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Comparing uncertainty data in epistemic and ontic sense used to decision making problem

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Abstract

In the paper aspect of comparability alternatives in decision making problem by imprecise or incomplete information is examined. In particular, new definitions of transitivity based on the measure of the intensity preference between pairs of alternatives in epistemic and ontic case is presented and its application to solve decision making problem is proposed.

Keywords: Epistemic/ontic order and transitivity, Selection alternatives, Interval-valued fuzzy relations.

1 Introduction

Many decision making processes take place in an environment, where the information is not precisely known. Then experts may feel more comfortable using an interval numbers rather than an exact crisp numerical values to represent their preferences. The interval values can be generated in the following circumstances: imprecise measurements, vague observations, values hidden deliberately or/and ranges of fluctuations.

In practical point of view, in information processing tasks, sets-intervals may have a disjunctive or a conjunctive reading ([7, 18]). In the first case the set is a disjunctive set (epistemic), i.e. set S contains an ill-known actual value of a point-valued quantity x, so we can write $x \in S$. It represents the epistemic state of an agent, hence does not exist per se. While sets representing collections of elements forming composite objects are be called conjunctive (ontic). A conjunctive set is the precise representation of an objective entity. An ontic set S is the value of a set-valued variable X, so we can write $X = S$. This distinction between what we call ontic vs. epistemic sets remains valid for sets of intervals, which represent uncertainty information. By imprecise or incomplete information present by intervals we have problem with comparability of interval values. Orders used for intervals and interval-valued fuzzy relations may be connected also with ontic and epistemic setting ([18, 16]). Mentioned comparability relations we may use to built adequate transitivity property of interval-valued fuzzy relations in epistemic, ontic or classic cases. This property is important because of its possible applications in the preference procedures. The final ranking of the alternatives must be based on consistent ruling as an inconsistent preference relation may lead to wrong conclusions. Usually, the consistency of a preference relation is characterized by transitivity. Therefore, the study of the transitivity of a preference relation is very important. Interval-valued fuzzy (reciprocal) preference relations (IVFRRs) can be considered as an appropriate representation format to capture experts' uncertain preference information. Because interval-valued fuzzy relations (IVFRs) ([46, 37, 24, 24, 25, 26, 27] or [38, 39]) as form a generalization of the concept of a fuzzy relation [45] are applied in group decision making [5, 31, 32, 41, 44]. The idea of a preference relation has been studied by many authors, not only in crisp or fuzzy environments [36, 6, 15], but also in the case of interval-valued fuzzy preference relations, have been studied by a range of authors [1, 31, 34, 42, 41, 44]. Especially, the assumption of reciprocity is often used for a preference relation in classical fuzzy environment [6] or their extensions [1, 43].

Our main goal is to examine certain aspect of decision making problem based on transitivity creating in epistemic and ontic cases by reciprocity property built by negation function, which means that instead of using classical negation

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in definition of reciprocity, we apply negation. Reciprocity appears in preference relations as a natural assumption. We will discuss the problem of comparability of intensities of preference among pairs of alternatives by defined relation in necessary (ontic) and possibly (epistemic) cases, which generate adequate transitivity property. This transitivity property of interval-valued fuzzy relations is used in decision making model. Thus these properties are important because of their possible applications in the preference procedures. Some papers connected with the mentioned problem through quaternary relations are ([20, 21] or [30]).

This work is composed of the following parts. Firstly, some concepts and results connected with comparability of interval values are presented (section II). Next, results of reciprocity property and measure of intensity of preference is presented (section III). Moreover, new transitivity properties are examined (section IV). Finally, an example of use of transitivity for the selection of alternatives has been shown (section V).

2 Comparability of interval values

Now, we consider set of intervals and in this set of intervals we compare its elements. We consider comparability of interval values in epistemic and ontic point of view and we recall comparability method in the classic case.

Firstly, we recall the notion of the family of interval values $L^I = \{[\underline{x}, \overline{x}] : \underline{x}, \overline{x} \in [0,1], \ \underline{x} \leq \overline{x}\}$, where are the top element given by $\mathbf{1} = [1,1]$ and the bottom element given by $\mathbf{0} = [0,0]$ and the union and intersection of any two elements is defined by $[\underline{x}, \overline{x}] \vee [\underline{y}, \overline{y}] = [\max(\underline{x}, \underline{y}), \max(\overline{x}, \overline{y})]$ and $[\underline{x}, \overline{x}] \wedge [\underline{y}, \overline{y}] = [\min(\underline{x}, \underline{y}), \min(\overline{x}, \overline{y})]$. We present following kind of comparability relations:

2.1 Ontic. Necessary relation

We define the following restricted case of comparability intervals, i.e. necessary relation, which we may interpreted as conjunctive (ontic) relation and present that in one interval is collection of true values of each variable smaller then or equal to all true values from second interval. If $x, y \in L^I, x^* \in x$ and $y^* \in y$, then $\forall_{x^*} \forall_{y^*} x \preceq_\nu y \Leftrightarrow \overline{x} \leq \underline{y}$, (1)

2.2 Epistemic. Possible relation

Possibility relation describes more general situation, which we may write $\exists_{x^*} \exists_{y^*} x \preceq_\pi y \Leftrightarrow \underline{x} \leq \overline{y}$, (2)
where $x, y \in L^I, x^* \in x$ and $y^* \in y$. Relation $\preceq_\pi$ is more suitable for the epistemic setting of the interval-valued fuzzy relations. So, if $[\underline{x}, \overline{x}]$ is an unprecise description of a variable x and $[\underline{y}, \overline{y}]$ is an unprecise description of a variable y, then $[\underline{x}, \overline{x}] \preceq_\pi [\underline{y}, \overline{y}]$ means that it is possible that the true value of x is smaller than or equal to the true value of y. The relation $\preceq_\pi$ thus has a possibility interpretation [17]. Properties of presented comparability relations were considered and presented in [35], i.e $\preceq_\pi$ is an interval order (complete and has Ferrers property) and $\preceq_\nu$ is antisymmetric and transitive in L^I. Moreover, structures $(IVFR(X), \preceq_\pi)$ and $(IVFR(X), \preceq_\nu)$ were examined.

2.3 Classic relation

Usually, in many papers (for example [2, 8, 12, 13, 33, 40]) devoted to intervals the following relation is called natural or classic order.

$$x \preceq y \Leftrightarrow \underline{x} \leq \underline{y} \ \text{ and } \ \overline{x} \leq \overline{y}, \tag{3}$$

where $x, y \in L^I$. The structure $(L^I, \preceq)$ is a complete bounded lattice. Moreover, we observe the following connections between the mentioned comparability relations:

Proposition 2.1. $x \preceq_\nu y \Rightarrow x \preceq y \Rightarrow x \preceq_\pi y$.

So considerations of necessary and possible order gives wider outlook than the description of the situation by classical order.

3 Interval-valued fuzzy relations

In this section we will recall the notion of interval-valued fuzzy relation, and we will focus on the specific case of reciprocal relations of preference relation defined on the basis of negation and interpretation of intensity of preference between pair of alternatives in epistemic and ontic cases.

Definition 3.1. (cf. [37, 46]) *An interval-valued fuzzy relation (IVFR) R between universes X, Y is a mapping $R: X \times Y \to L^I$ such that $R(x, y) = [\underline{R}(x, y), \overline{R}(x, y)]$, for all pairs $(x, y) \in X \times Y$. The class of all IVFRs between universes X, Y is denoted by $IVFR(X \times Y)$, or $IVFR(X)$ for $X = Y$.*

An approach that adds flexibility to represent uncertainty in decision making problems consists of using interval-valued fuzzy relations ([24], [39], [42]). An interval-valued fuzzy preference relation R on X is interpreted as an interval-valued fuzzy subset of $X \times X$, that is, $R : X \times X \to L^I$. The interval $R(x_i, x_j) = r_{ij} = [\underline{r}_{ij}, \overline{r}_{ij}]$ denotes the degree to which elements x_i and x_j are related (representing the degree of preference of x_i over x_j) in the relation R for all x_i, $x_j \in X$. We may consider properties of interval-valued fuzzy relation and their dependencies some of which we already knew ([22]), but in this paper we will use their generalization by mentioned comparability relations. For classical comparability relation we can find some facts for example in [2]. First considerations of properties of interval-valued fuzzy relations connected with possible and necessary comparability relations are in [35]. Possible and necessary property (π and ν property) we define in the following way, Relation $R \in IVFR(X)$ has
(i) possible property P (pos-property) if there exists at least one instance R^* of R that has property P.
(ii) necessary property P (nec-property) if every instance R^* of R it has property P.

Now, we introduce the crucial definition for preference relations, i.e. reciprocity property based on negation. It is why, we recall definition of interval-valued (IV) negation.

