Relationships between completeness of fuzzy quasi-uniform spaces

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Abstract

In this paper, we give a kind of Cauchy 1-completeness in probabilistic quasi-uniform spaces by using 1-filters. Utilizing the relationships among probabilistic quasi-uniformities, classical quasi-uniformities and Hutton [0,1]-quasi-uniformities, we show the relationships between their completeness. In fuzzy quasi-metric spaces, we establish the relationships between the completeness of induced probabilistic quasi-uniform spaces and both completeness of induced classical quasi-uniform spaces and induced Hutton [0,1]-quasi-uniform spaces.

Keywords: probabilistic quasi-uniformity; classical quasi-uniformity; Hutton [0,1]-quasi-uniformity; 1-filter; Cauchy 1-completeness.

1 Introduction

The theory of uniform structures is an important area of analysis and topology because it provides an appropriate context to link metrics with general topological structures. Quasi-uniformity is a uniformity structure which does not satisfy the symmetric condition. With the development of fuzzy topology, many mathematical structures have been generalized to the fuzzy case, such as fuzzy convergence structures [19, 20] and fuzzy convex structures [21, 22, 26, 27]. For uniformities, many researchers put forward various lattice-valued (quasi-)uniformities and obtain a series of interesting results: see e.g. Höhle’s probabilistic (quasi-)uniformity [14], Lowen’s (quasi-)uniformity [17], Hutton’s L-(quasi-)uniformity [12, 31], Shi’s pointwise (quasi-)uniformity [24, 25] and J. Gutiérrez García’s L-uniformity [3]. In [1], J. Gutiérrez García studied the relationships between the different notions of fuzzy (quasi-)uniformities. It is worth mentioning that Zhang [32] studied a comparison of various types of uniformities in fuzzy topology and then analyzed the relationships between several notions of lattice-valued (quasi-)uniformities in [33].

The completeness discussed by means of filters theory is an important content in uniform spaces. Lowen in [17, 18] studied the completeness of fuzzy uniform spaces based on prefilters. Höhle studied T-completeness of probabilistic uniform spaces based on T-filters in [15]. J. Gutiérrez García and M. A. De Prada Vicente in [2] studied the completeness of Hutton [0,1]-quasi-uniform spaces based on tight and stratified L-filters. In this paper, with the help of the idea of J. L. Sieber and W. J. Pervin [23], we propose a kind of completeness of probabilistic quasi-uniform spaces, which is called Cauchy 1-completeness based on 1-filters in the unit interval [0,1]. Inspired by the relationships between various types of lattice-valued (quasi-)uniformities, we want to discuss relationships between completeness of probabilistic quasi-uniformities and both completeness of classical quasi-uniformities and Hutton [0,1]-quasi-uniformities. Fuzzy (quasi-)metric spaces draw much attention in fuzzy mathematics. The usual concept of fuzzy (quasi-)metric spaces can date back to George and Veeramani [9, 10], which slightly modified the definition given by Kramosil and Michalek [16] who adapted the concept of probabilistic metrics to the fuzzy setting. Furthermore, the completeness of fuzzy (quasi-)metric spaces also has studied in [9, 10, 11]. What’s more, many authors associated to each fuzzy (quasi-)metric space a lattice-valued (quasi-)uniform space (such as [28, 29]). The paper is organized as follows. In section 2 we provide lattice theoretical environment and some concepts of lattice-valued quasi-uniformities used in this paper. Furthermore, we give a kind of Cauchy 1-completeness in probabilistic quasi-uniform spaces by using 1-filters. Section 3 and Section 4 are devoted to study the relationships between completeness of classical quasi-uniform spaces and...
probabilistic quasi-uniform spaces, and the relationships between completeness of probabilistic quasi-uniform spaces and Hutton \([0,1]\)-quasi-uniform spaces. Finally, in section 5, in the framework of fuzzy quasi-metric spaces, we establish the relationships between completeness of induced classical quasi-uniform spaces and induced probabilistic quasi-uniform spaces and the relationships between the completeness of induced probabilistic quasi-uniform spaces and induced Hutton \([0,1]\)-quasi-uniform spaces.

