Abstract

This paper is a study of fuzzy type theory (FTT) with partial functions. Out of several possibilities we decided to introduce a special value "∗" that represents "undefined". In the interpretation of FTT, this value lays outside of the corresponding domain. In the syntax it can be naturally represented by the description operator acting on the empty (fuzzy) set, because choosing an element from its kernel gives no result. Among many results, we will show that if, in a theory $T$, "∗" is defined or provable then $T$ is contradictory. We will also show completeness in the sense any consistent theory of FTT with partial functions has a model.

Keywords: Fuzzy type theory, EQ-algebra, partial function, description operator.

1 Introduction

In the applications of type theory to logical analysis of natural language (cf. [4, 10]) and elsewhere, e.g., in computer science, one often meets the requirement to deal with partial functions. They naturally raise, for example, when a mistake such as square root of a negative number or division by zero occurs, or when we want to express semantics of the expression "Czech president in 18th century", because there was no such president before 1918. There are several ways how partial functions can be introduced. For example, W. Farmer in his paper [6] discusses 8 possibilities which appear in the literature: non-denoting expressions as non-well-formed terms, functions represented as relations, total functions with unspecified value, many-sorted language, error values, non-existent values, partial valuation for terms and formulas, partial valuation for terms but total valuation for formulas. Each of these approaches provides a certain kind of solution but has also drawbacks. It seems that fully satisfactory solution does not exist. In this paper, we introduce a fuzzy type theory with partial functions. There are two basic solutions: either we can introduce a special functional value "∗" interpreted as "undefined", or we can modify the syntactic system to distinguish between general equality and equality of defined values. In this paper, we decided for the first option because the formal system of fuzzy type theory is fairly complicated and the second option would make it even more difficult to treat. We can also introduce "∗" into other types. This approach has one important advantage. Tichý in [17] gave a counterexample demonstrating that introducing partial functions leads to failure of the principle of $\lambda$-conversion. As pointed by Lepage in [9], however, by introducing "∗" as a special value, the counterexample of Tichý can be overcome and so, the $\lambda$-conversion is preserved.

Recall that in type theory, there is one specific formula which requires treatment with respect to partiality, namely the description operator $\iota_{\alpha(o)}$. In [11, 13], it is considered in such a way that its interpretation is just a partial function giving a value from the kernel of the corresponding fuzzy set if the latter is normal, and giving nothing otherwise. This suggests the idea that $\ast_o$ of type $o$ can be defined as the formula $\iota_{\alpha(o)} \cdot \lambda x_o \bot$ whose interpretation requires application of the description operator to the empty set. Such value is, of course, undefined and so, we get natural interpretation of $\ast_o$. Using it, we can define $\ast_\alpha$ as $\iota_{\alpha(\alpha)} \cdot \lambda x_\alpha \bot$, i.e., the description operator is here applied to a nowhere defined function. Similarly, we can introduce $\ast_\alpha$ for all types $\alpha$. When dealing with "undefined" as introduced above we must be aware that it is a well formed formula. Namely, when combined with the equality it must be treated as the valid formula and so, $(A_\alpha \equiv_\alpha \ast_\alpha)$ means that the formula $A_\alpha$ is equal to the element "undefined" (which itself is also a formula). But if
A\alpha is different from \ast\alpha then equality of these two formulas can by no means be true. Consequently, applying \ast\alpha to the function \(g_{\alpha\alpha} := \lambda x_\alpha \cdot (A_\alpha \equiv_\alpha x_\alpha)\) gives \(g_{\alpha\alpha} \ast_\alpha \equiv_\alpha \perp\). We will see that this approach has several important consequences.

In this paper, we will prove that the above outlined way enables us to introduce \(\ast\) consistently into FTT so that partial functions can be included. We will consider the most general variant of FTT based on the \(\Delta\)-algebras that are special algebras having structure of truth values that is natural for fuzzy type theory. The reason is that the basic operation in \(\Delta\)-algebras is a fuzzy equality and the implication is derived from it. It can also be demonstrated that every residuated lattice is an \(\Delta\)-algebra, but the converse does not hold. The paper is structured as follows: In Section 2 we introduce the \(\Delta\)-algebra of truth values and extend it by the special value "undefined". In Section 3 we introduce the syntax of the extended fuzzy type theory, namely introduce its axioms and prove various theorems characterizing the behavior of partial functions and show that it conforms to the results of Lapierre and Lepage presented in the papers [5, 9].

2  Truth values and fuzzy equality

2.1 Truth values

The truth degrees form a linearly ordered bounded good \(\Delta\)-algebra (see [11, 13]) \(\mathcal{E}_\Delta = \langle E, \land, \lor, \sim, 0, 1, \Delta \rangle\) where for all \(a, b, c, d \in E:\)

(E1) \( \langle E, \land, 1 \rangle \) is a commutative idempotent monoid (i.e. \(\land\)-semilattice). We put \(a \leq b\) iff \(a \land b = a\), as usual. Then 1 is the top and 0 the bottom element.

(E2) \( \langle E, \lor, 1 \rangle \) is a monoid, \(\lor\) is isotone w.r.t. \(\leq\).

(E3) \(a \sim a = 1\) \hspace{1cm} (reflexivity)

(E4) \( ((a \land b) \sim c) \lor (d \sim a) \leq c \sim (d \land b)\) \hspace{1cm} (substitution)

(E5) \( (a \sim b) \land (c \sim d) \leq (a \sim c) \sim (b \sim d)\) \hspace{1cm} (congruence)

(E6) \( (a \land b \land c) \sim a \leq (a \land b) \sim a\) \hspace{1cm} (monotonicity)

(E7) \(a \sim 1 = a\) \hspace{1cm} (goodness)

We define, \(a \rightarrow b = (a \land b) \sim a\), and \(\neg a = a \rightarrow 0\). It can be proved that \(a \leq b\) iff \(a \rightarrow b = 1\). Therefore, we can rewrite axioms (4)–(6) using \(\rightarrow\) instead of \(\leq\). The delta operation \(\Delta : E \rightarrow E\) has the following properties:

(ED1) \(\Delta 1 = 1\),

(ED2) \(\Delta a \leq a\),

(ED3) \(\Delta a \leq \Delta \Delta a\),

(ED4) \(\Delta(a \sim b) \leq \Delta a \sim \Delta b\),

(ED5) \(\Delta(a \land b) = \Delta a \land \Delta b\),

(ED6) \(\Delta(a \lor b) \leq \Delta a \lor \Delta b\),

(ED7) \(\Delta a \lor \neg \Delta a = 1\).

It should be emphasized that every residuated lattice is a good \(\Delta\)-algebra with fuzzy equality being the biresiduation \(a \sim b = (a \rightarrow b)\land(b \rightarrow a)\). An \(\Delta\)-algebra \(\mathcal{E}\) is prelinear if for all \(a, b \in E\), 1 is the unique upper bound of \(\{a \rightarrow b, b \rightarrow a\}\).

Lemma 2.1 ([5]). The following is equivalent:

(a) \(\mathcal{E}\) is prelinear.

(b) \( ((a \rightarrow b) \lor c) \leq ((b \rightarrow a) \lor c) \rightarrow c\)

If \(\mathcal{E}\) is prelinear then it is lattice ordered where the join is defined by \(a \lor b = ((a \rightarrow b) \land ((b \rightarrow a) \rightarrow a))\).

Convention 1. A fuzzy set is a function \(A : M \rightarrow E\). We will often write \(A \subseteq M\) if we want to emphasize that \(A\) is a fuzzy set on the universe \(M\).

The kernel of \(A \subseteq M\) is the set \(\text{Ker}(A) = \{m \in M \mid A(m) = 1\}\). The fuzzy set \(A\) is normal, if \(\text{Ker}(A) \neq \emptyset\) and subnormal otherwise. The set of all fuzzy sets on \(M\) is denoted by \(\mathcal{F}(M)\) and the set of normal ones by \(\mathcal{N} \mathcal{F}(M)\).
2.2 Extended algebra of truth values

To deal with partial functions, we will consider a special "truth value" * where * ∉ E and interpret it as undefined. Then we will introduce extended EQ\(_\Delta\)-algebra of truth values and denote it by \(\mathcal{E}_\Delta\). Its support is \(E^* = E \cup \{\ast\}\). We will extend the operations \(\sim, \land, \otimes\) and \(\Delta\) to the whole \(E^*\) as follows. The list of axioms is extended by

(EX1) \(a \sim \ast = 0, \quad a \in E\),

(EX2) \(x \land \ast = \ast, \quad x \in E^*\),

(EX3) \(x \otimes \ast = \ast, \quad x \in E^*\).

(EX3') \(\ast \otimes x = \ast, \quad x \in E^*\).

(EX4) \(\Delta \ast = \ast, \quad x \in E^*\).

Notice that by Axioms (EX2)–(EX4), the operations \(\land, \otimes\) and \(\Delta\) are Bochvar style \([3]\), i.e., the * collapses the result again to *. Taking that Axiom (E3) holds also for *, we obtain the following extended operations.

Corollary 2.2. Let \(a, b \in E\) and \(\circ \in \{\land, \otimes\}\). Then the following tables define the operations in the extended algebra \(\mathcal{E}_\Delta\):

<table>
<thead>
<tr>
<th>(a)</th>
<th>(\circ \ast)</th>
<th>(b)</th>
<th>(\circ \ast)</th>
<th>(\rightarrow)</th>
<th>(b)</th>
<th>(\ast)</th>
<th>(\lor)</th>
<th>(b)</th>
<th>(\ast)</th>
<th>(\Delta x)</th>
<th>(x)</th>
<th>(\sim x)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\ast)</td>
<td>(0)</td>
<td>(\ast)</td>
<td>(\ast)</td>
<td>(a \rightarrow b)</td>
<td>(0)</td>
<td>(\ast)</td>
<td>(\lor)</td>
<td>(b)</td>
<td>(\ast)</td>
<td>(a \Delta a)</td>
<td>(a)</td>
<td>(\sim a)</td>
</tr>
</tbody>
</table>

Note that 0 remains the bottom element for all \(a \in E\). We, at the same time, have \(\ast \leq 0\) but \(0 \not\leq \ast\).

Theorem 2.3. Let us replace \(\leq\) by \(\rightarrow\) in the axioms above. Then formulas (E3)–(E6), (ED2)–(ED6) give \(1\) for all \(x \in E^*\), (E7) and (ED7) give \(1\) for \(a \in E\).