Definition 3.2. [1] *An interval-valued (IV) negation is a function $N_{IV} : L^I \to L^I$ that is decreasing with respect to $\preceq$ with $N_{IV}(\mathbf{1}) = \mathbf{0}$ and $N_{IV}(\mathbf{0}) = \mathbf{1}$. An IV negation is said to be involutive if it fulfils $N_{IV}(N_{IV}(x)) = x$ for any $x \in L^I$.*

The important result links involutive IV negations (representable IV negation $N_{IV} = (N, N)$) with strong (involutive) fuzzy negations N:

Theorem 3.3. [11] *N_{IV} is an involutive IV negation if and only of there exists a strong (involutive) fuzzy negation N such that $N_{IV}([\underline{x}, \overline{x}]) = [N(\overline{x}), N(\underline{x})]$.*

Now, we can define general reciprocity property by using representable IV negation $N_{IV} = (N, N)$:

Definition 3.4. [34] *An Interval-Valued Fuzzy Reciprocal Relation (IVFRR) R on the set X is a matrix $R = (r_{ij})_{n \times n}$ with $r_{ij} = [\underline{R}(i, j), \overline{R}(i, j)]$, for all $i, j \in \{1, \ldots, n\}$, where $r_{ij} \in L^I$, $r_{ii} = [0.5, 0.5]$, $r_{ji} = N_{IV}(r_{ij}) = [N(\overline{R}(i, j)), N(\underline{R}(i, j))]$, for $i \neq j$, where N is a fuzzy negation, N_{IV} is an IV negation and $card(X) = n$.*

This is the reciprocity property based on negation and is a generalization of the reciprocity property introduced in [42], where N was a standard negation. The assumption $r_{ji} = 1 - r_{ij}$ for $i, j \in \{1, \ldots, n\}$, which stems from the reciprocity property, is rather strong and frequently violated by decision makers in real-life situations. This is why we use a fuzzy negation instead of the classical negation $N(x) = 1 - x$. Especially, if $\overline{R}(i, j) = \underline{R}(i, j) = r_{ij}$ for $i, j \in \{1, \ldots, n\}$, then an IVFRR reduces to a fuzzy reciprocal relation (it is also worth mentioning that IVFRRs may be built from the fuzzy ones using the concept of ignorance function [1]). The interval r_{ij} indicates the interval-valued reciprocal degree or intensity of the alternative x_i over alternative x_j and $\underline{R}(i, j)$, $\overline{R}(i, j)$ are the lower and upper limits of r_{ij}, respectively. Moreover, we may consider a relation measure intensity of preferences between pairs of alternatives in set X, $card(X) = n, n \in \mathbf{N}$. Some results we see in ([20], [21] or [28]). If we use mentioned comparability relations connected with epistemic and ontic settings, then we obtain the following interpretations of intensity of preferences between pair of alternatives:
1. Necessary preference relation $\preceq_\nu$ on X^2 defined as, $(x_i, x_j) \preceq_\nu (x_k, x_l) \Leftrightarrow R(i, j) \preceq_\nu R(k, l)$ means that the intensity of preferences between x_k and x_l is greater than the intensity of preferences between x_i and x_j for each values of $R(i, j)$ and $R(k, l)$, $i, j, k, l \in \{1, ..., n\}$;
2. Possible preference relation $\preceq_\pi$ on X^2 defined as, $(x_i, x_j) \preceq_\pi (x_k, x_l) \Leftrightarrow R(i, j) \preceq_\pi R(k, l)$, means that the intensity of preferences between x_k and x_l is greater than the intensity of preferences between x_i and x_j for at least one values of $R(i, j)$ and $R(k, l)$, $i, j, k, l \in \{1, ..., n\}$. By (1) and (2) we can write as following:
1. $(x_i, x_j) \preceq_\nu (x_k, x_l) \Leftrightarrow \overline{R}(i, j) \leq \underline{R}(k, l)$,
2. $(x_i, x_j) \preceq_\pi (x_k, x_l) \Leftrightarrow \underline{R}(i, j) \leq \overline{R}(k, l)$.

By additional reciprocity property we obtain

Proposition 3.5. *If R is the reciprocal relation created by involutive negation N on set of alternatives X, then*
1. $(x_i, x_j) \preceq_\nu (x_k, x_l) \Rightarrow (x_l, x_k) \preceq_\nu (x_j, x_i)$,
2. $(x_i, x_j) \preceq_\pi (x_k, x_l) \Rightarrow (x_l, x_k) \preceq_\pi (x_j, x_i)$.

Proof. If $(x_i, x_j) \preceq_\nu (x_k, x_l)$, then $\overline{R}(i, j) \leq \underline{R}(k, l)$ and by reciprocity we have $\overline{R}(i, j) = N(\underline{R}(j, i)), \overline{R}(k, l) = N(\underline{R}(l, k))$. Thus by involutive negation we obtain $N(\underline{R}(j, i)) \leq N(\underline{R}(l, k)) \Leftrightarrow \underline{R}(j, i) \geq \underline{R}(l, k) \Leftrightarrow (x_l, x_k) \preceq_\nu (x_j, x_i)$. Similarly, we can prove the second condition. □

Now, we consider preference between single alternatives. Then based on the necessary preference relation on X^2 we can define a "strictly preferred" relation on X. If we consider (x_i, x_j) with smaller intensity of preference to (x_j, x_i) we obtain $x_i \preceq'_\nu x_j \Leftrightarrow \overline{R}(i,j) \le \underline{R}(j,i)$. The relation $\preceq'_\nu$ means that x_j is strictly preferred over x_i ($r_{ij} = [1,1]$ means that x_i is absolutely preferred over x_j).