2 Preliminaries

In this paper, we use the unit interval \(I = [0,1]\) as the true table although most of the results are also valid in complete residuated lattices.

2.1 Lattice theoretical preliminaries

**Definition 2.1 ([23]).** A binary operation \(* : I \times I \rightarrow I\) is called a (left-)continuous t-norm if it satisfies the following conditions:

1. \(*\) is associative and commutative;
2. \(1\text{ is the unit, i.e.}, a * 1 = a\) for all \(a \in I\);
3. \(a * b \leq c * d\) whenever \(a \leq c\) and \(b \leq d\);
4. \(*\) is (left-)continuous.

In this paper, we always assume \(*\) is a left-continuous t-norm on \(I\). We say that the left-continuous t-norm \(*\) does not have nontrivial zero divisors, if \(\alpha * \beta \neq 0\) whenever \(\alpha, \beta \neq 0\).

If \(*\) is a left-continuous t-norm, since the map \(\alpha * (-) : I \rightarrow I\) preserves arbitrary joins for each \(\alpha \in I\), it has a right adjoint \(\alpha \rightarrow (-) : I \rightarrow I\) determined by the adjoint property \(\alpha * \beta \leq \gamma \iff \beta \leq \alpha \rightarrow \gamma\), \(\alpha, \beta, \gamma \in I\). Hence the implication \(\rightarrow\) is the binary operation on \(I\) given by \(\alpha \rightarrow \gamma = \bigvee \{\beta \in I \mid \alpha * \beta \leq \gamma\}\), \(\alpha, \gamma \in I\).

Three distinguished examples of left-continuous t-norm are \(\wedge, \cdot\) and \(\ast_L\) (the Lukasiewicz t-norm) which are given as:

\[\alpha \wedge \beta = \min\{\alpha, \beta\}, \quad \alpha \cdot \beta = \alpha\beta \quad \text{and} \quad \alpha \ast_L \beta = \max\{\alpha + \beta - 1, 0\};\]

\[\alpha \triangleleft \beta = \begin{cases} 1 & \text{if } \alpha \leq \beta, \\ \alpha & \text{if } \beta < \alpha \end{cases} \quad \text{and} \quad \alpha \triangleright \beta = \begin{cases} 1 & \text{if } \alpha \leq \beta, \\ \frac{\beta}{\alpha} & \text{if } \beta < \alpha \end{cases} \quad \text{and} \quad \alpha \triangleright L \beta = \min\{1 - \alpha + \beta, 1\},\]

for all \(\alpha, \beta \in I\). It is well-known and easy that \(* \leq \wedge\) for each left-continuous t-norm \(*\).

For a set \(X\), a binary map \(S_X(-,-) : I^X \times I^X \rightarrow I\) is defined by \(S_X(A, B) = \bigwedge_{x \in X} (A(x) \rightarrow B(x))\) for each \(A, B \in I^X\), where \(S_X(A, B)\) can be interpreted as the degree to which \(A\) is a subset of \(B\). It is called fuzzy inclusion order or subsethood degree of \(I\)-subsets. \(S_X(-,-)\) is also denoted as \(S(-,-)\).

2.2 Relationships among classical filters, 1-filters and \(I\)-filters

Below we collect some definitions and results regarding classical filters, 1-filters and \(I\)-filters for the unit interval \(I\), that will be needed later on.

**Definition 2.2 ([11]).** A nonempty subset \(\mathcal{F}\) of \(I^X\) is called a 1-filter on \(X\) provided it satisfies the following properties:

1. \((IF1)\) If \(A \in I^X\) such that \(\bigvee_{B \in \mathcal{F}} S(B, A) = 1\), then \(A \in \mathcal{F}\);
2. \((IF2)\) \(A_1 \wedge A_2 \in \mathcal{F}\) for all \(A_1, A_2 \in \mathcal{F}\);
3. \((IF3)\) \(\bigvee_{x \in X} A(x) = 1\) for all \(A \in \mathcal{F}\).

