *median filter* which is a nonlinear TI system quite similar to the three point *mean filter* of Example 1.1, and have a very concise representation in terms of comparison operators.
The median filter on three points is defined as

$$T(f)(n) \stackrel{\text{def}}{=} \text{Median}(f(n), f(n-1), f(n-2)),$$

which is clearly an isotone TI operator with a basis consisting of three maps $h_{r,s}$ for $r, s \in \{-2, -1, 0\}$ defined as

$$h_{r,s}(n) \stackrel{\text{def}}{=} \begin{cases} 0 & n = r, s \\ -\infty & otherwise. \end{cases}$$

Therefore, by the Strong Reconstruction Theorem (Equation 2) we have

$$\begin{aligned} T(f)(n) &= \sup_{r,s \in \{-2,-1,0\}} Er(f, h_{r,s})(n) \\ &= \sup \ \ (\inf(f(n), f(n-1)), \inf(f(n-1), f(n-2)), \inf(f(n-2), f(n))). \end{aligned}$$

It is interesting to note that the final expression is a *fuzzy extension* of the sum of products form of the Boolean Median expression

$$x_1x_2 + x_2x_3 + x_3x_1,$$

in which multiplication (Boolean $\wedge$) is replaced by (inf) and summation (Boolean $\vee$) is replaced by (sup). From this point of view the space of all isotone TI systems is the fuzzy extension of the space of isotone Boolean expressions (i.e. Boolean expressions that only contain positive Boolean variables and the $\wedge$ and $\vee$ operators (for more on this see [13, 30, 38])).

Note that both Equations 1 and 2 express the system as a limit of convolutions (in Equation 1 the limit is over a one element set!). In the rest of this paper our main objective is to show that the convolution operator can be expressed as a Hom functor in an enriched setup, and that the reconstruction is closely related to the enriched Yoneda lemma. In Section 2.1 we prove this completely for the case of TI systems. We introduce an approach to the theory of LSI operators in Section 2.2, where we show that the Haar fraction can also be expressed in the enriched setup. We postpone the rest of the discussion to Section 3.

2. The Main Meta-theory

In this section we generalize the basic setup discussed in the last section to a categorical framework. For all basic concepts and definitions that do not appear in the sequel we refer to the celebrated books of F. Borceux and G. M. Kelly [4, 19]. In Section 2.1 we introduce a general meta-model that covers the theory of TI systems as a special case. In Section 2.2 we introduce a step toward a meta-model that might be developed to a meta-theory of integration, where as a piece of evidence we show that this model covers the construction of the Haar fraction on a locally

compact group.
In what follows we assume that $\mathcal{V} = (\mathcal{V}, ., \div, I, a, l, r, c)$ is a symmetric closed monoidal category enriched over itself, with identity I, and natural isomorphisms a, l and r for associativity, left inverse and right inverse, respectively, for which $A.- \dashv - \div A$ for any $A \in \mathcal{V}$ and the base category $\mathcal{V}_0$ is complete and cocomplete. We also assume that $\mathsf{V} = [I, -]_{\mathcal{V}_0} : \mathcal{V}_0 \longrightarrow \mathsf{Set}$ is the base functor and we adapt the multiplicative notation in $\mathcal{V}$. Note that under these assumptions $\mathcal{V}$ is also complete and cocomplete as a $\mathcal{V}$–category, I is a dense generator for $\mathcal{V}$ (in the enriched sense) and consequently $\mathcal{V}$ is locally presentable as a $\mathcal{V}$–category since $- \div I$ is a right adjoint and preserves all colimits. Also, note that by the cocompleteness of $\mathcal{V}_0$, ${}^{(-)}I$ is the left adjoint of the base functor.
On the other hand, let $\mathcal{D}$ be a class of diagram–schemes and assume that $\mathcal{V}_f \subseteq \mathcal{V}_0$ is a small full subcategory such that $\mathcal{V}_0$ is a free $\mathcal{D}$–cocompletion of $\mathcal{V}_f$ in the sense that (see Theorem 5.35 of [19]),

- The totality of all $\mathcal{D}$–colimits constitute a density presentation for the inclusion $\mathsf{i} : \mathcal{V}_f \hookrightarrow \mathcal{V}_0$.
- For any $A \in \mathcal{V}_f$, the hom–functor $[A, -]$ preserves all $\mathcal{D}$–colimits.

Under these conditions i is full, faithful and dense (in the ordinary sense), and for any $\mathcal{D}$–cocomplete ordinary category $\mathcal{C}$, the left Kan extension

$$\mathrm{Lan}_{\mathsf{i}} : [\mathcal{V}_f, \mathcal{C}] \longrightarrow \mathcal{D}\text{–}\mathrm{Cocts}[\mathcal{V}_0, \mathcal{C}]$$

is an equivalence. Also, for each $V \in \mathrm{obj}(\mathcal{V}_0)$, the identity map

$$1 : [\mathsf{i}-, V] \longrightarrow [\mathsf{i}-, V]$$

as unit, exhibits V as the canonical colimit $[\mathsf{i}-, V] * \mathsf{i}$ and this colimit is preserved by any $[A, -]$ with $A \in \mathcal{V}_f$. Throughout the paper we assume that $\mathcal{D}$ contains all diagram–schemes of the canonical cones over $\mathcal{V}_f$, since, by changing $\mathcal{D}$ to the class of all diagram–schemes of the canonical cones, one obtains the same free cocompletion. Moreover, we consider a (small) commutative group $(G, +, -)$, with identity 0 and we focus on the product space $\Phi = \mathcal{V}_0^G$ equipped with the pointwise categorical structure of $\mathcal{V}_0$. It is easy to see that if we consider Φ as a category of functors between the discrete category G and $\mathcal{V}_0$, Φ inherits the completeness properties of $\mathcal{V}_0$ and has a dense $\mathcal{D}$–presentable full subcategory Φ_f. In this regard, we use the following definitions and lemmas.