Similarly, based on the possible preference relation we can define a "weakly preferred" relation on X, $x_i \preceq'_\pi x_j$ if and only if $\underline{R}(i,j) \le \overline{R}(j,i)$. Indifference between x_i and x_j we obtain in following cases:

$$x_i \sim x_j \quad \text{if} \quad r_{ij} = [0.5, 0.5] \text{ or } \ (\textit{neither} \ \ x_i \preceq'_\nu x_j \text{ or } \ x_j \preceq'_\nu x_i) \ \text{ or } \ (x_i \preceq'_\pi x_j \text{ and } \ x_j \preceq'_\pi x_i).$$

If we have a group of experts and we present their preferences by adequate relations. Then we also can compare intensity of preference of the same pairs of alternatives by different experts. For example, for two experts we obtain:
1. $R_1(i,j) \preceq_\nu R_2(i,j) \Leftrightarrow \overline{R}_1(i,j) \le \underline{R}_2(i,j)$,
2. $R_1(i,j) \preceq_\pi R_2(i,j) \Leftrightarrow \underline{R}_1(i,j) \le \overline{R}_2(i,j)$
it means that second expert with greater intensity prefers x_i over x_j, than first expert in necessary (1), possible (2) sense, respectively. From practical point of view more interesting will be for us the necessary preference relation, because the possible preference relation is lack of an antisymmetricity, which is important for ranking of alternatives. But we use both to create the transitivity property and later to model the decision making problem. In the mentioned model we will use aggregation functions, so we recall definition of them.

Definition 3.6. (cf. [4], [29]) *An operation $\mathcal{A} : (L^I)^n \to L^I$ is called an aggregation function if it is increasing with respect to the order $\preceq$ and $\mathcal{A}(\underbrace{\mathbf{0}, ..., \mathbf{0}}_{n\times}) = \mathbf{0}$ and $\mathcal{A}(\underbrace{\mathbf{1}, ..., \mathbf{1}}_{n\times}) = \mathbf{1}$.*

Special class of aggregation functions is representable aggregation functions.

Definition 3.7. (cf. [10], [14]) *Let $\mathcal{A} : (L^I)^2 \to L^I$ be an aggregation function. $\mathcal{A}$ is said to be a representable aggregation function if there exist two (real) aggregation functions $A_1, A_2 : [0,1]^2 \to [0,1]$, $A_1 \le A_2$ such that, for every $[x_1, x_2], [y_1, y_2] \in L^I$ it holds that $\mathcal{A}([x_1, x_2], [y_1, y_2]) = [A_1(x_1, y_1), A_2(x_2, y_2)]$.*

Operations $\wedge$ and $\vee$ in L^I are also representable aggregation functions in L^I, with $A_1 = A_2 = \min$ in the first case and $A_1 = A_2 = \max$ in the second. Moreover, we can observe other examples, such as:
(i) $\mathcal{A}_P([x_1, x_2], [y_1, y_2]) = [x_1 y_1, x_2 y_2]$ (the representable product),
(ii) $\mathcal{A}_Q([x_1, x_2], [y_1, y_2]) = [x_1 + y_1 - x_1 y_1, x_2 + y_2 - x_2 y_2]$ (the representable arithmetic product),
(iii) $\mathcal{A}_{mean}([x_1, x_2], [y_1, y_2]) = [\frac{x_1+y_1}{2}, \frac{x_2+y_2}{2}]$ the representable arithmetic mean,
(iv) $\mathcal{A}_{wmean}([x_1, x_2], [y_1, y_2]) = [w_1 x_1 + w_2 y_1, w_1 x_2 + w_2 y_2]$ (the representable weighted mean with $w_1 + w_2 = 1$, $w_1, w_2 \in [0,1]$),
(v) $\mathcal{A}_g([x_1, x_2], [y_1, y_2]) = [\sqrt{x_1 y_1}, \sqrt{x_2 y_2}]$ (the representable geometric mean),
(vi) $\mathcal{A}_{wg}([x_1, x_2], [y_1, y_2]) = [x_1^{w_1} y_1^{w_2}, x_2^{w_1} y_2^{w_2}]$ (the representable weighted geometric mean with $w_1 + w_2 = 1$, $w_1, w_2 \in [0,1]$),
(vii) $\mathcal{A}_{P,mean}([x_1, x_2], [y_1, y_2]) = [x_1 y_1, \frac{x_2+y_2}{2}]$ (the representable product-mean).

Let us look at representability is not the only possible way to build interval-valued aggregation functions in L^I. But the representable aggregations are often used in applications.

The following result characterizes representable aggregation functions which preserve reciprocity in group of considered relations.