Proof. This follows by straightforward verification using the operations defined above. \(\square\)

In [2], a special operation \(\sqcup\) was introduced that we will call star-box\(^1\)

<table>
<thead>
<tr>
<th>(x)</th>
<th>(\sqcup)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td>(0)</td>
</tr>
<tr>
<td>(\ast)</td>
<td>(0)</td>
</tr>
<tr>
<td>(1)</td>
<td>(\ast)</td>
</tr>
</tbody>
</table>

This operation is characterized by the following axioms that may extend axioms of the algebra of truth values:

(ESb1) \((\sqcup \, 1 \sim \ast) = 1\),

(ESb2) \((\sqcup x \sim \ast) \lor \neg \sqcup x = 1, \quad x \in E^*\).

As we do not need this operation in this paper, we will not consider (ESb1), (ESb2) as special axioms of the extended EQ-algebra.

Finally we introduce the following derived operations on \(E^*\):

\(?x = x \sim \ast, \quad !x = \sim ?x, \quad \downarrow x = x \sim 1, \quad \uparrow x = \sim !x \lor \downarrow x.\)

Their truth tables are the following (\(a \in E\)):

<table>
<thead>
<tr>
<th>(a)</th>
<th>(?)</th>
<th>(!)</th>
<th>(\uparrow)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td>0</td>
<td>1</td>
<td>(\ast)</td>
</tr>
<tr>
<td>(\ast)</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

The operation "?" is a test for undefined, "!" is a test for defined, \(\downarrow\) and \(\uparrow\) are star-0 and star-1 reinterpretation, respectively.

2.3 Fuzzy equality

A fuzzy equality \(\hat{=}\) on \(M\) is a binary fuzzy relation on \(M, \hat{=} : M \times M \to E\) that is reflexive, symmetric and \(\otimes\)-transitive. If \(m, m' \in M\) then we usually write \([m \hat{=} m']\) instead of \(\hat{=}(m, m')\). We say that \(\hat{=}\) is separated, provided that \([m \hat{=} m'] = 1\) iff \(m = m'\) holds for all \(m, m' \in M\). A fuzzy equality on truth values is the operation \(\sim\) from the EQ-algebra \(\mathcal{E}\). This fuzzy equality is separated. Let us now extend \(M\) to \(M' = M \cup \{\ast\}\) where \(* \not\in M\). Analogously as above, * represents the value "undefined" or "not known". Then we will extend the fuzzy equality \(\hat{=}\) defined on \(M\) to \(\hat{=} : M^* \times M^* \to E\) as follows\(^2\):

<table>
<thead>
<tr>
<th>(x)</th>
<th>(\hat{=} y)</th>
<th>(\ast)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(?)</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>(!)</td>
<td>(0)</td>
<td>1</td>
</tr>
</tbody>
</table>

\(^1\)"Star box" is a name of one origami art piece.

\(^2\)It seems unnecessary to use different symbols for fuzzy equality \(\hat{=}\) on \(M\) and its extension to \(M^*\).
where \( x, y \in M_\alpha \), i.e., the objects of \( M \) are functions \( h : M_\alpha \to M_\beta \) where \( M_\alpha, M_\beta \) are sets endowed with the corresponding fuzzy equalities \( \equiv_\alpha, \equiv_\beta \). We will also add an element \( *_{\beta \alpha} \not\in M \) to \( M \), i.e., \( M^* = M \cup \{ *_{\beta \alpha} \} \). Note that in this case, \( *_{\beta \alpha} \) is a nowhere defined function on \( M_\alpha \). Then we introduce the fuzzy equality \( \equiv : M^* \times M^* \to E \) by

\[
h \equiv h' = \bigwedge_{m \in M^*_\alpha} [h(m) \equiv_\beta h'(m)], \quad h, h' \in M^* (1)
\]

where \( \equiv_\beta \) is the fuzzy equality on the set \( M_\beta \). Note due to definition of \( \equiv \) above, we obtain \( 0 \) in \( E \) whenever there is \( m \in M^*_\alpha \) such that for \( h, h' \in M \), either \( h(m) = * \) or \( h'(m) = * \), i.e., one of the functional values is undefined. Consequently, also for functions \( h \in M \) we obtain \( [h \equiv *_{\beta \alpha}] = 0 \) and \( [\equiv_{\beta \alpha}] = 1 \). A function \( f : M_\alpha \to M_\beta \) is weakly extensional if for all \( m, m' \in M_\alpha, [m \equiv_\alpha m'] = 1 \) implies \( [f(m) \equiv_\beta f(m')] = 1 \).

### 3 Syntax of partial FTT

We will consider the basic FTT whose truth degrees form a linearly ordered good EQ\(\Delta\)-algebra introduced in [13]. This is the most general kind of FTT which can be extended to various specific versions including the linearly ordered IMTL-algebra originally introduced in [12].

#### 3.1 Syntax

The basic syntactical objects of FTT are classical — see [11], namely the concepts of type and formula. The atomic types are \( \epsilon \) (elements) and \( o \) (truth degrees). Complex types \( \langle \beta \alpha \rangle \) are formed from previously formed ones \( \beta \) and \( \alpha \). The set of all types is denoted by \( Types \). The language of FTT denoted by \( J \) consists of variables \( x_\alpha, \ldots \), special constants \( c_\alpha, \ldots \) (\( \alpha \in Types \)), auxiliary symbols \( \lambda \) and brackets. Formulas are formed of variables, constants (each of specific type), and the symbol \( \lambda \). Thus, each formula \( A \) is assigned a type (we write \( A_\alpha \)). The set of formulas of type \( \alpha \) is denoted by \( Form_\alpha \), the set of all formulas by \( Form \). Interpretation of a formula \( A_\beta \) is a function from the set of objects of type \( \alpha \) into the set of objects of type \( \beta \). Thus, if \( B \in Form_\beta \) and \( A \in Form_\alpha \) then \( (BA) \in Form_\beta \). Similarly, if \( A \in Form_\beta \) and \( x_\alpha \in J \), \( \alpha \in Types \), is a variable then \( \lambda x_\alpha A_\beta \in Form_\beta \), a formula whose interpretation is a function which assigns to each object of type \( \alpha \) an object of type \( \beta \) represented by the formula \( A_\beta \).

The set of formulas of type \( \alpha \), \( \alpha \in Types \), is denoted by \( Form_\alpha \). A set of all formulas of the language \( J \) is \( Form = \bigcup_{\alpha \in Types} Form_\alpha \). Specific constants always present in the language of FTT are the following.

(i) \( E_{\langle \alpha \rangle} \), \( \alpha \in \{ o, \epsilon \} \), (fuzzy equality)

(ii) \( C_{\langle \alpha o \rangle} \), \( \alpha \in \{ o, \epsilon \} \), (conjunction)

(iii) \( S_{\langle \alpha o \rangle} \), \( \alpha \in \{ o, \epsilon \} \), (strong conjunction)

(iv) \( D_{\langle o o \rangle} \), \( \alpha \in \{ o, \epsilon \} \), (delta)

(v) \( t_{\langle \alpha o \rangle} \), \( \alpha \in \{ o, \epsilon \} \), (description operator)

A variable \( x_\alpha \) is bound in a formula \( A_\beta \) if the latter has a well formed part \( \lambda x_\alpha B_\beta \). Otherwise \( x_\alpha \) is free. A formula \( A \) is closed if it does not contain free variables. A closed formula \( A_\alpha \) of type \( o \) is called a sentence.

#### Formal definitions

(i) Basic fuzzy equality \( \equiv_\langle \alpha o \rangle \equiv \lambda x_\alpha \lambda y_\alpha \cdot (E_{\langle \alpha o \rangle} y_\alpha) x_\alpha, \alpha \in \{ o, \epsilon \} \).

(ii) Strong conjunction \( \&_{\langle \alpha o \rangle} \equiv \lambda x_\alpha \lambda y_\alpha \cdot (C_{\langle \alpha o \rangle} y_\alpha) x_\alpha \).

(iii) Conjunction: \( \wedge_{\langle \alpha o \rangle} \equiv \lambda x_\alpha \lambda y_\alpha \cdot (S_{\langle \alpha o \rangle} y_\alpha) x_\alpha \).

(iv) Delta connective: \( \Delta_{\langle o o \rangle} \equiv \lambda x_\alpha D_{\langle o o \rangle} x_\alpha \).

(v) Implication: \( \Rightarrow_{\langle \alpha o \rangle} \equiv \lambda x_\alpha \lambda y_\alpha \cdot (x_\alpha \wedge y_\alpha) \equiv x_\alpha \).

(vi) Negation: \( \neg_{\langle o o \rangle} \equiv \lambda x_\alpha \lambda y_\alpha \cdot (\neg_{\langle o o \rangle} y_\alpha x_\alpha) \equiv x_\alpha \).

(vii) Disjunction: \( \lor_{\langle o o \rangle} \equiv \lambda x_\alpha \lambda y_\alpha \cdot (x_\alpha \Rightarrow y_\alpha) \wedge ((y_\alpha \Rightarrow x_\alpha) \Rightarrow x_\alpha) \).

(viii) General quantifier: \( (\forall x_\alpha) A_\alpha \equiv (\lambda x_\alpha A_\alpha \equiv \lambda x_\alpha \top) \).

(ix) Representation of truth and falsity: \( T_\alpha \equiv (\lambda x_\alpha x_\alpha \equiv \alpha) \equiv (\lambda x_\alpha x_\alpha \equiv \alpha) \equiv (\lambda x_\alpha x_\alpha \equiv \alpha) \equiv \top \).

Note that the element \( *_{\beta \alpha} \) represents a nowhere defined function.
(xi) Test of undefined: \(?_{\alpha\alpha} \equiv \lambda x_\alpha \cdot x_\alpha \equiv \alpha_\alpha\).
(xii) Test of defined: \(1_{\alpha\alpha} \equiv (\exists x_\alpha) x_\alpha \equiv ? x_\alpha\).
(xiii) Star-0 reinterpretation: \(1_{\alpha\alpha} \equiv (\forall x_\alpha) x_\alpha \equiv \top\).
(xiv) Star-1 reinterpretation: \(1_{\alpha\alpha} \equiv (\forall x_\alpha) x_\alpha \equiv \top\).
(xv) Existential quantifier: \((\exists x_\alpha) A_\alpha \equiv (\forall y_\alpha)((\forall x_\alpha) \Delta(A_\alpha \equiv y_\alpha) \Rightarrow \top y_\alpha)\) (\(y_\alpha\) does not occur in \(A_\alpha\)).

Convention 1. The fuzzy equality between formulas of the type \(\alpha\) is denoted by \(\equiv_{\alpha}\). We will sometimes omit the type at "\(\equiv\)" if it is clear from the arguments and no confusion can occur. Note also that the fuzzy equality between truth values is usually called fuzzy equivalence.