Definition 2.1. Let $\Delta_W \in \Phi$ be the constant functor with value W. A *point* $P^{g,V} \in \Phi$ with value $V \in \mathrm{obj}(\mathcal{V}_0)$ at coordinate $g \in G$ is defined (up to isomorphism) as

$$P^{g,V}(x) \stackrel{\mathrm{def}}{=} \begin{cases} V & x = g \\ -\infty & x \neq g. \end{cases}$$

Also, we define the map $\mathsf{M}^{g,\mathsf{f}} : P^{g,V} \longrightarrow P^{g,W}$ for $\mathsf{f} : V \longrightarrow W$ (up to isomorphism) as

$$\mathsf{M}^{g,\mathsf{f}}_{(x)} \stackrel{\mathrm{def}}{=} \begin{cases} \mathsf{f} & x = g \\ 1_{-\infty} & x \neq g. \end{cases}$$

Lemma 2.2. *For any* $A \in \mathrm{obj}(\Phi)$ *and* $\mathsf{f} \in \mathrm{map}(\Phi)$ *we have,*

a) $A = \coprod_{g\in G} P^{g, A_{(g)}}$.

b) $\coprod_{g\in G} A_{(g)} \simeq \prod_{W} W^{[A,\Delta_W]_\Phi}$.

c) $\mathsf{f} = (\mathsf{f}_g)_{g\in G} = \coprod_{g\in G} \mathsf{M}^{g,\mathsf{f}_g}$.

Proof. (a) and (c) are clear. For (b) assume that $(Q, (\mathsf{m}_g))$ is a representation of the coproduct $Q \overset{\text{def}}{=} \coprod_{g\in G} A_{(g)}$ and let $P \overset{\text{def}}{=} \prod_{W} W^{[A,\Delta_W]_\Phi}$.
Therefore, $\mathsf{m} = (\mathsf{m}_g) \in [A, \Delta_Q]_\Phi$ and by definition of P, we have a map $\mathsf{p} : P \longrightarrow Q$. On the other hand, by the definition of P, for each W in the base of the product, we have a morphism from $A_{(g)}$ to W, and consequently, by the universal properties of P and Q we have a map $\mathsf{q} : Q \longrightarrow P$. This shows that $\mathsf{p} = \mathsf{q}^{-1}$ and $Q \simeq P$. □

Lemma 2.3. *The mapping* $\coprod : \Phi \longrightarrow \mathcal{V}_0$ *that maps* $A \in \text{obj}(\Phi)$ *to* $\coprod_{g\in G} A_{(g)}$ *and* $\mathsf{f} = (\mathsf{f}_g)_{g\in G} : V \longrightarrow W$ *to the unique map* $\coprod_{g\in G} \mathsf{f}_g : \coprod_{g\in G} V_{(g)} \longrightarrow \coprod_{g\in G} W_{(g)}$ *is a functor.*

Proof. Clear by definitions. □

2.1. **The Pointwise Constructions.** As was mentioned earlier, here we consider the following maps for a fixed $D \in \Phi$ which, in their most general form, can be described as,

$$\mathsf{T}_D : \text{obj}(\Phi) \longrightarrow \text{obj}(\Phi), \quad \mathsf{T}_D(A)_{(z)} \overset{\text{def}}{=} \prod_{|J|<\infty} (\bigoplus_{j\in J} W_j)^{[A_{(x)}, \oplus_j (\Delta_{W_j} \div D_{(x^{-1}z)})]_\Phi},$$

$$\mathsf{H}_D : \text{obj}(\Phi) \longrightarrow \text{obj}(\Phi), \quad \mathsf{H}_D(B)_{(z)} \overset{\text{def}}{=} \coprod_{|J|<\infty}^{[\oplus_j (\Delta_{V_j} \cdot D_{(z^{-1}x)}), B_{(x)}]_\Phi} (\bigoplus_{j\in J} V_j),$$

where $z \in G$ is a fixed coordinate and $\oplus$ is a suitable associative bifunctor. To be more precise, note that $\mathsf{T}_D(A)_{(z)}$ is the product of all finite sums $\oplus_j W_j$ such that $W_j \in \text{obj}(\mathcal{V}_0)$ for all j and there exists morphisms $\mathsf{f}_x : A_{(x)} \longrightarrow \oplus_j (\Delta_{W_j} \div D_{(x^{-1}z)})$ for all $x \in G$.
The map H_D can be interpreted similarly and we use this notation throughout the paper. The following theorem answers the most basic question.

Theorem 2.4. *For any* $D \in \Phi$, *both* T_D *and* H_D *can be naturally extended to define functors on* Φ.