Theorem 3.8. [34] *If N is a strong fuzzy negation, $\mathcal{A} : (L^I)^2 \to L^I$ is a representable aggregation function with $\mathcal{A}([x_1, x_2], [y_1, y_2]) = [A_1(x_1, y_1), A_2(x_2, y_2)]$ for some aggregation functions $A_1, A_2 : [0,1]^2 \to [0,1]$. Then $\mathcal{A}$ preserves reciprocity if and only if $A_1 = A_2^N$, $A_1(0.5, 0.5) = A_2(0.5, 0.5) = 0.5$, where $A_2^N(x,y) = N(A_2(N(x), N(y)))$, $x, y \in [0,1]$.*

So, let us notice that in the proof of Theorem 3.8 we do not use any of the assumptions of the aggregation function. As a result the above theorem is true for any representable (decomposable [14]) function in L^I. As a consequence, since weighted means and the arithmetic mean are self-dual aggregation functions then they preserve reciprocity property by the classical fuzzy negations.

4 Transitivity property

As main aspect in this paper, transitivity property of interval-valued fuzzy relations is examined. This property is important because of its possible applications in the preference procedures. The transitivity assumption can be used to check for the judgmental consistency of the group decision making in the sense that if an alternative x_1 is preferred to or equivalent to alternative x_2, and x_2 is preferred to or equivalent to alternative x_3, then x_1 must be preferred to or equivalent to x_3. Therefore, the study of the transitivity of a preference relation is very important.

We may define transitivity in different way, by each our comparability relation. If we use comparability relations in presented earlier epistemic and ontic cases, then we obtain following adequate definitions of transitivity. The "strictly preferred" relation $\preceq_\nu$ naturally generate ν-transitivity (nec-transitivity) property and "weakly preferred" $\preceq_\pi$ naturally generate π-transitivity (pos-transitivity) property in the following form

Definition 4.1. (cf. [8, 35]) *Let $A, A_1, A_2 : [0,1]^2 \to [0,1]$ be aggregation functions. A relation $R \in IVFR(X)$ is*
(i) π-A-transitive, if $A(\underline{R}(x,y), \underline{R}(y,z)) \le \overline{R}(x,z)$.
(ii) ν-A-transitive, if $A(\overline{R}(x,y), \overline{R}(y,z)) \le \underline{R}(x,z)$.
(iii) $\mathcal{A}$-transitive, if $A_1(\underline{R}(x,y), \underline{R}(y,z)) \le \underline{R}(x,z), A_2(\overline{R}(x,y), \overline{R}(y,z)) \le \overline{R}(x,z)$,
where $\mathcal{A} = [A_1, A_2]$.

As a results we have

Proposition 4.2. *Let $A, A_1, A_2 : [0,1]^2 \to [0,1]$ be aggregation functions and $R \in IVFR(X)$.*
(1) If $\underline{R}$ is A-transitive, then R is π-A-transitive.
(2) If R is $\mathcal{A}$-transitive, then R is π-A_1-transitive.
(3) If $\mathcal{A} = [A_1, A_2]$, $A_1 \le A_2$, A_1 is isotonic and R is ν -A_2 transitive, then is π-A_1-transitive and $\mathcal{A}$-transitive.

Proof. We prove the last case. If R is A_2 transitive, i.e. $A_2(\overline{R}(x,y), \overline{R}(y,z)) \le \underline{R}(x,z)$, then by $A_1 \le A_2$ we have

$$A_1(\underline{R}(x,y), \underline{R}(y,z)) \le A_1(\overline{R}(x,y), \overline{R}(y,z)) \le A_2(\overline{R}(x,y), \overline{R}(y,z)) \le \underline{R}(x,z) \le \overline{R}(x,z).$$

So R is $\mathcal{A}$-transitive. □

For two operations, one less than or equal to other, transitivity by bigger operation implies transitivity by smaller operation.

Theorem 4.3. *Let $A_1, A_2 : [0,1]^2 \to [0,1]$ and $A_1 \le A_2$ be aggregation functions. If $R \in IVFR(X)$ is π-A_2-transitive (ν-A_2-transitive), then R is π-A_1-transitive (ν-A_1-transitive).*

Proof. By π-A_2-transitivity of R we have, $A_2(\underline{R}(x,y), \underline{R}(y,z)) \le \overline{R}(x,z)$. If $A_1 \le A_2$, then by isotonicity of aggregations we obtain, $A_1(\underline{R}(x,y), \underline{R}(y,z)) \le A_2(\underline{R}(x,y), \underline{R}(y,z))$. So $A_1(\underline{R}(x,y), \underline{R}(y,z)) \le \overline{R}(x,z)$, i.e. R is is π-A_1-transitive. By similar way we may prove ν-A_1-transitivity. □

Directly from definition we observe the following implications:
(i) $\nu - A$-transitivity of $R \Rightarrow A$-transitivity of $\overline{R}$.
(ii) A-transitivity of $\underline{R} \Rightarrow \pi - A$-transitivity of R.