The set of all 1-filters on \(X\) is denoted by \(\mathcal{F}_1(X)\). For a 1-filter on \(X\), the axiom \((IF1)\) means \(B \in \mathcal{F}\) whenever \(A \in \mathcal{F}\) with \(A \leq B\). For each \(x \in X\), let \([x] \subseteq I^X\) be \([x] = \{A \in I^X \mid A(x) = 1\}\). Then \([x]\) is a 1-filter and \([x]\) is usually called the point 1-filter of \(x\).
Definition 2.3 ([13]). A nonempty subset \( B \subseteq I^X \) is called a 1-filter base on the set \( X \) if it satisfies the following conditions:

\[
\begin{align*}
(B1) & \quad \bigvee_{B \in B} S(B,C \land D) = 1 \text{ for all } C, D \in B; \\
(B2) & \quad \bigvee_{x \in X} C(x) = 1 \text{ for all } C \in B.
\end{align*}
\]

Definition 2.4 ([13]). Let \( X \) be a set. A map \( v : I^X \to I \) is called an \( I \)-filter on \( X \) if it satisfies the following properties:

\[
\begin{align*}
(F0) & \quad v(1_X) = 1; \\
(F1) & \quad \text{If } f_1, f_2 \in I^X, f_1 \leq f_2 \text{ then } v(f_1) \leq v(f_2); \\
(F2) & \quad v(f_1) \land v(f_2) \leq v(f_1 \land f_2) \text{ for each } f_1, f_2 \in I^X; \\
(F3) & \quad v(1_I) = 0.
\end{align*}
\]

An \( I \)-filter \( v \) is said to be stratified if it satisfies the following additional axiom:

\[
(F4) \quad \alpha * v(f) \leq v(\alpha * f) \text{ for all } \alpha \in I \text{ and } f \in I^X.
\]

An \( I \)-filter \( v \) is said to be tight if it satisfies the following important axiom:

\[
(F5) \quad v(\alpha * 1_X) = \alpha \text{ for all } \alpha \in I.
\]

From [31], we have the following results between classical filters and 1-filters.

Lemma 2.5. Let \( \mathcal{F}_I(X) \) be the set of all classical filter on \( X \) and \( \mathbb{F}_I(X) \) be the set of all 1-filter on \( X \). The following statements hold:

1. The order-preserving mapping \( \omega : \mathcal{F}_I(X) \to \mathbb{F}_I(X) \) is given by \( \omega(F) = \{ A \in I^X \mid \bigvee_{F \in F} \bigwedge_{x \in X} A(x) = 1 \} \), for each \( F \in \mathcal{F}_I(X) \). Then \( \omega(F) \) is a 1-filter.

2. The order-preserving mappings \( [ ] \), \( \iota : \mathbb{F}_I(X) \to \mathcal{F}_I(X) \) are respectively given by \( [F] = \{ u \in 2^X \mid \chi_u \in F \} \) and \( \iota(F) = \{ \sigma_r(A) \mid A \in F, r \in [0,1) \} \), for each \( F \in \mathbb{F}_I(X) \), where \( \sigma_r(A) = \{ x \in X \mid A(x) > r \} \). Then \( [F] \) and \( \iota(F) \) are classical filters.

Lemma 2.6. Let \( F \) be a classical filter on \( X \) and \( \mathbb{F} \) be a 1-filter. Then

\[
\begin{align*}
(1) & \quad \iota(\omega(F)) = F; \\
(2) & \quad \omega(\iota(F)) \supseteq F; \\
(3) & \quad [\omega(F)] = F; \\
(4) & \quad \omega([F]) \subseteq F.
\end{align*}
\]

The lemma state that these adjoint connections hold, i.e., \( F \dashv \omega \dashv [ ] \).

Definition 2.7. \( F \) is a 1-filter. \( F \) is called an induced 1-filter if there exists a classical filter \( F \) such that \( F = \omega(F) \).

It is easy to check the following results.

Lemma 2.8. \( F \) is an induced 1-filter if and only if \( \iota(F) = [F] \).

Furthermore, \( \omega(\iota(F)) = F \).

Next, we will mention the relationship between 1-filters and stratified and tight \( I \)-filters.