Proof. To begin, we define T_D on Hom–sets. Hence, let $A, B \in \Phi$ and $\mathsf{f} : A \longrightarrow B$ be a morphism. Then define,

$$\mathsf{T}_D(\mathsf{f})_{(z)} : \prod_{|J|<\infty} (\bigoplus_{j\in J} W_j)^{[A_{(x)}, \oplus_j (\Delta_{W_j} \div D_{(x^{-1}z)})]_\Phi} \longrightarrow \prod_{|J|<\infty} (\bigoplus_{j\in J} W_j)^{[B_{(x)}, \oplus_j (\Delta_{W_j} \div D_{(x^{-1}z)})]_\Phi}.$$

But, since for each $x \in G$ we have a map $\mathsf{f}_x : A_{(x)} \longrightarrow B_{(x)}$, the diagram of $\mathsf{T}_D(B)$ is a subdiagram of $\mathsf{T}_D(A)$. Therefore, we have a unique map

$$\mathsf{T}_D(\mathsf{f}) : \mathsf{T}_D(A) \longrightarrow \mathsf{T}_D(B).$$

Now, it is easy to check that T_D is a functor on Φ. For H_D one may follow the same procedure. $\square$

Proposition 2.5. *If* $\oplus$ *is preserved by* $(\div)$ (*resp.* $(.)$) *then*

$$\mathsf{T}_D(A)_{(z)} = \coprod_{x\in G} A_{(x)}.D_{(x^{-1}z)} \stackrel{\text{def}}{=} \dot{\mathsf{T}}_D(A)_{(z)}$$

$$(\textit{resp.}\ \ \mathsf{H}_D(A)_{(z)} = \prod_{x\in G} B_{(x)} \div D_{(z^{-1}x)} \stackrel{\text{def}}{=} \dot{\mathsf{H}}_D(A)_{(z)}).$$

Proof. By considering the adjunction $V.- \dashv - \div V$ in $\mathcal{V}$ and Lemma 2.2(b) we have,

$$\mathsf{T}_D(A)_{(z)} = \prod_W W^{[A_{(x)},\Delta_W \div D_{(x^{-1}z)}]_\Phi} = \prod_W W^{[A_{(x)}.D_{(x^{-1}z)},\Delta_W]_\Phi} \simeq \coprod_{x\in G} A_{(x)}.D_{(x^{-1}z)}.$$

The same result can be proved for $\dot{\mathsf{H}}_D$ dually. $\square$

The following two theorems show the importance of the case emphasized in the last proposition when the bifunctor $\oplus$ is omitted.

Theorem 2.6. *For any* $D \in \Phi$ *we have* $\dot{\mathsf{T}}_D \dashv \dot{\mathsf{H}}_D$.

Proof. We should show that for any $A, B \in \text{obj}(\Phi)$ there is a natural isomorphism

$$d : [\dot{\mathsf{T}}_D(A), B] \simeq [A, \dot{\mathsf{H}}_D(B)],$$

which may be reformulated as

$$d : \prod_{z\in G} [\mathsf{T}_D(A)_{(z)}, B_{(z)}] \simeq \prod_{z\in G} [A_{(z)}, \mathsf{H}_D(B)_{(z)}].$$

But, considering the definitions of $\dot{\mathsf{T}}_D$ and $\dot{\mathsf{H}}_D$, this leads us to show the following isomorphism,

$$d : \prod_{z\in G}\prod_{x\in G} [A_{(x)}.D_{(x^{-1}z)}, B_{(z)}] \simeq \prod_{z\in G}\prod_{x\in G} [A_{(z)}, B_{(x)} \div D_{(z^{-1}x)}].$$

To show the isomorphism, let x, z be two arbitrary elements of G. Then, by adjunction we have

$$e_{(x,z)} : [A_{(x)}.D_{(x^{-1}z)}, B_{(z)}] \simeq [A_{(x)}, B_{(z)} \div D_{(x^{-1}z)}].$$

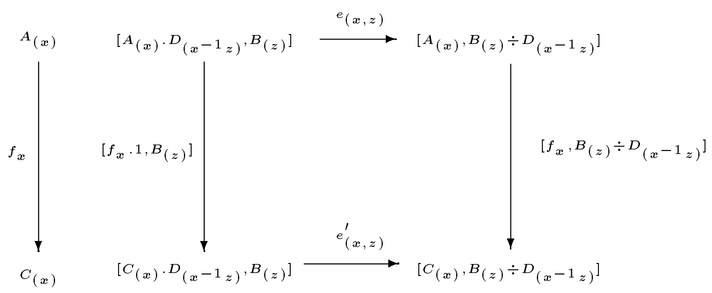

Figure 1. See the proof of Theorem 2.6.

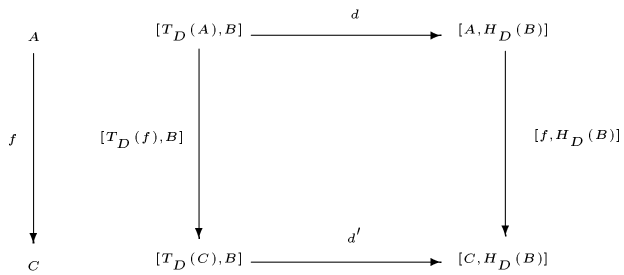

FIGURE 2. See the proof of Theorem 2.6.

However, this shows that there exists a unique map from the left product to the right one. Now, the isomorphism is established if we consider the same map from the right hand side to the left hand one and the universal property of the product. For the naturality of d, consider $A, C \in \mathrm{obj}(\Phi)$ and $\mathrm{map}(\Phi) \ni \mathsf{f} : A \longrightarrow C$, as well as $x, z \in G$. By the fact that $e_{(x,z)}$ is a natural isomorphism, we have the commutative diagram of Figure 1. Then, it is easy to see that the diagram of Figure 2 is also commutative by the universal property of product. Therefore, d is natural with respect to A, and it is easy to check that it is also natural with respect to B in the same way. □

Theorem 2.7. *There exist natural isomorphisms* $\tilde{a}, \tilde{l}$, $\tilde{r}$ *and* $\tilde{c}$ *such that for any* $D \in \mathrm{obj}(\Phi)$, $\dot{\Phi} = (\Phi, \dot{\mathsf{T}}_D, \dot{\mathsf{H}}_D, P^{0,I}, \tilde{a}, \tilde{l}, \tilde{r}, \tilde{c})$ *is a symmetric closed monoidal category.*