### 3.2 Axioms and inference rules

#### Fundamental axioms.

Let \(\alpha, \beta \in \text{Types}\).

- (FT-fund1) \((\exists x_\alpha) [x_\alpha]\).
- (FT-fund2) \(\Delta(x_\alpha \equiv y_\alpha) \Rightarrow (f_{\beta\alpha} x_\alpha \equiv y_\alpha)\).
- (FT-fund3) \((\forall x_\alpha)(f_{\beta\alpha} x_\alpha \equiv g_{\beta\alpha} x_\alpha) \Rightarrow (f_{\beta\alpha} \equiv g_{\beta\alpha})\).
- (FT-fund4) \((f_{\beta\alpha} \equiv g_{\beta\alpha}) \Rightarrow (f_{\beta\alpha} x_\alpha \equiv g_{\beta\alpha} x_\alpha)\).
- (FT-fund5) \((x_\alpha B_\beta) A_\alpha \equiv (\forall \alpha (B_\beta \Rightarrow A_\alpha))\), \(\alpha, \beta \in \text{Types}\) where \(C_\beta\) is obtained from \(B_\beta\) by replacing all free occurrences of \(x_\alpha\) in it by \(A_\alpha\), provided that \(A_\alpha\) is substitutable to \(B_\beta\) for \(x_\alpha\) (lambda conversion).

- (FT-fund6) \((x_\alpha \equiv y_\alpha) \equiv (y_\alpha \equiv x_\alpha)\).
- (FT-fund7) \((x_\alpha \equiv y_\alpha) \land (y_\alpha \equiv z_\alpha) \Rightarrow (x_\alpha \equiv z_\alpha)\).

#### Axioms of truth degrees

As usual in fuzzy logic, we have two kinds of conjunction, namely the "ordinary" conjunction \(\land\) and the strong conjunction \(\&\). Let \(\bigcirc \in \{\land, \&\}\).

- (FT-tval1) \(\top\).
- (FT-tval2) \((x_\alpha \land y_\alpha) \equiv (y_\alpha \land x_\alpha)\).
- (FT-tval3) \((x_\alpha \lor y_\alpha) \equiv (y_\alpha \lor x_\alpha)\).
- (FT-tval4) \(!A_\alpha \Rightarrow ((A_\alpha \equiv \top) \equiv A_\alpha)\).
- (FT-tval5) \(!A_\alpha \equiv (!A_\alpha \equiv \top)\), \(\alpha \in \text{Types}\).
- (FT-tval6) \((A_\alpha \equiv A_\alpha) \equiv \top\), \(\alpha \in \text{Types}\).
- (FT-tval7) \(!((A_\alpha \equiv A_\alpha) \equiv \top)\), \(\alpha \in \text{Types}\).
- (FT-tval8) \((A_\alpha \equiv \top) \equiv A_\alpha\).
- (FT-tval9) \(((\top \land A_\alpha) \equiv A_\alpha)\).
- (FT-tval10) \(!A_\alpha \Rightarrow ((A_\alpha \equiv \bot) \equiv \bot)\).
- (FT-tval11) \((A_\alpha \land A_\alpha) \equiv A_\alpha\).
- (FT-tval12) \(((x_\alpha \land y_\alpha) \equiv z_\alpha) \land (t_\alpha \equiv x_\alpha) \Rightarrow (z_\alpha \equiv (t_\alpha \land y_\alpha))\).
- (FT-tval13) \((x_\alpha \equiv y_\alpha) \land (z_\alpha \equiv t_\alpha) \Rightarrow (x_\alpha \equiv z_\alpha) \equiv (y_\alpha \equiv t_\alpha)\).
- (FT-tval14) \((x_\alpha \equiv (y_\alpha \land z_\alpha)) \Rightarrow (x_\alpha \equiv y_\alpha)\).
- (FT-tval15) \((x_\alpha \Rightarrow y_\alpha) \Rightarrow (x_\alpha \land z_\alpha \Rightarrow y_\alpha \land z_\alpha)\).
- (FT-tval16) \((x_\alpha \Rightarrow y_\alpha) \Rightarrow (z_\alpha \land x_\alpha \Rightarrow z_\alpha \land y_\alpha)\).
- (FT-tval17) \((x_\alpha \Rightarrow y_\alpha) \Rightarrow (y_\alpha \Rightarrow x_\alpha \Rightarrow z_\alpha) \Rightarrow z_\alpha)\).

#### Axioms of delta

- (FT-delta1) \((g_{\alpha\alpha}(\Delta x_\alpha) \land g_{\alpha\alpha}(\Delta x_\alpha)) \equiv (\forall y_\alpha) g_{\alpha\alpha}(\Delta y_\alpha)\).
- (FT-delta2) \((\Delta A_\alpha \equiv B_\alpha) \equiv (\Delta A_\alpha \lor \Delta B_\alpha)\).
- (FT-delta3) \((\Delta A_\alpha \lor B_\alpha) \Rightarrow \Delta A_\alpha \lor \Delta B_\alpha\).
- (FT-delta4) \((\Delta \uparrow A_\alpha \lor \Delta \downarrow A_\alpha) \equiv \top\).
Theorem 3.3. It is consistent if \( T \) is a theory and \( A_o \) a formula then \( T \vdash A_o \) means that \( A_o \) is provable in \( T \). A theory \( T \) is contradictory if \( T \vdash \bot \). Otherwise it is consistent. The concepts of extension and conservative extension of a theory \( T \) are classical.

Axioms of star

(FT-B1) \( (\bot \equiv \ast_o) \equiv \bot \),
(FT-B2) \( (\top \equiv \ast_o) \equiv \bot \),
(FT-B3) \( A_o \circ \ast_o \equiv \ast_o \),
(FT-B4) \( \Delta \ast_o \equiv \ast_o \),
(FT-B5) \( (A_o \equiv \ast_o \vee \neg(A_o \equiv \ast_o)) \equiv \top \), \( \alpha \in \{o, \epsilon\} \).

Axioms of quantifiers

(FT-quant1) \( \Delta(\forall x_o)(A_o \Rightarrow B_o) \Rightarrow (A_o \Rightarrow (\forall x_o)B_o) \), \( x_o \) is not free in \( A_o \),
(FT-quant2) \( (\forall x_o)(A_o \Rightarrow B_o) \Rightarrow ((\exists x_o)A_o \Rightarrow B_o) \), \( x_o \) is not free in \( B_o \),
(FT-quant3) \( (\forall x_o)(A_o \lor B_o) \Rightarrow ((\forall x_o)A_o \lor B_o) \), \( x_o \) is not free in \( B_o \).

Axioms of descriptions

(FT-descr1) \( \iota_{(\alpha o o)}(E_{(\alpha o) o o}) \equiv y_o \), \( \alpha \in \{o, \epsilon\} \),
(FT-descr2) \( (\forall x_o)(\neg \Delta(f_{\alpha o o}x_o \equiv \top)) \Rightarrow (\iota_{(\alpha o o)}f_{\alpha o} \equiv \ast_o) \), \( \alpha \in \{o, \epsilon\} \).

Remark 3.1. The list of axioms above may probably be redundant. To simplify them, however, will be the task of some future paper.

Remark 3.2. It follows from formal definitions (xi) and (xii) that Axioms (FT-B1), (FT-B2) and (FT-B5) can alternatively be written in the following form:

(FT-B1') \( \neg \bot \),
(FT-B2') \( \neg \ast_o \),
(FT-B1''') \( \ast_o \equiv \bot \),
(FT-B2'') \( \neg \ast_o \).

(FT-B5') \( (?A_o \lor !A_o) \equiv \top \), \( \alpha \in \{o, \epsilon\} \).

Recall that a formula \( A_o \) is crisp if \( \vdash A_o \lor \neg A_o \). Hence, both \(?A_o\) as well as \(?A_o\) are crisp.

A theory \( T \) is a set of formulas of type \( o \) (determined by a subset of special axioms, as usual). Provability is defined as usual. If \( T \) is a theory and \( A_o \) a formula then \( T \vdash A_o \) means that \( A_o \) is provable in \( T \). A theory \( T \) is contradictory if \( T \vdash \bot \). Otherwise it is consistent. The concepts of extension and conservative extension of a theory \( T \) are classical.

The inference rules remain unchanged:

\( A_o \equiv A'_o \in \text{Form}_o \) and \( B_o \in \text{Form}_o \) be formulas. Then we infer from them a formula \( B'_o \) which comes from \( B_o \) by replacing one occurrence of \( A_o \) by \( A'_o \), provided that the occurrence of \( A_o \) in \( B_o \) is not an occurrence of a variable immediately preceded by \( \lambda \).

\( (N) \) Let \( A_o \in \text{Form}_o \) be a formula. Then from \( A_o \equiv \top \) infer \( \Delta A_o \equiv \top \).

A theory \( T \) is a set of formulas of type \( o \) (determined by a subset of special axioms, as usual). Provability is defined as usual. If \( T \) is a theory and \( A_o \) a formula then \( T \vdash A_o \) means that \( A_o \) is provable in \( T \). A theory \( T \) is contradictory if \( T \vdash \bot \). Otherwise it is consistent. The concepts of extension and conservative extension of a theory \( T \) are classical.

Theorem 3.3. (a) \( (\exists x_o)(x_o o) \equiv \alpha \in \text{Types} \).
(b) \( (A_o \equiv \top) \vdash A_o \).
(c) \( A_o, (A_o \equiv B_o) \vdash B_o \), \( \alpha \in \text{Types} \). (Rule (EMP))
(d) \( \vdash A_o \equiv A_o \), \( \alpha \in \text{Types} \).
(e) \( \vdash (A_o \equiv B_o) \equiv (B_o \equiv A_o) \).
(f) \( (A_o \equiv \top), (A_o \Rightarrow B_o) \vdash B_o \). (Modus Ponens I)
(g) \( \vdash A_o \equiv \top \). Then \( \vdash !A_o \iff \vdash A_o \equiv \top \).
(h) \( !A_o, A_o, (A_o \Rightarrow B_o) \vdash B_o \). (Modus Ponens II)
(i) \( !A_o \vdash (A_o \Rightarrow \bot) \Rightarrow A_o \).
Proof. (a) By Axiom (FT-B2'), we have ⊢ ∅. Therefore, (∃x_o)l.o, and so, (a) follows from this and Axiom (FT-fund1) by induction on types.
(b) follows from Axiom (FT-B2') using Rule (R).
(c) follows immediately from the assumptions using Rule (R).
(d) follows from Axioms (FT-tval6) and (FT-tval1) by Rule (EMP).
(e) is proved by induction on the complexity of types. For α ∈ {o, ɛ}, this follows from Axiom (FT-fund6) and the axioms of truth values (see [14, Lemma 13(g)]).