Lemma 2.9 ([3]). Every 1-filter \( F \) on \( X \) induces a stratified and tight \( I \)-filter \( v_F \) by \( v_F(f) = \bigvee_\{ \alpha \in I \mid \alpha \to f \in F \} \), for all \( f \in I^X \).

Lemma 2.10 ([3]). Every stratified and tight \( I \)-filter \( v \) on \( X \) determines a 1-filter \( \mathbb{F}_v \) by \( \mathbb{F}_v = \{ f \in I^X \mid v(f) = 1 \} \).

J. Gutiérrez García established a relationship between 1-filters and \( I \)-filters by using some properties of the characteristic value in [3].
2.3 \((X, \mathcal{U}), (X, \mathcal{U})\) and \((X, \mathcal{U})\)

In this part, we now describe briefly the definition and some properties about classical quasi-uniformity \(U\), probabilistic quasi-uniformity \(\mathcal{U}\) and Hutton \([0, 1]\)-quasi-uniformity \(\mathcal{U}\).

**Definition 2.11** ([4], [23]). A nonempty subset \(U \subseteq I^{X \times X}\) is called a probabilistic quasi-uniformity on \(X\), if it is a 1-filter and still satisfies the following conditions:

\((IU0)\) \(U \in \mathcal{U}\) implies \(U(x, x) = 1\) for all \(x \in X\).

\((IUC)\) \(U \in \mathcal{U}\) implies that there exist \(V \in \mathcal{U}\) such that \(V \circ V \subseteq U\).

For a probabilistic quasi-uniform space \((X, \mathcal{U})\), \(N^U_x\) is the 1-filter generated by the set \(N^U_x = \{U(-, x) \mid U \in \mathcal{U}\}\). Therefore, \(\tau\mathcal{U} = \{A \in \mathcal{I}^\mathcal{I} \mid \forall x \in X, A(x) \subseteq \bigvee_{U \in \mathcal{U}} S(U(-, x), A)\}\) is generated \(I\)-topology by \(N^U_x\). Next, we will introduce the definition of Hutton \([0, 1]\)-quasi uniformities from J. Gutiérrez Garcia [3]. Let \(X\) be a set and \((I, \leqslant)\) be a complete lattice.

**Remark 2.12.** As pointed out in [3], each arbitrary join-preserving element \(\phi \in (I^X)^{I^X}\) is completely determined by the collection of \(I\)-set \(\{\phi(\alpha \ast 1_{\{x\}}) \mid \alpha \in (0, 1], x \in X\}\).

**Definition 2.13** ([3]). Let \(X\) be a set and \((\{0, 1\}, \leqslant)\) be a complete lattice. A Hutton \([0, 1]\)-quasi-uniformity on \(X\) is a nonempty subset \(\mathcal{U}\) of \(\mathcal{H}_1(X)\) such that

\((HU1)\) if \(\phi_1 \in \mathcal{U}\), \(\phi_1 \leqslant \phi_2\) and \(\phi_2 \in \mathcal{H}_1(X)\) then \(\phi_2 \in \mathcal{U}\),

\((HU2)\) if \(\phi_1, \phi_2 \in \mathcal{U}\), there exist \(\phi \in \mathcal{U}\) such that \(\phi \leqslant \phi_1\) and \(\phi \leqslant \phi_2\),

\((HU3)\) if \(\phi \in \mathcal{U}\), there exist \(\psi \in \mathcal{U}\) such that \(\psi \circ \psi \leqslant \phi\) (where \(\circ\) denotes the usual composition of functions).

The pair \((X, \mathcal{U})\) is called a Hutton \([0, 1]\)-quasi-uniform space such that \(X\) is a set and \(\mathcal{U}\) is a Hutton \([0, 1]\)-quasi-uniformity on \(X\). A nonempty subset \(\mathcal{B}\) of \(\mathcal{U}\), is a base for \(\mathcal{U}\) if for each \(\phi \in \mathcal{U}\), there exists \(\varphi \in \mathcal{B}\) such that \(\varphi \leqslant \phi\).