Proof. In Theorem 2.6 we proved that $\dot{\mathsf{T}}_D \dashv \dot{\mathsf{H}}_D$. Therefore, in what follows, we prove that $\dot{\Phi}$ is a symmetric monoidal category and we adopt the notation

$$A \otimes B \stackrel{\mathrm{def}}{=} \dot{\mathsf{T}}_A(B).$$

First, we address ourselves to the definition of the natural isomorphisms. In this regard, for $\tilde{a}$ and any $A, B, C \in \mathrm{obj}(\Phi)$, we may define

$$\tilde{a}_{ABC} : \dot{\mathsf{T}}_A(\dot{\mathsf{T}}_B(C)) \longrightarrow \dot{\mathsf{T}}_{\dot{\mathsf{T}}_A(B)}(C),$$

which implies that for any $z \in G$ one should have,

$$(\tilde{a}_{ABC})_z : \coprod_{x \in G} \coprod_{y \in G} A_{(x)}.(B_{(y)}.C_{(y^{-1}x^{-1}z)}) \longrightarrow \coprod_{x \in G} \coprod_{y \in G} (A_{(y)}.B_{(y^{-1}x)}).C_{(x^{-1}z)}.$$

However, for each element in the base of the left diagram, there exists a map to an element in the base of the right diagram, which is the component of the natural isomorphism a corresponding to the objects $A_{(x)}, B_{(y)}$ and $C_{(y^{-1}x^{-1}z)}$. Therefore, by the universal property of coproduct, one obtains a unique map which is what we consider as $(\tilde{a}_{ABC})_z$. Also, $\tilde{a}$ is a natural isomorphism, since for any $z \in G$,

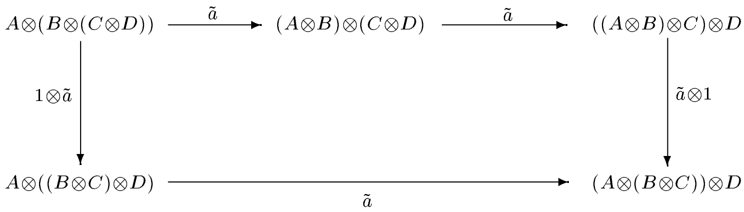

FIGURE 3. Associativity of $\tilde{a}$ where $A \otimes B \overset{\text{def}}{=} \dot{\mathsf{T}}_{A}(B)$ (See Theorem 2.7).

$(\tilde{a}_{ABC})_z$ is a natural isomorphism for any $A, B, C \in \text{obj}(\Phi)$. Now, we note that the associativity of $\tilde{a}$ is expressed as the commutativity of the diagram of Figure 3. The commutativity of this diagram has been already checked in [28] where the relationships between the monoidal structures of Φ and cohomology groups of G are studied. In order to define $\tilde{l}_A$ for each $A \in \text{obj}(\Phi)$ as,

$$\tilde{l}_A : \dot{\mathsf{T}}_{P^{0,I}}(A) \longrightarrow A,$$

we have to define for each $z \in G$ a map,

$$(\tilde{l}_A)_z : \coprod_{x \in G} P^{0,I}_{(x)}.A_{(x^{-1}z)} \longrightarrow A_{(z)}.$$

Again, as in the case of $\tilde{a}$, for each element of the left diagram, it is enough to find a map to an element of the right diagram. Note that, if $x = 0$, then $(l_A)_z$ is exactly what we are looking for. On the other hand, if $x \neq 0$, then by the following adjunction

$$[(-\infty).A_{(z)}, (-\infty)] \simeq [(-\infty), A_{(z)} \div (-\infty)]$$

we obtain the map

$$(-\infty).A_{(x^{-1}z)} \longrightarrow (-\infty) \longrightarrow A_{(z)}.$$

Thus, from the universal property of coproduct, we have a unique map which can be defined as $(\tilde{l}_A)_z$. It is again clear that $\tilde{l}$ is a natural isomorphism, since $(\tilde{l}_A)_z$ is a natural isomorphism for each $z \in G$.
To define $\tilde{r}$ (which is similar to $\tilde{l}$) we should define

$$(\tilde{r}_A)_z : \coprod_{x \in G} A_{(x)}.P^{0,I}_{(x^{-1}z)} \longrightarrow A_{(z)}$$

properly, for each $z \in G$. Again, in order to obtain the necessary maps between the diagrams, we note that in the case of $x = 0$, $(r_A)_z$ itself is the desired map, and if $x \neq 0$ then, since $\mathcal{V}$ is symmetric and monoidal, we have the following map,

$$A_{(x)}.(-\infty) \longrightarrow (-\infty).A_{(x)} \longrightarrow (-\infty) \longrightarrow A_{(z)}.$$

Thus, from the universal property of coproduct, we obtain a unique natural map which can be considered as $(\tilde{r}_A)_z$. This defines $\tilde{r}$ as a natural isomorphism.
To show that $\dot{\Phi}$ is a monoidal category, we should verify the commutativity of the

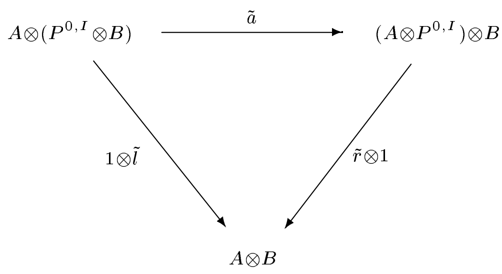

FIGURE 4. See the proof of Theorem 2.7.

diagram depicted in Figure 4. To do this, we define

$$P_{x,y,z} \stackrel{\text{def}}{=} \begin{cases} -\infty & y = 0 \\ A_{(x)}.B_{(x^{-1}z)} & y \neq 0, \end{cases}$$

which shows that $(A \otimes B)_{(z)} = \coprod_{y\in G} \coprod_{x\in G} P_{x,y,z}$. This reduces the commutativity of the diagram in Figure 4 to the commutativity of the diagram in Figure 5, which is clear since $\mathcal{V}$ is a monoidal category.