Let α = γβ.

\[ (L.1) \vdash (∀x β)(f γβ \equiv g γβ x β) \equiv (∀x β)(f γβ x β \equiv g γβ x β) \]

(instance of (c))

\[ (L.2) \vdash (f γβ x β \equiv g γβ x β) \equiv (g γβ x β \equiv f γβ x β) \]

(inductive assumption)

\[ (L.3) \vdash (f γβ \equiv g γβ \equiv f γβ) \equiv (g γβ \equiv f γβ) \]

(L.1, L.2 using Rule (R) and formal definition (xv))

(f)

\[ (L.1) \vdash A_o \equiv \top \quad \text{(assumption)} \]

\[ (L.2) \vdash \top \Rightarrow B_o \quad \text{(L.1, assumption, Rule (R))} \]

\[ (L.3) \vdash (\top \land B_o) \equiv \top \quad \text{(L.2, definition of ⇒)} \]

\[ (L.4) \vdash \top \land B_o \quad \text{(L.3, Rule (EMP))} \]

\[ (L.5) \vdash \top \land B_o \equiv B_o \quad \text{(Axiom (FT-tval8))} \]

\[ (L.6) \vdash B_o \quad \text{(L.4, L.5, Rule (EMP))} \]

(g) From the assumption and Axiom (FT-tval4) we obtain \( \vdash (A_o \equiv \top) \equiv A_o \) using Modus Ponens I. The rest follows from this using Rule (EMP).

(h)

\[ (L.1) \vdash A_o \equiv \top \quad \text{(assumption, Axiom (FT-tval5), Rule (EMP))} \]

\[ (L.2) \vdash (A_o \equiv \top) \equiv A_o \quad \text{(L.1, Axiom (FT-tval4, Modus Ponens I))} \]

\[ (L.3) \vdash A_o \equiv \top \quad \text{(L.2, Rule (EMP))} \]

\[ (L.4) \vdash B_o \quad \text{(L.3, assumption, Modus Ponens I)} \]

(i) Let us consider the following instance of Axiom (FT-tval12):

\[ \vdash ((A_o \land \perp) \equiv A_o) \Rightarrow (A_o \equiv (A_o \land \perp)). \]

Realizing that \( !A_o \vdash (A_o \land \perp) \equiv A_o \) and \( !A_o \vdash (A_o \land \perp) \equiv \perp \), we obtain \( \vdash (A_o \Rightarrow \perp) \Rightarrow A_o \) using Rule (R), Axioms (FT-tval6) and (FT-tval8).

Remark 3.4. Note that in the proof above and most of the following ones we silently applied also symmetry of \( \equiv \) due to item (c).

In the subsequent subsections we will prove a series of lemmas and theorems characterizing various properties of FTT with partial functions. Some of them are slight but necessary modifications of the properties of the standard FTT. Sometimes, however, we do not repeat the latter and refer to the properties whose proofs need not be modified by saying “using properties of FTT”. We must suppose that the reader is acquainted with the papers [12, 13] because otherwise the paper would be too long.

3.3 Logical connectives and ”undefined”

The following is the basic theorem characterizing definability of equivalence.

Theorem 3.5. Let α ∈ \( \text{Types} \). Then

\( (a) \vdash (A_o \equiv A_o), \)

\( (b) \vdash (\ast_o \equiv \ast_o), \)

\( (c) !A_o \vdash ! (A_o \equiv \ast_o), \)

\( (d) A_o \equiv B_o \vdash ! (A_o \equiv B_o), \)

\( (e) !A_o, !B_o \vdash ! (A_o \equiv B_o), \)

\( (f) !A_o \vdash ! !A_o, \)

\( (g) ?A_o \vdash ! ?A_o. \)
Proof. (a) follows from Axioms (FT-B2') and (FT-tval6) using Rule (R).
(b) is an instance of (a).
(c) By the assumption \( \vdash (A_o \equiv *_o) \equiv \bot \). As \( \vdash \bot \) by Axiom (FT-B1'), we obtain (c) using Rule (R).
(d) follows from (a) and the assumption using Rule (R).
(e) From the assumption, Axiom (FT-tval5) and using Rule (EMP) we obtain \( \vdash !A_o \equiv \top \) and \( \vdash !B_o \equiv \top \). Then (e) follows from this and Axiom (FT-B7) using Modus Ponens I.
(f) By the assumption, \( \vdash (A_o \equiv *_o) \equiv \bot \). Then (f) follows from this and (d).
(g) follows from (b) using the assumption and Rule (R).

Corollary 3.6. If \( A_o \) is an axiom of FTT then \( \vdash !A_o \).

Proof. If \( A_o \) is one of Axiom (FT-B2) or (FT-tval7) then the corollary trivially follows. Otherwise, all axioms have the form of equivalence and so, the corollary follows from Theorem 3.7 (d).

Theorem 3.7. Let \( \vdash A_o \). Then \( \vdash !A_o \).

Proof. By induction on the length of the proof.
(a) If \( A_o \) is an axiom then lemma follows from Corollary 3.6.
(b) Let \( A_o \) follow from \( A'_o \) and \( B_o \equiv C_o \) using Rule (R). Then \( \vdash !A'_o \) by the inductive assumption. This means that \( \vdash \neg (A'_o \equiv *_o) \). Then we obtain \( \vdash \neg (A_o \equiv *_o) \) using Rule (R) which means that \( \vdash !A_o \).
(c) Let \( A_o := \Delta B_o \equiv \top \) be obtained from \( B_o \equiv \top \) using Rule (N). Then we have \( \Delta B_o \equiv \top \vdash ! (\Delta B_o \equiv \top) \) by Theorem 3.7 (d).

Theorem 3.8. For all \( \alpha \in \text{Types} \), \( \vdash !A_o \lor ?A_o \).

Proof. For \( \alpha \in \{o, \epsilon\} \), this follows from Axiom (FT-B5).
Induction step for \( \alpha = \gamma \beta \). Using the properties of FTT, the inductive assumption and the definition of fuzzy equality between complex types, we can write the following sequence of provable formulas:

\[
\begin{align*}
\vdash \neg (A_{\gamma \beta}x_{\beta} \equiv *_{\beta}) &\Rightarrow (\exists x_{\beta}) \neg (A_{\gamma \beta}x_{\beta} \equiv *_{\beta}) \\
\vdash \neg (A_{\gamma \beta}x_{\beta} \equiv *_{\beta}) \land (A_{\gamma \beta}x_{\beta} \equiv *_{\beta}) &\Rightarrow (\exists x_{\beta}) \neg (A_{\gamma \beta}x_{\beta} \equiv *_{\beta}) \lor (A_{\gamma \beta}x_{\beta} \equiv *_{\beta}) \\
\vdash (\exists x_{\beta}) (A_{\gamma \beta}x_{\beta} \equiv *_{\beta}) &\lor (\forall x_{\beta}) (A_{\gamma \beta}x_{\beta} \equiv *_{\beta}) \\
\vdash \neg (\forall x_{\beta}) (A_{\gamma \beta}x_{\beta} \equiv *_{\beta}) &\lor (\exists x_{\beta}) (A_{\gamma \beta}x_{\beta} \equiv *_{\beta}) \\
\vdash !A_o \lor ?A_o
\end{align*}
\]

Note that the previous reasoning is sound because all the formulas are defined.

It follows from this theorem that both \( ?A_o \) as well as \( !A_o \) are crisp and so, the following is provable (cf. [13]):

\( \vdash ?A_o \equiv \Delta ?A_o \), and \( \vdash !A_o \equiv \Delta !A_o \).

Lemma 3.9. If \( \vdash A_o \) then \( \vdash \Delta A_o \).

Proof. (a) \( \vdash !A_o \) (Assumption, Theorem 3.7 (d))
(b) \( \vdash (A_o \equiv \top) \equiv A_o \) (Assumption, L.1, Axiom (FT-tval4), MP II)
(c) \( \vdash \Delta A_o \equiv \top \) (Assumptions, L.2, Rule (EMP))
(d) \( \vdash \Delta A_o \) (Rule (N))
(e) \( \vdash \Delta A_o \) (L.4, Axiom (FT-tval1), Rule (EMP))

Lemma 3.10. The following holds for all \( \alpha \in \text{Types} \):

(a) \( \vdash ?*_{\alpha} \).
(b) \( \vdash !?*_{\alpha} \).
(c) \( \vdash \neg !*_{\alpha} \).

Proof. (a) follows from formal definition (xi) and Theorem 3.7 (d).
(b) follows from (a) and Theorem 3.7 (a).
(c) follows from (a), formal definition (xii) and the fact that \( A_o \vdash \neg \neg A_o \).
Lemma 3.11. Let $T$ be a consistent theory and $u_a \notin J(T)$ be a new constant. Then $T \vdash (\exists x_a)\Delta B_o$ iff $T' = T \cup \{ B_o[x_a/u_a] \}$ is a conservative extension of $T$.

Theorem 3.12. (a) $A_o \equiv \top \vdash (\forall x_o)A_o$. (generalization I)
(b) $\neg A_o, A_o \vdash (\forall x_o)A_o$. (generalization II)
(c) $\square A_o, A_o \equiv B_o \vdash \square B_o$ where $\square \in \{ \top, ?, \downarrow \}$.

Proof. (a) and (b) follow analogously as items (b) and (c) from the proof of Lemma 3.11 using Modus Ponens I, II.
(c) follows from the assumption $\vdash A_o \equiv B_o$ using Rule (R).

Theorem 3.13. Let $\square \in \{ \land, \ltimes \}$. Then
(a) $(A_o \equiv \top), (B_o \equiv \top) \vdash A_o \lor B_o$,
(b) $A_o, A_o, !B_o, B_o \vdash A_o \lor B_o$,
(c) $(A_o \implies B_o), (B_o \implies A_o) \vdash A_o \equiv B_o$,
(d) $A_o, !B_o \vdash ! (A_o \land B_o)$,
(e) $\vdash ! (A_o \land B_o) \equiv ! (A_o \lor !B_o)$,
(f) $! A_o, !B_o \vdash ! (A_o \lor !B_o)$.