A Hutton \([0, 1]\)-quasi-uniformity \(\mathcal{U}\) is said to be stratified if it has a base \(\mathcal{B}\) which satisfies:

\((HU4)\) if \(\forall \varphi \in \mathcal{B}\), \(\forall \alpha \in I\), \(\forall x \in X\), \(\alpha \ast \varphi(1_{\{x\}}) \leqslant \varphi(\alpha \ast 1_{\{x\}})\).

2.4 Cauchy 1-completeness of probabilistic quasi-uniformity

In this part, we use the idea of Sieber and Pervin in [23] to give the Cauchy 1-completeness of probabilistic quasi-uniform spaces by 1-filters used in this paper.

**Definition 2.14.** Let \((X, \mathcal{U})\) be a probabilistic quasi-uniform space. A 1-filter \(\mathcal{F}\) is called a Cauchy 1-filter if and only if for every \(U \in \mathcal{U}\) there exists a point \(x \in X\) such that \(U(-, x) \subseteq \mathcal{F}\).

If \(\mathcal{F}\) is an induced 1-filter, then \(\mathcal{F}\) is called an induced Cauchy 1-filter if and only if for every \(U \in \mathcal{U}\) there exists a point \(x \in X\) such that \(U(-, x) \subseteq \mathcal{F}\). It is easy to know that \(N^U_x\) is a Cauchy 1-filter and we call it the neighborhood 1-filter of \(x\). If \(N^U_x \subseteq \mathcal{F}\), we call \(\mathcal{F}\) converges to \(x\).

**Definition 2.15.** A probabilistic quasi-uniform space \((X, \mathcal{U})\) will be said to be Cauchy 1-complete if and only if each Cauchy 1-filter converges.

Furthermore, a probabilistic quasi-uniform space \((X, \mathcal{U})\) is said to be induced Cauchy 1-complete if each induced Cauchy 1-filter converges. Now we recall the definition about the completeness of classical quasi-uniform space \((X, \mathcal{U})\) in [23].

**Definition 2.16** ([23]). A filter \(\mathcal{F}\) in a quasi-uniform space \((X, \mathcal{U})\) will be called a Cauchy filter if and only if for every \(u \in \mathcal{U}\) there exists a point \(z \in X\) such that \(u[z] \in \mathcal{F}\), where \(u[z] = \{y \in X \mid (y, z) \in u\}\).

From general topology, we obtain that for a quasi-uniform space \((X, \mathcal{U})\), \(N^U_z\), the neighborhood of \(z\), is generated by the set \(N^U_z = \{u[z] \mid u \in \mathcal{U}\}\). If \(N^U_z \subseteq \mathcal{F}\), we call \(\mathcal{F}\) converges to \(z\).

**Definition 2.17** ([23]). A quasi-uniform space \((X, \mathcal{U})\) will be said to be complete if and only if every Cauchy filter converges.
3 Relationship between the completeness of \((X, \mathcal{U})\) and \((X, \mathcal{U})\)

In this section, we will discuss the relationships between the completeness of classical quasi-uniform spaces and Cauchy 1-completeness of probabilistic quasi-uniform spaces. Next, we firstly introduce two functors \(\Phi, \Psi\) about classical quasi-uniformities and probabilistic quasi-uniformities.

**Proposition 3.1** ([11, 33]). \(\text{Let } (X, \mathcal{U})\) be a classical quasi-uniform space and \((X, \mathcal{U})\) be a probabilistic quasi-uniform space. Let \(\Phi(\mathcal{U})\) be the probabilistic quasi-uniformity generated by \(\{1_u \mid u \in \mathcal{U}\}\). Let \(\Psi(\mathcal{U})\) be the classical quasi-uniformity generated by \(\{\Psi(U) \mid U \in \mathcal{U}\}\), where \(\Psi(U) = \{(x, y) \in X \times X \mid U(x, y) = 1\}\). Then:

1. \(\Psi(\Phi(U)) = \mathcal{U}\);
2. \(\Phi(\Psi(U)) \subseteq \mathcal{U}\);
3. \(\Psi\) is a right adjoint of \(\Phi\).