Now, we prove that $\dot{\Phi}$ is symmetric with respect to a properly defined natural isomorphism

$$\tilde{c}_{A,B} : \dot{\mathsf{T}}_A(B) \longrightarrow \dot{\mathsf{T}}_B(A)$$

that, as before, is naturally defined by considering the following maps,

$$c_{B_{(x)},A_{(x^{-1}z)}} : B_{(x)}.A_{(x^{-1}z)} \longrightarrow A_{(x^{-1}z)}.B_{(x)}.$$

Next, we must verify the commutativity of diagrams in Figure 6 which is reduced, by definition, to the commutativity of the same diagrams for each $x, y, z \in G$ in $\mathcal{V}$. However, this is clear since $\mathcal{V}$ is symmetric as in the case of $\tilde{l}$ and $\tilde{r}$. □

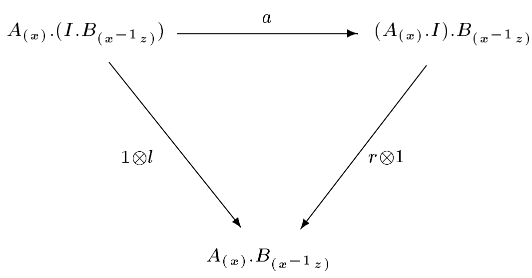

FIGURE 5. See the proof of Theorem 2.7.

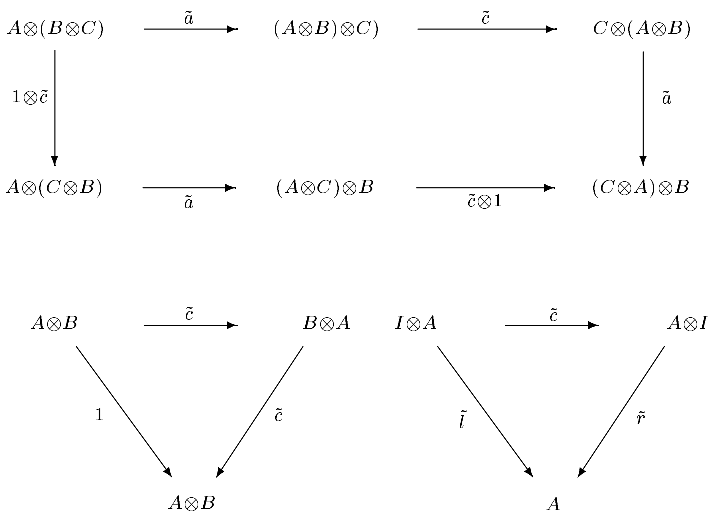

FIGURE 6. See the proof of Theorem 2.7.

Definition 2.8. For any $\dot{\Phi}$–functor $\mathsf{F} : \dot{\Phi} \longrightarrow \dot{\Phi}$, we define the *kernel* of F, $Ker(\mathsf{F})$, to be the category of elements of the Set–functor $[P^{0,I}, F-]_{\Phi}$.

Theorem 2.9. *Let* $\mathsf{F} : \dot{\Phi} \longrightarrow \dot{\Phi}$ *be a* $\dot{\Phi}$*–functor such that for any* $D \in \Phi$, F *preserves* $\dot{\mathsf{H}}_D$*, the internal* Hom *of* $\dot{\Phi}$*, and* $Ker(\mathsf{F})$ *is a* $\mathcal{D}$*–type diagram–scheme. Then* F *has a representation as a* $\mathcal{D}$*–colimit of representables as*

$$\mathsf{F}(A) \simeq \ \mathrm{Colim}_{(D,d) \in Ker(F)} \ \dot{\mathsf{H}}_D(A).$$

Proof. First, note that by the Yoneda lemma, the category of representable $\dot{\Phi}$–functors over F is the dual of $Ker(\mathsf{F})$. Hence,

$$[P^{0,I}, F-]_{\Phi} \simeq \mathrm{Colim}_{(D,d) \in Ker(F)} \ [D, -]_{\Phi} \simeq \mathrm{Colim}_{(D,d) \in Ker(F)} \ [P^{0,I}, \dot{\mathsf{H}}_D(-)]_{\Phi}.$$