Furthermore, the converse relation, which may be interested in point of view of decision making model, preserves transitivity property.

Proposition 4.4. *If A is the commutative aggregation function, then $R \in IVFR(X)$ is $\pi - A$-transitive ($\nu - A$-transitive) if and only if R^t is $\pi - A$-transitive ($\nu - A$-transitive), where $R^t(x,y) = R(y,x)$ for $x,y \in X$.*

Proof. Let R be $\pi - A$-transitive. For $x,y,z \in X$ by commutativity of A we have

$$A(\underline{R}^t(x,y), \underline{R}^t(y,z)) = A(\underline{R}(y,x), \underline{R}(z,y)) = A(\underline{R}(z,y), \underline{R}(y,x)) \le \overline{R}(z,x) = \overline{R}^t(x,z).$$

Thus R^t is $\pi - A$-transitive. Similarly we may prove for $\nu - A$-transitive property. □

Aggregation behaves similarly to the converse relation and also preserve transitivity but by adequate assumption. We will need to use the concept of domination, so A dominates B ($A \gg B$), if $A(B(x,y), B(z,t)) \ge B(A(x,z), A(y,t))$.

Proposition 4.5. *Let A, A_1, A_2 be aggregation functions, such that $A_2 \gg A$, $A_1 \gg A$ and $R_i \in IVFR(X)$, $i = \{1,...,n\}$, $n \in \mathbf{N}$.*
1. If (R_i) is a family of $\nu - A$-transitive relations, then $\mathcal{A}(R_i)$ is $\nu - A$-transitive for $\mathcal{A} = [A_2, A_2]$.
2. If (R_i) is a family of $\pi - A$-transitive relations, then $\mathcal{A}(R_i)$ is $\pi - A$-transitive for $\mathcal{A} = [A_1, A_1]$.

Proof. If A_2 dominates A and (R_i) is a family of $\nu - A$-transitive relations then

$$A(A_2(\overline{R}_1(x,y),...,\overline{R}_n(x,y)),A_2(\overline{R}_1(y,z),...,\overline{R}_n)(y,z)) \leq A_2(A(\overline{R}_1)(x,y),\overline{R}_1(y,z)),...,A(\overline{R}_1)(x,y),\overline{R}_n)(y,z))) \leq A_2(\underline{R}_i(x,z)).$$

The proof of $\pi - A$-transitivity is similar. What finished the proof. □

For selection of alternatives in decision making problem we will use the $\nu - A$-transitivity and $\pi - A$-transitivity properties of reciprocal relation. Because the reciprocal relation often is not transitive, thus we present algorithm built by $\nu - A$-transitivity and $\pi - A$-transitivity relations for given reciprocal relation R. After being inspired by the algorithm the Floyd-Warshall of compute the T-transitivity closure, we propose:

Procedure ν-*A-transitivity*:
Inputs: $X = \{x_1, \ldots, x_n\}$ set of alternatives, $i,j,k \in \{1,...,n\}$; Aggregation function $A \leq min$; R interval-valued fuzzy reciprocal relation;
Output: $\nu - A$-transitive interval-valued fuzzy relation R_ν^A;
For all i For all j For all k
If $A(\overline{R}(i,k),\overline{R}(k,j)) \geq \underline{R}(i,j)$, then
Begin $\underline{R}(i,j) := A(\overline{R}(i,k),\overline{R}(k,j))$
$\overline{R}(i,j) = max(A(\overline{R}(i,k),\overline{R}(k,j)),\overline{R}(i,j))$ END
ENDFOR ENDFOR ENDFOR
End.
Procedure π-*A-transitivity*:
Inputs: $X = \{x_1, \ldots, x_n\}$ set of alternatives, $i,j,k \in \{1,...,n\}$; Aggregation function A; R interval-valued fuzzy reciprocal relation;
Output: $\pi - A$-transitive interval-valued fuzzy relation R_π^A;
For all i For all j For all k
If $A(\underline{R}(i,k),\underline{R}(k,j)) \geq \overline{R}(i,j)$, then
Begin $\overline{R}(i,j) := A(\underline{R}(i,k),\underline{R}(k,j))$ END
ENDFOR ENDFOR ENDFOR
End

5 Decision making problem. Selection of methods of treatment for ovarian cancer

Our above results allow to perform the following application in decision making problem. We focus on the problem of Ovarian cancer. For example, we may observe that each year, more than 22,000 women in the U.S. are diagnosed with ovarian cancer and around 14,000 will die. According to the American Cancer Society, it is the 8th most common cancer among women in the United States. Tragically is that, the overall 5-year survival rate is only 46 percent in most developed countries (it is lower for more advanced stages). However, according to the National Cancer Institute, if diagnosis is made early, before the tumor has spread and treatment will begin, then the 5 year survival rate is 94 percent. The interesting method of diagnostic which we may find in [19] is based on Polish patients. In this paper we will consider the decision making problem during the selection of methods of treatment for ovarian cancer.