Proof. (a) can be proved from the instance $\vdash (\top \land \top) \equiv \top$ of Axiom (FT-tval8), Axiom (FT-tval1), the assumptions and Rule (R).
(b) is proved analogously using Modus Ponens II and also Axiom (FT-tval4).
(c) follows using the property $\vdash (A_o \implies B_o) \land (B_o \implies A_o) \equiv (A_o \equiv B_o)$ proved in Lemma 3.11(i), (a) and modus ponens I.
(d) is proved in the same way as Theorem 3.11(e) from axiom (FT-B3).
(e) By the provable properties of FTT, we have $\vdash (A_o \equiv *_o) \equiv ((A_o \land B_o) \equiv (_o \land B_o))$ and $\vdash (B_o \equiv *_o) \equiv ((B_o \land A_o) \equiv (_o \land A_o))$.
We apply Axiom (FT-B3), Rule (R), the provable properties of (A_o \implies B_o) = (B_o \implies A_o) and (A_o \equiv B_o) \land (B_o \equiv C_o) \equiv (A_o \equiv (B_o \lor C_o))$ and Modus Ponens II we obtain $\vdash ! ((A_o \land B_o) \equiv *_o) \implies ! (A_o \equiv *_o) \lor ! (B_o \equiv *_o)$ which is (e).
(f) follows from Theorem 3.11(e) and (d).

Lemma 3.14. (a) $! A_o, ! B_o \vdash A_o \implies (A_o \lor B_o)$,
(b) $\vdash \top \lor \top$, $\vdash \top$,
(c) $\vdash ! A_o$,
(d) $\vdash (\top \implies *_o) \equiv \bot$, $\vdash (*_o \implies *_o) \equiv \bot$, $\vdash (_o \equiv *_o) \equiv \top$,
(e) $! A \vdash ! A_o \equiv A_o$,
(f) $\vdash \top \equiv *_o \equiv \top$.

Proof. (a) follows from the standard properties of FTT using Modus Ponens II.
(b) follows from $\vdash \top \equiv \top, \vdash \top$, definition of $\top$, (a) and the standard properties of FTT.
(c) follows from the instance of axiom (FT-tval12): $\vdash (_o \equiv \bot) \equiv *_o$ and $(A \equiv \top) \equiv *_o \equiv ((A \equiv \top) \land \bot))$ axioms and Theorems 3.13 and 3.12.
(d) follows from the definition of implication and Theorems 3.13 and 3.12.
(e), (f) follow from axioms and the formal definitions of the corresponding operators.

Theorem 3.15. (a) $\vdash !(*_o \lor *_o)$ and $\vdash (*_o \lor *_o) \equiv \bot$,
(b) $\vdash !(*_o \land *_o)$ and $\vdash (*_o \land *_o) \equiv \bot$,
(c) $! A_o, ! B_o \vdash ! (A_o \lor B_o)$.

Proof. (a) Using formal definitions (v), (vi) and Lemma 3.11(b), (d), (f) we obtain the sequence of provable formulas $\vdash (*_o \lor *_o) \equiv \top$, $\vdash (\top \implies *_o) \equiv \bot$, $\vdash (\bot \land \bot) \equiv \bot$, and $\vdash (*_o \lor *_o) \equiv \bot$. Then (a) follows from this and Axiom (FT-B1) using Rule (R).
The second part follows from the definition of $\lor$ and the proved properties of $\equiv$ and $\land$.
The proof of (b) is analogous.
(c) follows from Theorem 3.13(d) and (f).

Lemma 3.16. $\vdash (f \beta \equiv g \beta \equiv \gamma)(f \beta x_o)(g \beta x_o) \equiv \delta x_o (f \gamma x_o)(g \gamma x_o)$.

Proof. The proof is analogous to the proof of Lemma 3.11 using Axioms (FT-fund3), Generalization II, (FT-fund4) and Theorem 3.12(c).

Theorem 3.17. (a) $\vdash (\forall x_o)B_o \equiv (B_o, x_o[A_o] \equiv \top)$, (substitution I)
(b) $B_o \vdash (\forall x_o)B_o \equiv (B_o, x_o[A_o] \equiv \top)$, (substitution II)
Lemma 3.19. 
(a) \( \forall \alpha \in \{a, b, c, d, e\}, \text{this follows from Axiom (FT-fund7)} \) and the axioms of truth values (see \[\text{Lemma 13(h)}\]).

(b) for \( \alpha \in \{a, b, c, d, e\} \), this follows from Axiom (FT-fund7) and the axioms of truth values (see \[\text{Lemma 13(h)}\]).

Let \( \alpha = \gamma \beta \).

Proof. (a) The proof is based on the following instance of Axiom (FT-fund4), \( (\lambda x_\alpha B_\alpha \equiv \lambda x_\alpha \top) \Rightarrow ((\lambda x_\alpha B_\alpha) A_\alpha \equiv (\lambda x_\alpha \top) A_\alpha) \) which reduces to \( (\forall x_\alpha) B_\alpha \Rightarrow (B_\alpha[x_\alpha \equiv A_\alpha] \equiv \top) \). As we cannot, in general, assure that \( \vdash B_\alpha[x_\alpha \equiv A_\alpha] \), we cannot apply Axiom (FT-tval4).

(b) is a corollary of (a) and Axiom (FT-tval5) using Rule (R).

(c) For \( \alpha \in \{a, b, c, d, e\} \), this follows from Axiom (FT-fund7) and the axioms of truth values (see \[\text{Lemma 13(h)}\]).

(d) is proved analogously as in \[\text{Lemma 13(h)}\], provided that the rules of substitution II and generalization II are applied.

\[\square\]

Lemma 3.18. 
(a) \( \forall \alpha \in \{a, b, c, d, e\}, \text{this follows from Axiom (FT-fund7)} \) and the axioms of truth values (see \[\text{Lemma 13(h)}\]).

(b) ?A_\alpha \vdash \exists \Delta A_\alpha.

Proof. (a)

\[\begin{align*}
& (L.1) \vdash !A_\alpha, \text{(Assumption, Theorem \[\text{Lemma 15(f)}\])} \\
& (L.2) \vdash (A_\alpha \equiv \top) \equiv A_\alpha, \text{(Assumptions, L.1, Axiom (FT-tval4), MP II)} \\
& (L.3) \vdash A_\alpha \equiv \top, \text{(Assumptions, L.2, (EMP))} \\
& (L.4) \vdash \Delta A_\alpha \equiv \top, \text{(Rule (N))} \\
& (L.5) \vdash !\Delta A_\alpha, \text{(Theorem \[\text{Lemma 15(b)}\])} \\
& (L.6) \vdash !A_\alpha; \text{(Theorem \[\text{Lemma 15(b)}\])} \\
& (L.7) \vdash (f_\gamma \equiv \gamma \beta, g_\gamma \equiv h_\gamma \beta) \Rightarrow (f_\gamma \equiv h_\gamma \beta, g_\gamma \equiv h_\gamma \beta) \equiv \top, \text{(L.6, using Lemma \[\text{Lemma 14(f)}\])}
\end{align*}\]

\[\square\]

Lemma 3.19. 
(a) \( \exists z_\alpha(z_\alpha \equiv y_\alpha) \).

(b) \( \vdash y_\alpha \equiv (\exists z_\alpha(z_\alpha \equiv y_\alpha)) \).

(c) \( \vdash (\exists z_\alpha)z_\alpha \).

(d) \( \vdash (\forall y_\alpha)z_\alpha \equiv \bot \).

(e) If \( r_\alpha \) is a constant then \( \vdash (\forall x_\alpha)r_\alpha \equiv r_\alpha \) and \( \vdash (\exists x_\alpha)r_\alpha \equiv r_\alpha \).

\[\square\]
Proof. (a) follows from the reflexivity of \( \equiv \) and 3-substitution. The proof of (b) is the same as in \[1\] where the provable property \( A_\alpha \Rightarrow (B_\alpha \Rightarrow C_\alpha) \vdash A_\alpha \& B_\alpha \Rightarrow C_\alpha \) must be used. (c) follows from (a) when putting \( y_0 := \top \). (d) is obtained from formal definition (viii) and axioms (FT-fund4) and (FT-fund5) by putting \( z_0 := \bot \). Similarly (e) is obtained using quantifier axioms and the properties of fuzzy equality.

The deduction theorem holds for the defined formulas. Hence, we have the following formulation.

**Theorem 3.20** (Deduction theorem). Let \( T \) be a theory and \( A_\alpha \in \text{Form}_\alpha \) a closed formula such that \( T \vdash !A_\alpha \). Then \( T \cup \{ A_\alpha \} \vdash B_\alpha \iff T \vdash \Delta A_\alpha \Rightarrow B_\alpha \), for every formula \( B_\alpha \in \text{Form}_\alpha \) such that \( T \vdash !B_\alpha \).

Proof. As \( A_\alpha, B_\alpha \) are assumed to be defined, the proof is verbatim repetition of the proof from \[12\], Theorem 17.

**Theorem 3.21.** (a) If \( T \vdash !s_\alpha \) then \( T \) is contradictory.

(b) If \( T \vdash s_\alpha \) then \( T \) is contradictory.

Proof. (a) The assumption means that \( T \vdash (s_\alpha \equiv s_\alpha) \equiv \bot \). This together with \( \vdash (s_\alpha \equiv s_\alpha) \equiv \top \) gives \( T \vdash \bot \) which means that \( T \) is contradictory.

(b) follows from Theorem 3.11 and (a).

**Theorem 3.22.** (a) \( \vdash (\forall x_\alpha) s_\alpha \equiv \bot \),

(b) \( \vdash (\exists x_\alpha) s_\alpha \equiv \bot \),

(c) Let \( T \vdash (\exists x_\alpha) A_\alpha x_\alpha \). Then \( T \vdash (\forall x_\alpha) A_\alpha x_\alpha \equiv \bot \).

(d) Let \( T \vdash (\forall x_\alpha) A_\alpha x_\alpha \). Then \( T \vdash (\exists x_\alpha) A_\alpha x_\alpha \equiv \bot \).

Proof. (a) By formal definition (viii), we have \( \vdash (\forall x_\alpha) s_\alpha \equiv (\lambda x_\alpha s_\alpha \equiv \lambda x_\alpha \top) \). Then from axiom (FT-fund4), lambda conversion and substitution I we conclude that \( \vdash (\forall x_\alpha) s_\alpha \equiv (s_\alpha \equiv \top) \). From this and Axiom (FT-B2) we obtain (a) using Rule (R).

(b) From formal definition (xv) we obtain \( \vdash (\exists x_\alpha) s_\alpha \Rightarrow ((\forall x_\alpha) \Delta (s_\alpha \Rightarrow y) \Rightarrow y) \). Then we apply Lemma 3.11(d), (e), generalization II and Lemma 3.11(d).