If we assume that G is a colimit of the representable $\dot{\Phi}$–functors over F (i.e. the right hand side), then by density of Φ_f it is sufficient to show that $[V, \mathsf{G}(A)]_{\Phi} \simeq [V, \mathsf{F}(A)]_{\Phi}$ for any $A \in \Phi$ and $V \in \Phi_f$. This can be verified as follows:

$$\begin{aligned}
[V, \mathsf{G}(A)]_{\Phi} \quad & \simeq [V, \mathrm{Colim}_{(D,d) \in Ker(F)} \ \dot{\mathsf{H}}_D(A)]_{\Phi} \simeq \mathrm{Colim}_{(D,d) \in Ker(F)} \ [V, \dot{\mathsf{H}}_D(A)]_{\Phi} \\
& \simeq \mathrm{Colim}_{(D,d) \in Ker(F)} \ [P^{0,I}, \dot{\mathsf{H}}_V(\dot{\mathsf{H}}_D(A))]_{\Phi} \\
& \simeq \mathrm{Colim}_{(D,d) \in Ker(F)} \ [P^{0,I}, \dot{\mathsf{H}}_D(\dot{\mathsf{H}}_V(A))]_{\Phi} \simeq [P^{0,I}, \mathsf{F}(\dot{\mathsf{H}}_V(A))]_{\Phi} \\
& \simeq [P^{0,I}, \dot{\mathsf{H}}_V(\mathsf{F}(A))]_{\Phi} \simeq [V, \mathsf{F}(A)]_{\Phi}.
\end{aligned}$$

□

In the following examples we consider some important special cases.

Example 2.10. Consider $G = (\mathbf{R}, +)$ and $\mathcal{V} = (\mathbf{R}, +, \leq)$ with two universal bounds $+\infty$ and $-\infty$ (see Section 1). Then $\dot{\mathsf{T}}$ is the Minkowski addition and

$\dot{\mathsf{H}}$ is the morphological erosion operator.
Also, in this case, it is clear that being a $\dot{\Phi}$–functor is equivalent to the concept of being an isotone translation invariant operator, where we have

$$Ker(\mathsf{F}) = \{A \in \dot{\Phi} \mid \mathsf{F}(A)_{(0)} \geq 0\}.$$

Moreover, note that, in this case Theorem 2.9 yields the Strong Reconstruction Theorem for isotone TI operators (Equation 2).
It is also interesting to consider $\mathcal{V} = (\mathbf{R}^{+}, ., \leq)$, where (.) is the ordinary multiplication, with 0 as the universal lower bound and an auxiliary universal upper bound $+\infty$. Then we have

$$\dot{\mathsf{T}}_{D}(A)_{(z)} = \sup_{x} (A_{(x)}.D_{(-x+z)}), \quad \dot{\mathsf{H}}_{D}(B)_{(z)} = \inf_{x} (B_{(x)} \div D_{(-z+x)}).$$

These relations, along with the corresponding reconstruction theorem, can be considered as a multiplicative version of the theory of TI systems.
Moreover, it should be noted that the same setup can be used for discrete spaces where $G = (\mathbf{Z}, +)$ or $G = (\mathbf{Z}_{n}, +)$.

2.2. **The Uniform Constructions.** In this section we consider the following maps for a fixed $D \in \Phi$,

$$\tilde{\mathsf{T}}_{D} : \Phi \longrightarrow \mathcal{V}, \quad \tilde{\mathsf{T}}_{D}(A) = \prod_{|J|<\infty} (\bigoplus_{j\in J} W_{j})^{[A_{(x)}, \oplus_{j}(\Delta_{W_j} \div D_{(x^{-1}z_j)})]_{\Phi}},$$

$$\tilde{\mathsf{H}}_{D} : \Phi \longrightarrow \mathcal{V}, \quad \tilde{\mathsf{H}}_{D}(B) = \coprod_{|J|<\infty}^{[\oplus_{j}(\Delta_{V_j} . D_{(z_j^{-1}x)}), B_{(x)}]_{\Phi}} (\bigoplus_{j\in J} V_{j}),$$

where $\oplus$ is a suitable associative bifunctor. Again, at first, we consider the most simple case for which we have,

$$\ddot{\mathsf{T}}_{D}(A) = \prod_{j} W_{j}^{[A_{(x)}, \Delta_{W_j} \div D_{(x^{-1}z_j)}]_{\Phi}} \quad \text{and} \quad \ddot{\mathsf{H}}_{D}(B) = \coprod_{j}^{[\Delta_{V_j} . D_{(z_j^{-1}x)}, B_{(x)}]_{\Phi}} V_{j}.$$

In this regard we prove the following theorem.

Theorem 2.11. *If the tensor of $\mathcal{V}$ is preserved by coproducts, then there exist a natural composition law and an identity map such that $[D, B] \stackrel{\text{def}}{=} \ddot{\mathsf{H}}_{D}(B)$, as the internal* Hom*, turns Φ into a $\mathcal{V}$–category.*

Proof. We first introduce the composition map

$$M : [D, B] \;.\; [A, D] \longrightarrow [A, B]$$

as follows:

$$M : \coprod_{i}^{[(\Delta_{V_i} \,.\, D_{(z_i^{-1}x)}), B_{(x)}]_{\Phi}} V_{i} \,.\, \coprod_{j}^{[(\Delta_{W_j} \,.\, A_{(z_j^{-1}x)}), D_{(x)}]_{\Phi}} W_{j} \longrightarrow \coprod_{k}^{[(\Delta_{U_k} \,.\, A_{(z_k^{-1}x)}), B_{(x)}]_{\Phi}} U_{k}.$$

Therefore, M may be rewritten as

$$M : \coprod_{i,j}^{[(\Delta_{V_i} \,.\, D_{(z_i^{-1}x)}), B_{(x)}]_{\Phi} \times [(\Delta_{W_j} \,.\, A_{(z_j^{-1}x)}), D_{(x)}]_{\Phi}} (V_{i} \,.\, W_{j}) \longrightarrow \coprod_{k}^{[(\Delta_{U_k} \,.\, A_{(z_k^{-1}x)}), B_{(x)}]_{\Phi}} U_{k}.$$