**Lemma 3.2.** \(\text{Let } (X, \mathcal{U})\) be a classical quasi-uniform space, \(\mathcal{F}\) be a classical filter on \(X\) and \(x_0 \in X\). Then:

1. \(\mathcal{F}\) is a Cauchy filter in \((X, \mathcal{U})\) if and only if \(\omega(\mathcal{F})\) is a Cauchy 1-filter in \((X, \Phi(\mathcal{U}))\).
2. \(\mathcal{F}\) converges to \(x_0\) in \((X, \mathcal{U})\) if and only if \(\omega(\mathcal{F})\) converges to \(x_0\) in \((X, \Phi(\mathcal{U}))\).

**Proof.** (1) Necessity: Let \(u \in \mathcal{U}\) and \(1_u \in \Phi(\mathcal{U})\). Since \(\mathcal{F}\) is a Cauchy filter in \((X, \mathcal{U})\), there exists \(x \in X\) such that \(u[x] \in \mathcal{F}\). Hence, \(\bigvee_{F \in \mathcal{F}} \bigwedge_{y \in u[x]} 1_u(y, x) = 1\). Therefore, \(1_u(\cdot, x) \in \omega(\mathcal{F})\). Sufficiency: Let \(u \in \mathcal{U}\). Then \(1_u \in \Phi(\mathcal{U})\). Since \(\omega(\mathcal{F})\) is a Cauchy 1-filter in \((X, \Phi(\mathcal{U}))\), there exists \(x \in X\) such that \(1_u(\cdot, x) \in \omega(\mathcal{F})\). Then there is \(\bigvee_{F \in \mathcal{F}} \bigwedge_{y \in u[x]} 1_u(y, x) = 1\). So we can find \(F_r \in \mathcal{F}\) satisfying \(\bigwedge_{y \in u[x]} 1_u(y, x) > r\) for all \(r \in (0, 1)\). When \(y \in F_r\), we have \((y, x) \in u\), namely, \(y \in u[x]\). Hence, \(F_r \subseteq u[x]\). Therefore, \(u[x] \in \mathcal{F}\).

(2) Necessity: Let \(u \in \mathcal{U}\), \(1_u \in \Phi(\mathcal{U})\) and \(1_u(\cdot, x_0) \in \mathcal{N}_{x_0}^{\Phi(\mathcal{U})}\). Since \(\mathcal{F}\) converges to \(x_0\) in \((X, \mathcal{U})\), we have \(\mathcal{N}_{x_0}^{\Phi(\mathcal{U})} \subseteq \omega(\mathcal{F})\). If \(\omega(\mathcal{F})\) converges to \(x_0\) in \((X, \Phi(\mathcal{U}))\), we have \(\mathcal{N}_{x_0}^{\Phi(\mathcal{U})} \subseteq \omega(\mathcal{F})\). Furthermore, when \(1_u(\cdot, x_0) \in \mathcal{N}_{x_0}^{\Phi(\mathcal{U})}\), we have \(1_u(\cdot, x_0) \in \omega(\mathcal{F})\). Then there is \(\bigvee_{F \in \mathcal{F}} \bigwedge_{y \in u[x_0]} 1_u(y, x_0) = 1\). So we can find \(F_r \in \mathcal{F}\) satisfying \(\bigwedge_{y \in u[x_0]} 1_u(y, x_0) > r\) for all \(r \in (0, 1)\). When \(y \in F_r\), we have \((y, x_0) \in u\). Then there is \(y \in u[x_0]\). Hence, \(F_r \subseteq u[x_0]\). Therefore, \(u[x_0] \in \mathcal{F}\).

**Lemma 3.3.** \(\text{Let } (X, \mathcal{U})\) be a probabilistic quasi-uniform space, \(\mathcal{F}\) be an induced 1-filter and \(x_0 \in X\). Then:

1. \(\mathcal{F}\) is a Cauchy 1-filter in \((X, \mathcal{U})\) if and only if \(\iota(\mathcal{F})\) is a Cauchy filter in \((X, \Psi(\mathcal{U}))\);
2. \(\mathcal{F}\) converges to \(x_0\) in \((X, \mathcal{U})\) if and only if \(\iota(\mathcal{F})\) converges to \(x_0\) in \((X, \Psi(\mathcal{U}))\).