Generally, in a multi-expert decision making problem we have a set of alternatives described on set of criteria, a set of experts and each of the latter provides his/her opinions on the former set of alternatives by given criteria. It is well known that, depending on the context and/or the level of knowledge of the experts, in some decision making problems it may occur that it is difficult to express the preferences using precise numerical values. That is why we use interval values. Moreover, it can also happen, when there are alternatives pairwise compared, that experts are not sure if they prefer one alternative or another.

In our considerations the set of alternatives will be contained with methods of treatment for ovarian cancer after of surgery, i.e. chemotherapy, radiotherapy and nintedanib.

The kind of treatment depends on many factors, including the type of ovarian cancer, its stage and grade, as well as the general health of the patient. These values create the set of criteria.

We consider an interval-valued fuzzy relation on X (set of alternatives) which represents the expert's opinion of each alternative over another one, i.e. preferences of given methods of treatment of the patient. The preferences will be represented with respect to m experts, mathematically these are relations $R_1, \ldots, R_m \in IVFR(X)$. To find the solution the best method (order of methods) we apply modified voting method similar to [1], where was used ignorance function to create interval-valued fuzzy relation relations and strict, indifference and incomparability relations to selection, but we will propose use of both epistemic and ontic transitivity and a linear order generated by aggregation functions $\leq_{K_{1,2}}$ ([3]) defined in the following way: $x \leq_{K_{1,2}} y$ if and only if $K_1(\underline{x}, \overline{x}) < K_1(\underline{y}, \overline{y})$ or ($K_1(\underline{x}, \overline{x}) = K_1(\underline{y}, \overline{y})$ *and* $K_2(\underline{x}, \overline{x}) \leq K_2(\underline{y}, \overline{y})$) for two continuous aggregation functions, such that, for all $x, y \in L^I$, the equalities $K_1(\underline{x}, \overline{x}) = K_1(\underline{y}, \overline{y})$ and $K_2(\underline{x}, \overline{x}) = K_2(\underline{y}, \overline{y})$ hold if and only if $x = y$.

It is worth to mention that at the beginning of algorithm it may be checked if $R_1, \ldots, R_n$ are reciprocal with respect to the Sugeno family of fuzzy (strong) negations $N_S^\lambda(x) = \frac{1-x}{1+\lambda x}$, where $\lambda \in (-1, \infty)$ (we use for $\lambda = 0.5$). If the answer is positive we may apply the presented in this paper results in order to consider the appropriate aggregation function to aggregate these relations and obtain aggregated result.

Next, we present the five steps to solve decision making problem using IVFRs on set of alternatives X from a given interval-valued fuzzy preference relations (we omit problem of generate interval-valued fuzzy relation from fuzzy relation presenting empirical values as fuzzy values):

Procedure *Selection*:
Inputs: $X = \{chemotherapy,\ radiotherapy,\ nintedanib\}$ set of alternatives, $i, j, k \in \{1, ..., n\}$; IV aggregation functions $\mathcal{A}, \mathcal{B}$; R interval-valued fuzzy aggregated preference relation; linear order $\leq_{K_{1,2}}$; representable IV negation N_{IV};
Output: x_i selection;
S1. Aggregate interval-valued fuzzy preference relations presenting opinions of experts on alternatives $\to R$ (see Theorem 3.8).
S2. Normalization of relation R, for $i \neq j$

$$R^*_{ij} = \begin{cases} N_{IV}(R_{ji}) & \text{if } R_{ij} \succeq_\nu R_{ji} \text{ or } R_{ij} \geq_{K_{1,2}} R_{ji}, \\ R_{ij} & \text{else.} \end{cases} \tag{4}$$

For $i = j$ $R^*_{ij} = [0.5, 0.5]$.
S3. Execute Procedure $\pi - A$-*transitivity* and $\nu - A - transitivity$.
S4. Calculate $M = \mathcal{A}(R^A_\nu, R^A_\pi)$; **S5.** For $m = 1$ to $n+1$, find $x_{selection} = arg\max_i(\mathcal{B}_{1 \leq j \neq i \leq n}(M_{ij}))$, for $\mathcal{B} \succeq \vee$ and using a linear order $\leq_{K^m_{1,2}}$;
End.

Obtained the order give / suggest the best order of methods which needs for given patient and depends on criteria connected with him.