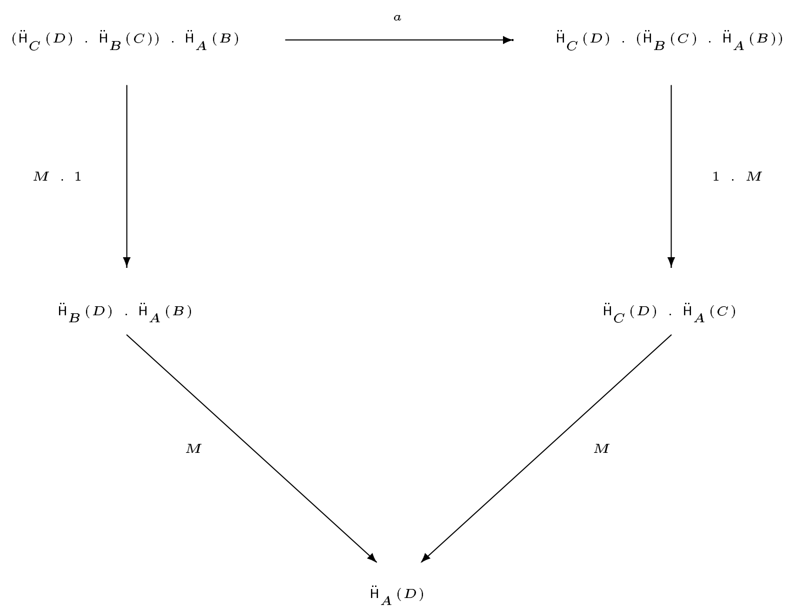

FIGURE 7. See the proof of Theorem 2.11.

To introduce M, it is enough to show that the left diagram is a subdiagram of the right one. A typical element of the left diagram is $V_i.W_j$ which is indexed by the maps,

$$\mathsf{f}_x : (V_i \ . \ D_{(z_i^{-1}x)}) \longrightarrow B_{(x)}$$

$$\mathsf{g}_x : (W_j \ . \ A_{(z_j^{-1}x)}) \longrightarrow D_{(x)}.$$

Hence, we have the following composition of maps,

$$\mathsf{f}_x \circ (1_{V_i} \ . \ g_{z_i^{-1}x}) : V_i \ . \ W_j \ . \ A_{(z_k^{-1}x)} \longrightarrow B_{(x)}.$$

By the universal property of coproduct, we get a unique map (up to isomorphism) which is defined to be M. Now, we introduce the map J_A, for each $A \in \mathrm{obj}(\Phi)$, as, $J_A : I \longrightarrow [A, A]$. By definition,

$$[D, B] = \coprod_i^{[(\Delta_{V_i} \ . \ A_{(z_i^{-1}x)}), A_{(x)}]_\Phi} V_i = \coprod_{z\in G} \coprod_i^{[(\Delta_{V_i} \ . \ A_{(z^{-1}x)}), A_{(x)}]_\Phi} V_i = \coprod_{z \in G} Er(A, A)_{(z)}.$$

On the other hand we have

$$I = \coprod_{z \in G} T_{(z)},$$

where $T_{(x)}$ is as follows:

$$T_{(x)} \stackrel{\text{def}}{=} \begin{cases} I & x = 0 \\ -\infty & x \neq 0. \end{cases}$$

Note that to introduce J, it is enough to introduce a map such as

$$\delta : I \longrightarrow Er(A, A)_{(0)}$$

which is easily defined by the universal property of the coproduct and the map l_x of the monoidal category $\mathcal{V}$. Therefore J is almost unique.
Next, we check commutativity of diagrams. At first, we consider the diagram of Figure 7. Since the tensor of $\mathcal{V}$ is preserved by coproducts, it is clear that

$$(\ddot{\mathsf{H}}_C(D) \, . \, \ddot{\mathsf{H}}_B(C)) \, . \, \ddot{\mathsf{H}}_A(B)$$

can be expressed as a coproduct of $((V_i \, . \, W_j) \, . \, U_k)$'s over a diagram which is the product of the corresponding three diagrams in Set, in which V_i, W_j and U_k are the typical elements of these diagrams. Now, we may note that each $(V_i \, . \, W_j)$ can be considered as a node of the base diagram of $\ddot{\mathsf{H}}_B(D)$ as was discussed in the definition of the composition map M. Hence, $((V_i \, . \, W_j) \, . \, U_k)$ may be considered as a node of the base diagram of $\ddot{\mathsf{H}}_B(D) \, . \, \ddot{\mathsf{H}}_A(B)$, which (again as it was discussed in the definition of the composition map M) shows that it is also a node of the base diagram of $\ddot{\mathsf{H}}_A(D)$. Therefore, by the universal property of coproduct, the diagram depicted in Figure 7 is commutative.

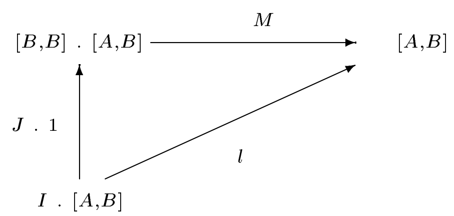

FIGURE 8. See the proof of Theorem 2.11.

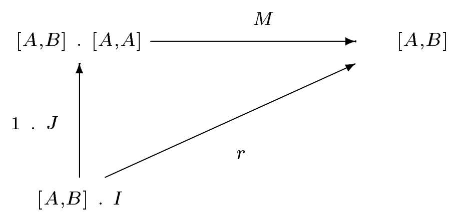

FIGURE 9. See the proof of Theorem 2.11.