(c) Using Lemma 3.11, we can extend the language of FTT by a new constant \( u_\alpha \) and obtain from the assumption \( T' \vdash !A_\alpha u_\alpha \) which is equivalent to \( T' \vdash A_\alpha u_\alpha \equiv s_\alpha \). Using generalization, the quantifier properties and the fact that \( T' \) is a conservative extension of \( T \) we obtain \( T \vdash (\forall x_\alpha) A_\alpha x_\alpha \equiv (\forall x_\alpha) s_\alpha \) which implies (c) using (a).

(d) From the assumption we have \( T \vdash A_\alpha x_\alpha \equiv s_\alpha \). From this and \( T \vdash (\exists x_\alpha) A_\alpha x_\alpha \equiv (\exists x_\alpha) A_\alpha x_\alpha \) we have \( T \vdash (\exists x_\alpha) A_\alpha x_\alpha \equiv (\exists x_\alpha) s_\alpha \) using rule (R). This gives (d) using (b).

It also follows from (a) that if a formula \( A_\alpha x_\alpha \) is undefined for some \( x_\alpha \) then its closure is false.

**Lemma 3.23.** For all \( \alpha, \beta \in \text{Types} \), \( \vdash \ast_{\beta \alpha} s_\alpha \equiv \ast_{\beta \beta} \).

Proof. This follows from formal definition (x) using \( \lambda \)-conversion.

By this lemma, the nowhere defined function at the “undefined” point gives again the value “undefined”.

### 3.4 Description operator and the “undefined”

The description operator \( \iota_{\alpha(\alpha)} \) for \( \alpha \in \{ \alpha, \epsilon \} \) must be taken as a special constant with the proper definition of its interpretation. In fact, it can be applied to a formula \( A_\alpha \) provided that \( \vdash (\exists x_\alpha) \Delta (A_\alpha x_\alpha) \). We can extend the description operator to complex types \( \beta \alpha \) as follows:

\[
\iota_{(\alpha \beta)(\alpha \beta)} h_{\alpha \beta}) := \lambda x_\alpha \cdot \iota_{\beta(\alpha \beta)} \cdot \lambda z_\beta (\exists x_\beta) (\Delta (h_{\alpha \beta})(\Delta f_{\beta \alpha} x_\alpha \equiv z_\beta)).
\]

The following theorem characterizes the basic properties of the description operator.

**Theorem 3.24** \([11]\). (a) \( \vdash \iota_{\gamma(\alpha)} (E_{(\alpha \gamma)} y_\gamma) \equiv y_\gamma \) holds for every type \( \gamma \in \text{Types} \).

(b) \( \vdash \Delta (p_{\alpha \alpha} \equiv q_{\alpha \alpha}) \Rightarrow (\iota_{\alpha(\alpha \alpha)} p_{\alpha \alpha} \equiv \iota_{\alpha(\alpha \alpha)} q_{\alpha \alpha}) \).

(c) \( \vdash (\exists y_x) (\Delta (p_{\alpha \alpha} x_\alpha \equiv (E_{(\alpha \alpha)} y_\alpha x_\alpha)) \Rightarrow p_{\alpha \alpha}(\iota_{\alpha(\alpha \alpha)} p_{\alpha \alpha}))) \).

(d) \( \vdash (\exists x_\alpha) \Delta A_\alpha x_\alpha \Rightarrow \iota_{\alpha(\alpha \alpha)} A_{\alpha \alpha} \).

By (a), the description operator can be extended to all types. The statement (d) says that if a fuzzy set interpreting the formula \( A_{\alpha \alpha} \) has a non-empty kernel then the element \( \iota_{\alpha(\alpha \alpha)} A_{\alpha \alpha} \) belongs to it.

**Theorem 3.25.** For all types \( \gamma \),

\[
(\forall x_\gamma) (\neg \Delta (A_{\alpha \gamma} x_\gamma \equiv \top)) \vdash \iota_{\gamma(\alpha \gamma)} A_{\alpha \gamma} \equiv \gamma \ast_\gamma.
\]
Proof. By induction on the length of type. For $\gamma \in \{\alpha, \epsilon\}$ (4) follows from axiom (FT-descr2).

Let $\gamma = \beta \alpha$. Let the inductive assumption hold.

(L.1) $\vdash \lambda x_\alpha \cdot \lambda x_\beta \cdot \lambda z_\beta (\exists f_\beta)(\Delta(A_{\alpha(\beta)}f_\beta \equiv \top))$

(definition (3))

(L.2) $\vdash (\forall f_\beta)(\neg \Delta(A_{\alpha(\beta)}f_\beta \equiv \top))$

(assumption)

(L.3) $\vdash \delta_{\alpha(\beta)}f_\beta \equiv \bot$

(L.2, properties of FTT)

(L.4) $\vdash \delta(A_{\alpha(\beta)}f_\beta) \&(f_\beta x_\alpha \equiv z_\beta) \equiv \bot$

(L.3, properties of FTT)

(L.5) $\vdash (\exists f_\beta)(\delta(A_{\alpha(\beta)}f_\beta) \&(f_\beta x_\alpha \equiv z_\beta)) \equiv \bot$

(L.4, properties of FTT)

Note that (L.6) states that some formula $\lambda z_\beta C_{\alpha\beta}z_\beta$ represents a first-order fuzzy set. Hence, using (L.1) we continue as follows:

(L.7) $\vdash \lambda x_\alpha \cdot \lambda x_\beta \cdot \lambda z_\beta (\exists f_\beta)(\Delta(A_{\alpha(\beta)}f_\beta \equiv \top))$

(L.1, L.6, $\lambda$-conversion, ind. assumption)

(L.8) $\vdash \lambda x_\alpha \cdot \lambda x_\beta \cdot \lambda z_\beta (\exists f_\beta)(\Delta(A_{\alpha(\beta)}f_\beta \equiv \top))$

(L.7, def. of $\lambda x_\alpha$)

\[\square\]

Corollary 3.26. For all types $\alpha \in \text{Types}$:

(a) Let $\vdash ?A_{\alpha \alpha}$. Then $\vdash \lambda x_\alpha A_{\alpha \alpha} \equiv *\alpha$.

(b) $\vdash \lambda x_\alpha A_{\alpha \alpha} \equiv *\alpha$.}

Proof. (a)

(L.1) $\vdash A_{\alpha \alpha} \equiv *\alpha$

(Assumption, definition of "$\equiv"$

(L.2) $\vdash \lambda x_\alpha A_{\alpha \alpha} \equiv *\alpha$

(L.1, properties of FTT)

(L.3) $\vdash A_{\alpha \alpha} \equiv *\alpha$

(L.2, Axiom (FT-fund4))

(L.4) $\vdash (\forall x_\alpha)(\lambda x_\alpha A_{\alpha \alpha} \equiv *\alpha)$

(Axiom (FT-B2))

(L.5) $\vdash (\forall x_\alpha)(\lambda x_\alpha A_{\alpha \alpha} \equiv *\alpha)$

(L.3, L.4, Rule (R))

(L.6) $\vdash (\forall x_\alpha)(\lambda x_\alpha A_{\alpha \alpha} \equiv *\alpha)$

(L.5, rule (N), property of $\Delta$, generalization II)

(L.7) $\vdash \lambda x_\alpha A_{\alpha \alpha} \equiv *\alpha$

(L.6, Theorem 3.26)

The rule of generalization (Theorem 3.13) was in line L.6 used correctly because the initial formula is defined.

(b) is a special case of (a) using Lemma 3.11(a).

\[\square\]

4 Semantics of partial FTT

The extended general frame for FTT with partial functions is the following tuple: $\mathcal{M}^* = \langle (M^*_{\alpha\alpha}, \hat{\sim}\alpha)_{\alpha \in \text{Types}}, \delta_{\Delta}, I_{\alpha}, I_{\epsilon} \rangle$ so that the following holds:

(i) We put $M^*_{\alpha\alpha} = E^*$ where the latter is the support of the extended EQ-algebra $\mathcal{E}_{\Delta}^*$. Furthermore, let $*\alpha \notin M_{\epsilon}$. Then we put $M^*_{\alpha\alpha} = M_{\epsilon} \cup \{*\alpha\}$. For all $\alpha = \gamma\beta$, the function $\hat{\sim}\alpha : M^*_{\alpha\alpha} \rightarrow M^*_{\beta\beta}$, where $\hat{\sim}\alpha(m_{\beta\beta}) = \hat{\sim}\beta$ for all $m_{\beta\beta} \in M^*_{\beta\beta}$, represents "undefined". For all types $\gamma\beta$ we put $M^*_{\gamma\beta} \subseteq (M^*_{\alpha\alpha})^{M_{\beta\beta}}$, where we require that $\hat{\sim}\beta \in M^*_{\beta\beta}$.

(ii) The $\delta_{\Delta}^*$ is the extended algebra of truth degrees (EQ-algebra). We assume that the sets $M^*_{\alpha\alpha}$, $M^*_{\alpha\alpha\alpha}$ contain all the operations discussed in Subsection 3.1.2.

(iii) $\hat{\sim}\alpha : M^*_{\alpha\alpha \alpha} \rightarrow E$ is a fuzzy equality on $M^*_{\alpha\alpha}$ for every $\alpha \in \text{Types}$. We define: $\hat{\sim}\alpha := \sim$ and $\hat{\sim}\epsilon$ is the fuzzy equality on $M^*_{\epsilon}$ given explicitly. The fuzzy equality $\hat{\sim}\beta\alpha$ for complex types is defined in (1).

(iv) $I_{\alpha} : \mathcal{F}(M_{\alpha\alpha}) \rightarrow M_{\alpha\alpha}$, $I_{\epsilon} : \mathcal{F}(M_{\epsilon\epsilon}) \rightarrow M_{\epsilon\epsilon}$ are partial functions interpreting the basic description operators.

Let $B \subseteq M_\alpha$ and $C \subseteq M_\epsilon$. Then

$$I_{\epsilon}(B) = \begin{cases} a_B & \text{if } B \text{ is normal,} \\ *\epsilon & \text{otherwise,} \end{cases}$$

and

$$I_{\alpha}(C) = \begin{cases} x_C & \text{if } C \text{ is normal,} \\ *\alpha & \text{otherwise.} \end{cases}$$

Remark 4.1. The elements $a_B, x_C$ in the formulas above are assumed to be certain specific elements from the kernels of $B$ and $C$ determined on the basis of some special requirements. Note that the functions $I_{\epsilon}, I_{\alpha}$ resemble the defuzzification operations used in the applications of fuzzy modeling.