Example 5.1. *To testing of our algorithm we used opinions of preference methods of treatment of given patients presented by four experts from one of Polish medical center. These preference information are present in four interval-valued fuzzy relations $R_1, \ldots, R_4 \in IVFR(X)$. After aggregation of interval-valued fuzzy relations $R_1, \ldots, R_4$, where $X = \{chemotherapy,\ radiotherapy,\ nintedanib\}$. These relations represent the preferences, i.e. the experts opinion for each methods over another one for the given patient in one of the Polish medical center. $R \in IVFR(X)$ after aggregation and normalization by (4) with $N_{IV} = (N_S^{0.5}, N_S^{0.5})$ is reciprocal relation represented by the matrix:*

$$R = \begin{bmatrix} [0.5000, 0.5000] & [0.0910, 0.4853] & [0.6911, 0.9897] \\ [0.4142, 0.8694] & [0.5000, 0.5000] & [0.1687, 0.9676] \\ [0.0069, 0.2296] & [0.0218, 0.7666] & [0.5000, 0.5000] \end{bmatrix}$$

and by the Procedure $\pi - A$-transitivity for A arithmetic mean we obtain:

$$R^A_\pi = \begin{bmatrix} [0.5000, 0.5000] & [0.0910, 0.4853] & [0.6911, 0.9897] \\ [0.4142, 0.8694] & [0.5000, 0.5000] & [0.1687, 0.9676] \\ [0.0069, 0.2535] & [0.0218, 0.7666] & [0.5000, 0.5000] \end{bmatrix}$$

and by the Procedure $\nu - A$-transitivity for $A = min$ we obtain:

$$R^A_\nu = \begin{bmatrix} [0.5000, 0.5000] & [0.7666, 0.7666] & [0.7666, 0.7666] \\ [0.5000, 0.5000] & [0.7666, 0.7666] & [0.5000, 0.5000] \\ [0.5000, 0.5000] & [0.5000, 0.5000] & [0.5000, 0.5000] \end{bmatrix}.$$

By $\mathcal{A}_{mean}$ in S4. we have

$$M = \begin{bmatrix} [0.5000, 0.5000] & [0.4288, 0.62595] & [0.72885, 0.87815] \\ [0.4571, 0.6847] & [0.6333, 0.6333] & [0.33435, 0.7338] \\ [0.25345, 0.37675] & [0.2609, 0.6333] & [0.5000, 0.5000] \end{bmatrix},$$

then for $\mathcal{B} = \mathcal{A}_{mean}$ in S5. we obtain, $x_1 = [0.578825, 0.75205]$, $x_2 = [0.483825, 0.68355]$, $x_3 = [0.38045, 0.56665]$ *Finally by using the linear orders* $\leq_{K_{1,2}}$ *for different four pair of aggregations* (K_1, K_2) *we imply,* $x_1 \succ x_3 \succ x_2$. *Thus for given patient will be the best firstly use chemotherapy.*

Presented algorithm can support the decision of young, inexperienced doctors to choose the right treatment for the individual patient and in the future can be expanded with new innovative treatments. Moreover, we compared presented above algorithm with its modification, where we used in S4. three kind transitivity (epistemic, ontic and classic) and we observe that our proposition give better results (according to the accepted expert opinion).

6 Conclusions

We presented some method of selection of alternatives in decision making problem by using relation intensity of preference in ontic and epistemic cases, where we use new idea of reciprocity property. In future we would like to study more properties and classification of this reciprocity. Moreover, we will consider in presented algorithms IV aggregations defined with respect to other order (linear, necessary or possibly). Also in the future it will be interesting to examine existing $\pi - A$-transitive and $\nu - A$-transitive closures (similar to [23, 9]).

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8 Appendix

In the Matlab code we find transitivity relation in epistemic and ontic cases, so we may present as the code snippet:

```
%------------------------------------------------------------
%Computes A-transitivity / epistemic case
%------------------------------------------------------------
function [R] = A_trans_epistemic(R)
[rows, cols] = size(R);
n = rows;
n = cols;
for x=1:n for y=1:n for z=1:n
beta1_d=R{x,z}(1,1);
beta2_d=R{z,y}(1,1);
beta3_g=R{x,y}(1,2);  t_d=(beta1_d+beta2_d)/2;
if t_d>beta3_g, R{x,y}(1,2)=t_d;
end
end end end
%------------------------------------------------------------
%Computes A-transitivity / ontic case
%------------------------------------------------------------
function [R] = A_trans_ontic(R)
[rows, cols] = size(R);
n = rows;
n = cols;
for x=1:n for y=1:n for z=1:n
beta1_g=R{x,z}(1,2);
beta2_g=R{z,y}(1,2);
beta3_d=R{x,y}(1,1); t_d=min(beta1_g,beta2_g);
if t_d>beta3_d, R{x,y}(1,1)=t_d; R{x,y}(1,2)=t_d;
end
end end end
```

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