Next, we consider the diagram of Figure 8, for commutativity. We shall prove that there is only one map from I . $[A,B]$ to $[A,B]$. To see this, we first note that

$$\begin{aligned} [B,B]\ .\ [A,B] &= \coprod_j^{[(\Delta_{W_j}\ .\ B_{(z_j^{-1}x))}, B_{(x)}]_\Phi} W_j\ .\ \coprod_i^{[(\Delta_{V_i}\ .\ A_{(z_i^{-1}x))}, B_{(x)}]_\Phi} V_i \\ &= \coprod_{j,i}^{[(\Delta_{W_j}\ .\ B_{(z_j^{-1}x))}, B_{(x)}]_\Phi \times [(\Delta_{V_i}\ .\ A_{(z_i^{-1}x))}, B_{(x)}]_\Phi} (W_j\ .\ V_i), \end{aligned}$$

$$I\ .\ [A,B] = \coprod_{x,i}^{[(\Delta_{V_i}\ .\ A_{(z_i^{-1}x))}, B_{(x)}]_\Phi} (T_{(x)}.V_i),$$

and that in the monoidal category $\mathcal{V}$ there is the map $l_{B_{(x)}} : I\ .\ B_{(x)} \longrightarrow B_{(x)}$ which shows that for $z_j = 0$ this map is the index of the node I in the base diagram of $[B,B]$. Consequently, from each node of the diagram of I . $[A,B]$, such as $T_{(x)}\ .\ V_i$, there is a map to one of the nodes of the diagram of $[B,B]$. $[A,B]$, namely $I\ .\ V_i$. But, using the same reasoning as the definition of the composition map, the base diagram of $[B,B]$. $[A,B]$ is a subdiagram of $[A,B]$. Therefore, by the universal property of the coproduct, there is only one map from I . $[A,B]$ to $[A,B]$, which shows that the diagram is commutative.
The commutativity of the right inverse diagram (Figure 9) can be checked in the same way. □

As a special important example we have,

Example 2.12. It is possible to consider the setup of Example 2.10 when the bifunctor $\oplus$ is omitted and compute the corresponding functors. However, if we reverse the order we come across another important concept.
Let $G = (\mathbf{R}, +)$ and $\mathcal{V} = (\mathbf{R}^+, ., \geq)$. Then

$$\ddot{\mathsf{H}}_D(B) = \inf_z\ (\sup_x\ (B_{(x)} \div D_{(-z+x)}))$$

in the usual order of real numbers. In what follows we show that this can be considered as a simplified version of the Haar fraction!
To see this, let $\oplus = +$ be the ordinary addition of real numbers. Then we obtain the Haar fraction as follows:

$$\ddot{\mathsf{H}}_D(B) = \inf_{c_j, z_j}\ \{\Sigma_j\ c_j \mid \quad \forall\, x\ \ B_{(x)} \leq \Sigma_j\ c_j D_{(-z_j+x)}\} = (B : D),$$

where everything is expressed in terms of the usual order of real numbers.

3. Concluding Remarks

As it has been shown in this paper, the fuzzy extension of positive Boolean functions can give rise to a very general meta-system-theory based on the theory of enriched categories. The generalization not only covers the classical case of TI systems but also seems to be rich enough to prepare the basic steps toward a general meta-theory for harmonic analysis on locally compact groups (or something very similar). This, in a way, suggests that, above all, everything is expressed in terms of *objects* and *maps*, and what we are concerned with, or what fixes our intuition about the whole theory, is how one is treating the maps as basic objects. In the more algebraic context, maps are usually treated as functions, mainly because of their operational properties, while in the set theoretic approach one prefers to treat maps as generalized sets where this in a way justifies the prefix *fuzzy* in the literature. From this point of view one may find some connections to the logical aspects of the subject that will not be discussed here [1, 2, 4, 5, 6, 7, 30, 38].

On the other hand, there seems to be so many nice problems from the other way round. Among these, a reference to the results of [14] related to opening, closing and granulometries and their generalizations to the categorical setup seems to be among the most interesting problems, which are definitely related to the theory of enriched monads [3, 4, 19, 21] (see [31] for some connections to computer science). Also, it is instructive to mention that, considering some recent results supporting the existence of some connections between mathematical morphology and evidence theory ([18] and references therein), the idea of formalizing these connections also seems to be an interesting project.

From a purely categorical point of view, we believe that one of the most interesting problems is to construct more general limits for which the whole setup is valid. This, on the one hand, is important since the second half of the construction for a generalized Haar measure needs a general categorical limit that also satisfies basic properties of a topological limit in some sense. Also, on the other hand, it is quite fortunate that some basic steps towards generalizing limit constructions in the enriched framework has already been taken [5, 6, 7, 20]. Above all, another interesting aspect is to study the consequences of results of this paper when the enriched category is chosen to be something other than a totally ordered subgroup of the real numbers.

One may also introduce some basic and interesting problems from a system-theoretic point of view. As a fundamental problem, we would like to mention the *design problem* for TI systems that may also include a study of such systems with feedback. As another important problem, one may ask about a generalization of some operational results in Boolean algebras and their consequences in the categorical setup. Among these, it is interesting to single out the Hadamard-Walsh transform as one of the most basic ones that ought to be studied.

Strictly speaking, it seems that based on some recent developments throughout different branches of mathematics, the structure of a complete closed monoidal category is a good candidate to be thought of as a generalization of the concept of a Banach space (in the sense discussed in this article). Therefore, it is natural to ask about the limits of the generalized theories, to gain something new that is still effective in a sense. This, a hard work to be done, may definitely influence many different branches of mathematics if it turns out to be fruitful enough, and presents a vast arena for continuing research that may unify and prove the effectiveness of well-developed aspects of fuzzy set theory.

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