Interpretation of formulas in a frame $\mathcal{M}$ is defined using an assignment $p$ of elements from $\mathcal{M}$ to variables, i.e., it is a function from the set of all variables of the language $J$ to elements from $\mathcal{M}^*$ in keeping with the corresponding types. Given an assignment $p$, we define a new assignment $p' = p \setminus x_\alpha$ which equals to $p$ for all variables except for $x_\alpha$. The set of all assignments over $\mathcal{M}$ is denoted by $\text{Asg}(\mathcal{M})$. 
Interpretation of variables and constants

(i) If \( x_\alpha \) is a variable then \( \mathcal{M}_p(x_\alpha) = p(x_\alpha) \in M_\alpha^* \).

(ii) If \( c_\alpha \) is a constant then \( \mathcal{M}_p(c_\alpha) \in M_\alpha^* \) is a specific element. As special cases, we define the following:

\[
\begin{align*}
\mathcal{M}_p(E_{(\alpha\omega)_\alpha}) & \equiv \sim, \\
\mathcal{M}_p(E_{(\alpha\omega)\alpha}) & \equiv \uparrow, \\
\mathcal{M}_p(C_{(\alpha\omega)_\alpha}) & \equiv \land, \\
\mathcal{M}_p(S_{(\alpha\omega)_\alpha}) & \equiv \ominus, \\
\mathcal{M}_p(D_{(\alpha\omega)_\alpha}) & \equiv \Delta.
\end{align*}
\]

(iii) Interpretation of the description operator. \( \mathcal{M}_p(t_{(\alpha\omega)_\alpha}) = I_\alpha \) and \( \mathcal{M}_p(t_{(\alpha\omega)\alpha}) = I_\alpha \).

Interpretation of complex formulas

(i) Interpretation of a formula \( B_{\beta\alpha}A_\beta \), which is of type \( \beta \), is \( \mathcal{M}_p(B_{\beta\alpha}A_\beta) = \mathcal{M}_p(B_{\beta\alpha})(\mathcal{M}_p(A_\alpha)) \).

(ii) Interpretation of a formula \( \lambda x_\alpha A_\beta \), which is of type \( \beta A_\alpha \) is a function \( \mathcal{M}_p(\lambda x_\alpha A_\beta) = F : M_\alpha \rightarrow M_{\beta\alpha} \), which assigns to each \( m_\alpha \in M_\alpha \) the element \( F(m_\alpha) = \mathcal{M}_p(A_\beta) \) determined by an assignment \( p' \) such that \( p' = p\setminus x_\alpha \) and \( p'(x_\alpha) = m_\alpha \). It follows from this definition that \( F \) is weakly extensional w.r.t "\( =_\alpha \)" and "\( =_\beta \)".

A model of a theory \( T \) is a general frame \( \mathcal{M} \) for which \( \mathcal{M}_p(A_\alpha) = 1 \) holds for all axioms \( A_\alpha \) of \( T \) and all assignments \( p \in \text{Asg}(\mathcal{M}) \). A formula \( A_\alpha \) is true in the theory \( T, T \models A_\alpha \) if it is true in the degree 1 in all its models.

By a straightforward verification we can prove that interpretation of the connectives \( \land, \land, \lor, \lor, \rightarrow, \Delta, \equiv \) in \( \mathcal{M}^* \) is equal to the corresponding operations \( \sim, \land, \land, \lor, \lor, \rightarrow, \Delta, \equiv \) introduced in Sections 4.2 and 4.3.

**Lemma 4.2.**

(a) \( \mathcal{M}_p((\forall x_\alpha)A_\alpha) = \left\{ \{ \mathcal{M}_p(A_\alpha) \mid p' = p\setminus x_\alpha, p'(x_\alpha) \in M_\alpha^* \right\} \),

(b) \( \mathcal{M}_p((\exists x_\alpha)A_\alpha) = \left\{ \{ \mathcal{M}_p(A_\alpha) \mid p' \neq p\setminus x_\alpha \in M_\alpha^* \cup \mathcal{M}_p(A_\alpha) \neq *_\alpha \right\} \),

\( \mathcal{M}_p((\exists x_\alpha)A_\alpha) = \left\{ \mathcal{M}_p(A_\alpha) \mid p' = p\setminus x_\alpha, p'(x_\alpha) \in M_\alpha^* \cup \mathcal{M}_p(A_\alpha) \neq *_\alpha \rightarrow a \} \)

**Proof.** Ad (a): \( \mathcal{M}_p((\forall x_\alpha)A_\alpha) = \mathcal{M}_p(\lambda x_\alpha A_\alpha) \equiv \lambda x_\alpha \top = \bigwedge \{ \mathcal{M}_p(A_\alpha) \mid p' = p\setminus x_\alpha, p'(x_\alpha) \in M_\alpha^* \} \)

\( \mathcal{M}_p((\exists x_\alpha)A_\alpha) = \bigwedge \{ a \in E \mid \mathcal{M}_p(A_\alpha) \rightarrow a \} \)

which is (b), because if \( \mathcal{M}_p(A_\alpha) = *_\alpha \) for some \( p' = p\setminus x_\alpha \) then the implication \( \mathcal{M}_p(A_\alpha) \rightarrow a = 1 \) for arbitrary \( a \in E \), and so, it is ignored by the corresponding infimum operation, and if \( \mathcal{M}_p(A_\alpha) = *_\alpha \) for all assignments \( p' \) then interpretation of the existential quantifier reduces to \( \bigwedge \{ a \mid a \in E \} = 0 \).

5 Canonical model of partial FTT

The canonical model of EQ-FTT with partial functions can be obtained by extension of the canonical model for the basic FTT (see [4, 3]). We will denote it by \( \mathcal{T}^p \). In the same way as in FTT, we define a special function \( \mathcal{Y} \), whose domain and range are equivalence classes of formulas.

5.1 Extension of theories

Let \( T \) be a theory and \( \sim \) be the canonical equivalence on formulas defined in the standard way: \( A_\alpha \approx B_\alpha \) iff \( T \vdash A_\alpha \equiv B_\alpha \), \( \alpha \in \text{Types} \). The corresponding equivalence class is denoted by \( |A_\alpha| \).

**Definition 5.1.** Let \( T \) be a theory. We say that:

(i) \( T \) is maximally consistent if each its extension \( T', T' \supset T \) is inconsistent.

(ii) \( T \) is linear if for every two formulas \( A_\alpha, B_\alpha \) such that \( T \vdash ! A_\alpha \) and \( T \vdash ! B_\alpha \) or \( T \vdash B_\alpha \rightarrow A_\alpha \).

(iii) \( T \) is extensionally complete if for every closed formula of the form \( A_{\beta\alpha} \equiv B_{\beta\alpha} \), \( T \not\vdash A_{\beta\alpha} \equiv B_{\beta\alpha} \) it follows that there is a closed formula \( C_\alpha \) such that \( T \vdash ! C_\alpha, T \not\vdash A_{\beta\alpha}C_\alpha \equiv B_{\beta\alpha}C_\alpha \).

**Theorem 5.2 ([3]).** Every consistent theory \( T \) can be extended to a maximally consistent linear theory.

\(^3\)Recall that the description operator represents, in fact, a defuzzification operation (cf. [4], Chapter 3).
Now we define the canonical frame.

**Theorem 5.3.** Every consistent theory \( T \) can be extended to an extensionally complete consistent theory \( T \).

**Proof.** The proof is obtained by small modification of the proof of \([8] \text{ Theorem 12}\). Namely, we have to confine to defined formulas. Let us denote \( \pi = \text{Card}(J(T)) \). Let \( K_{\pi} \) be a well ordered set of new constants of type \( \alpha \in \text{Types} \). \( \text{Card}(K_{\pi}) \leq \pi \) and put \( K = \bigcup_{\alpha \in \text{Types}} K_{\alpha} \) and \( J^+(T) = J(T) \cup K \). Therefore, we will consider closed formulas \( A_{\alpha}, B_{\alpha} \) of the language \( J(T) \) such that \( T \vdash !A_{\alpha} \wedge !B_{\alpha} \). If \( T \vdash !A_{\alpha} \wedge !B_{\alpha} \) and enumerate them by ordinal numbers. Theorem 12(e) then assures us that \( T \vdash ![A_{\alpha} \equiv B_{\alpha}] \).

In the proof, we construct a sequence of theories \( T_{\mu}, \mu \leq \pi \) and a sequence of special sets of formulas \( \Psi_{\mu} \), such that:

(a) \( T_0 = T \) and \( T_{\mu} \subset T_{\nu} \) for \( \nu < \mu \),

(b) \( \Psi_{\mu} \subset \Psi_{\mu} \) and \( T_{\mu} \not\models \psi_{n} \).

The construction proceeds by transfinite recursion. We suppose that \( T_{\mu} = \bigcup_{\nu < \mu} T_{\nu} \) and \( \Psi_{\mu} = \bigcup_{\nu < \mu} \Psi_{\nu} \) are already constructed.

Let \( D \in \Psi_{\mu}. \) Furthermore, let \( A_{\beta_{\alpha}} \equiv B_{\beta_{\alpha}} \) be the first not yet processed formula of this form, and let \( c_{\alpha} \in K \) be the first not yet used constant (both in the given well ordering). We distinguish two cases:

(i) Let \( T_{\mu} \models D \cap (A_{\beta_{\alpha}} \equiv B_{\beta_{\alpha}}) \). Then we put \( \Psi_{\mu} = \Psi_{\mu} \cup \{A_{\beta_{\alpha}} \equiv B_{\beta_{\alpha}}\} \) and put \( T_{\mu} = T_{\mu} \cup \{A_{\beta_{\alpha}} \equiv B_{\beta_{\alpha}}\} \).

(ii) Let \( T_{\mu} \not\models D \cap (A_{\beta_{\alpha}} \equiv B_{\beta_{\alpha}}) \). Then we put \( \Psi_{\mu} = \Psi_{\mu} \cup \{D \cap (A_{\beta_{\alpha}} \equiv B_{\beta_{\alpha}}) \mid D \in \Psi_{\mu}\} \) and \( T_{\mu} = T_{\mu} \).

Now we must show that in both cases \( T_{\mu} \not\models \psi_{n} \). This done analogously as in \([8]\). In case (ii), we must show that the assumption \( T_{\mu} \models D \cap (A_{\beta_{\alpha}} \equiv B_{\beta_{\alpha}}) \) leads to contradiction; indeed, replacing \( c_{\alpha} \) by a variable not occurring in \( D \) and \( A_{\beta_{\alpha}}, B_{\beta_{\alpha}} \) and further, using (FT-quant3) and Lemma \( (\text{A}0\text{A}) \) we obtain \( T_{\mu} \models D \cap (A_{\beta_{\alpha}} \equiv B_{\beta_{\alpha}}) \).

To show, finally, that \( T_{\mu} \) is extensionally complete, let \( A_{\beta_{\alpha}} \equiv B_{\beta_{\alpha}} \) be a formula processed in step \( \mu \) and \( T_{\mu} \not\models A_{\beta_{\alpha}} \equiv B_{\beta_{\alpha}} \). Then it had to be processed as the case (ii), i.e. \( T_{\mu} \not\models D \cap (A_{\beta_{\alpha}} \equiv B_{\beta_{\alpha}}) \), otherwise we get a contradiction. But the previous reasoning, together with Theorem \( \text{A}1\text{A}(\text{b}) \), leads to \( T_{\mu} \not\models (A_{\beta_{\alpha}} \equiv B_{\beta_{\alpha}}) \) for some constant \( c_{\alpha} \).

\( \square \)

### 5.2 Extended canonical algebra of truth values

Now we will put \( T^*E = \{A \mid A \in \text{Form}_{\pi}, T \vdash !A\} \), and \( T^*E^* = T^*E \cup \{*o\} \), where \( *o \) is defined in \((2)\). It follows from the definition of “\( \approx \)” that \( |*o| = \{A \mid A \in \text{Form}_{\pi}, T \vdash ?A\} \).

**Theorem 5.4.** Let \( T \) be a consistent linear extensionally complete theory. Then the algebra \( T^*E^* = (T^*E^*, T^*E \cap \text{Card}(T^*\pi), T^*\gamma, T^*\Delta, T^*I) \) is an extended linearly ordered good \( E\text{Q}\Delta\text{-algebra}. \)

**Proof.** We must prove that the operations behave in accordance with the extended operations due to Subsection \( \text{A}2\text{A} \). The proof of the standard operations on \( T^*E \) and linearity of the algebra \( T^*E \) is the same as in \([8]\). Let us now check the extended operations due to Axioms (EX1)–(EX4). First note the following: if \( T \vdash !A_0 \) then \( A_0 \in \{*o\} \).

On the other hand, if \( T \vdash !A_0 \) then \( |A_0| \in \{*o\} \). Indeed, otherwise \( A_0 \in \{*o\} \) which would mean that \( T \vdash A_0 \equiv *o \) and, consequently, \( T \) would be contradictory. We thus conclude that \( T^*E^* \) is not degenerated.

Then the extended operations in \( T^*E^* \) due to Corollary \( \text{A}2\text{A} \) are obtained as follows:

(i) \( |A_0| \cap \{*o\} = |A_0| \cap |A_1| \) and \( |A_0| \cap |A_1| \) is closed on \( T^*E \). Furthermore, \( |A_0| \) is closed on \( T^*E \). Indeed, \( |A_0| \models T^*I \) by Axiom (FT-val6). Finally, \( T \vdash !A_0 \) implies \( T \vdash (A_0 \equiv *o) \) which gives \( |A_0| \models \{*o\} \) for all \( |A_0| \in T^*E \).

(ii) \( |A_0| \models \{*o\} \) by Axiom (FT-B3), where \( \models \in \{\wedge, \vee\} \).

(iii) \( T^*\Delta \models |A_0| \models \{*o\} \) by Axiom (FT-B4).

\( \square \)

### 5.3 Canonical frame and completeness

Now we define the canonical frame.

**Definition 5.5.** Let \( T \) be a consistent, linear and extensionally complete theory. Then the canonical frame is \( \mathcal{M}^* = (\{M_{\alpha}^*, \xi_{\alpha}\}_{\alpha \in \text{Types}}, \ \text{Type}_{\alpha}, I_{\alpha}, I_{\xi}) \) where:

1. \( M_{\alpha}^* = \{A(x) \mid A \in \text{Form}_{\pi}, A \text{ closed}, \alpha \in \text{Types}\} \) and \( \alpha \in \text{Types} \). The construction of \( \gamma \) proceeds inductively:
   
   (i) If \( \alpha = \alpha \) then \( A(x) \models A(x) \) and \( \gamma(*o) = |*o| = |\lambda x o \| \).

   (ii) If \( \alpha = \epsilon \) then \( A(x) \models A(x) \models A_{\pi} \models A_{\gamma} \models |*o| = |\lambda x o \| \).

   (iii) If \( \alpha = \beta \gamma \) then we put \( \gamma(A_{\beta \gamma}) \subseteq M_{\gamma} \times M_{\beta} \) which is a relation consisting of couples \( \gamma(A_{\beta \gamma}) \subseteq M_{\beta} \cap \text{Form}_{\gamma} \). As a special case, \( \gamma(* \gamma) = \gamma(* \gamma) \subseteq M_{\beta} \cap \text{Form}_{\gamma} \).
(2) The $\mathcal{D}$ is the extended EQ-$\Delta$-algebra from Theorem 4.6.

(3) Let $T \vdash A_{\infty}$ and $\forall(A_{\infty}) = \{ (\forall(B_o) \text{ and } \forall(A_{\infty}B_o)) \mid \forall(B_o) \in M^*_o \}$ be a fuzzy set on $M^*_o$. Then we put

$$I(\forall(A_{\infty})) = \begin{cases} |\epsilon_{s_{\infty}}A_{\infty}|, & \text{ if } \vdash (\exists x_o)(A_{\infty}x_o), \\ \forall(s_o), & \text{ if } \vdash (\forall x_o)(\neg \Delta(A_{\infty}x_o \equiv \top)). \end{cases}$$

Similarly for a fuzzy set $\forall(A_{\infty})$ on $M^*_o$ we put

$$I(\forall(A_{\infty})) = \begin{cases} |\epsilon_{s_{\infty}}A_{\infty}|, & \text{ if } \vdash (\exists x_o)(A_{\infty}x_o), \\ \forall(s_o), & \text{ if } \vdash (\forall x_o)(\neg \Delta(A_{\infty}x_o \equiv \top)). \end{cases}$$

Finally we will prove the theorem saying that every consistent theory of FTT with partial functions has a model.

**Theorem 5.6** (Completeness of FTT with partial functions). *Let $T$ be a theory, the special axioms of which have the form $A_0 \equiv \top$. Then $T$ is consistent iff it has a general model $\mathcal{M}$.***

**Proof.** All axioms of FTT are defined by Corollary 4.6. Moreover, using Axiom (FT-B2') and Rule (R) we can prove that also $T \vdash A_0$, holds for all special axioms $A_0$ of $T$. Using Theorems 5.2 and 5.3 we extend $T$ into a linear and extensionally complete theory. Then this theorem follows from the construction of the canonical frame above using the same arguments as in [5]. \qed

## 6 Partial functions

In this section we will briefly touch the results introduced and analyzed by Lapierre and Lepage in [3, 6].

The following definitions characterize the basic concepts introduced in these papers:

(i) **Total function:** $\text{TotF}_{\epsilon_{\beta}\alpha} \equiv \lambda f_{\beta\alpha} \cdot (\forall x_o)((x_o \Rightarrow (f_{\beta\alpha}x_o)))$,

(ii) **Partial function:** $\text{PartF}_{\epsilon_{\beta}\alpha} \equiv \lambda f_{\beta\alpha} \cdot (\exists x_o)((x_o \& \neg(f_{\beta\alpha}x_o)))$,

(iii) **Strict function:** $\text{Str}tF_{\epsilon_{\beta}\alpha} \equiv \lambda f_{\beta\alpha} \cdot (\forall x_o)((x_o \Rightarrow (f_{\beta\alpha}x_o)))$,

(iv) **Non-strict function:** $\text{NStr}tF_{\epsilon_{\beta}\alpha} \equiv \lambda f_{\beta\alpha} \cdot (\exists x_o)(\neg(x_o \& \neg(f_{\beta\alpha}x_o)))$.

Note that all these predicates are crisp, i.e., any function is either total or partial, or either strict or non-strict.

We will also introduce a special ordering: $\prec_{\epsilon_{\alpha}\alpha} \equiv \lambda x_o \lambda y_o \cdot \neg x_o \lor \Delta(x_o \equiv y_o)$, and it can be proved that this relation is crisp and is indeed an ordering. On the basis of it, we can define monotonous function:

$$\text{MonF}_{\epsilon_{\beta}\alpha} \equiv \lambda f_{\beta\alpha} \cdot (\forall y_o)((\forall x_o)((x_o \prec y_o) \Rightarrow (f_{\beta\alpha}x_o \prec f_{\beta\alpha}y_o))).$$

(5)

Lapierre and Lepage gave many arguments in favor of the idea to consider all the functions to be monotones in the sense of (5).

**Lemma 6.1.** *Let $T$ be a consistent theory, $f_{\beta\alpha} \in \text{Form}_{\beta\alpha}$ be a formula such that $T \vdash \text{TotF}f_{\beta\alpha} \& \neg\text{NStr}tF f_{\beta\alpha}$ and, moreover, the following is provable: $T \vdash (\exists x_o)(\Delta(x_o \land \neg \Delta(f_{\beta\alpha}x_o \equiv f_{\beta\alpha}s_o)))$. Then $T \not\vdash \text{MonF} f_{\beta\alpha}$.*

According to this lemma, monotonicity excludes also improper functions having the properties from Lemma 6.1. This result is in accordance with the results of [6].

To obtain a special case of FTT in accordance with the papers [3, 6], we might extend the list of its axioms by the following one:

(FT-fund6) $(\forall f_{\beta\alpha}) \text{MonF} f_{\beta\alpha}, \quad \alpha, \beta \in \text{Types}.$

It seems, however, that this axiom should not belong among logical axioms of FTT and should be added only to special theories. The reason is that it would restrict FTT too much. For example, the operation $\equiv$ considered in Subsection 5.2 and in [6] could not be included because it is not monotonous.

## 7 Conclusions

In this paper, we studied possibilities how partial functions can be introduced into fuzzy type theory. We have chosen the most general kind of FTT based on EQ-algebra of truth degrees, see [6].

From several options how a special value “undefined” (denoted by $*$) can be introduced, we decided to represent it by an element laying outside of the given domain (for each specific type). In our construction, we used the fact that the description operator $\epsilon_{\alpha\alpha}$ gives no result when applied to formulas representing subnormal fuzzy sets. Hence, we defined $s_o$ as the formula $\epsilon_{s_{\infty}} \cdot \lambda x_o \perp$ and accordingly the other values “undefined”. This made it possible to have “undefined” inside the set Form of all the formulas without necessity to extend the language by new constant with special properties. Further research will be focused on formalization and deeper elaboration of the theory of partial functions using the means of the fuzzy type theory.
Acknowledgement

This paper was supported by the grant 16-19170S of GAČR, Czech Republic.

References