**Proof.** (1) Sufficiency: Let \(U \in \mathcal{U}\). Then \(\Psi(U) \in \Psi(\mathcal{U})\). Since \(\iota(\mathcal{F})\) is a Cauchy filter in \((X, \Psi(\mathcal{U}))\), there exists \(x \in X\) such that \(\Psi(U)[x] \in \iota(\mathcal{F}) = \mathcal{F}\). Let me denote \(\Psi(U)[x] = A\). Furthermore, we have \(\chi_A \in \mathcal{F}\). Hence,

\[
\bigvee_{B \in \mathcal{F}} S(B, U(\cdot, x)) \geq \bigwedge_{y \in A} (\chi_A(y) \rightarrow U(y, x)) = \bigwedge_{y \in A} U(y, x) = 1.
\]

Therefore, \(U(\cdot, x) \in \mathcal{F}\).

Necessity: Let \(u \in \Psi(\mathcal{U})\). Then there exists \(U \in \mathcal{U}\) such that \(\Psi(U) \subseteq u\). Since \(\mathcal{F}\) is a Cauchy 1-filter in \((X, \mathcal{U})\), there is some \(x \in X\) satisfying \(U(\cdot, x) \in \mathcal{F}\). On account of \(\omega(\iota(\mathcal{F})) = \mathcal{F}\), it follows that \(U(\cdot, x) \in \omega(\iota(\mathcal{F}))\). Then there is \(\bigvee_{G \in \iota(\mathcal{F})} \bigwedge_{y \in u[x]} U(y, x) = 1\). So we can find \(G_r \in \iota(\mathcal{F})\) such that \(\bigwedge_{y \in u[x]} U(y, x) > r\) for any \(r \in [0, 1]\). When \(y \in G_r\), we have \(U(y, x) > r\) for any \(r \in [0, 1]\). Hence, \(y \in \Psi(U)[x]\), where \(\Psi(U)[x] = \{y \in X \mid U(y, x) = 1\}\). Furthermore, there is \(G_r \subseteq \Psi(U)[x] \subseteq u[x]\). Therefore, \(u[x] \in \iota(\mathcal{F})\).

(2) Sufficiency: Let \(U \in \mathcal{U}\), \(\Psi(U) \in \Psi(\mathcal{U})\) and \(1_u \in \mathcal{N}_{x_0}^{\Psi(\mathcal{U})}\). Since \(\iota(\mathcal{F})\) converges to \(x_0\) in \((X, \Psi(\mathcal{U}))\), we have \(\mathcal{N}_{x_0}^{\Psi(\mathcal{U})} \subseteq \iota(\mathcal{F})\). Let me denote \(\Psi(U)[x_0] = A\). For \(A \in \mathcal{N}_{x_0}^{\Psi(\mathcal{U})}\), there is \(A \in \iota(\mathcal{F})\). Furthermore, we obtain \(\chi_A \in \mathcal{F}\). Hence,

\[
\bigvee_{B \in \mathcal{F}} S(B, U(\cdot, x)) \geq \bigwedge_{y \in A} (\chi_A(y) \rightarrow U(y, x_0)) = \bigwedge_{y \in A} U(y, x_0) = 1.
\]
Therefore, \( U(-, x_0) \in \mathcal{F} \).

Necessity: Let \( U \in \mathcal{U} \) and \( \Psi(U)[x_0] \in \mathcal{N}_{x_0}^{\mathcal{U}} \). Since \( \mathcal{F} \) converges to \( x_0 \) in \((X, \mathcal{U})\), we have \( \mathcal{N}_{x_0}^{\mathcal{U}} \subseteq \mathcal{F} \). For \( U(-, x_0) \in \mathcal{N}_{x_0}^{\mathcal{U}} \), there is \( U(-, x_0) \in \mathcal{F} \). On account of \( \omega(\emptyset(\mathcal{F})) = \emptyset \), it follows that \( U(-, x_0) \in \emptyset(\emptyset(\mathcal{F})) \). Then there is \( \bigvee_{G \in \emptyset(\emptyset(\mathcal{F}))} \bigwedge_{y \in G} U(y, x_0) = 1 \). So we can find \( G_r \in \emptyset(\mathcal{F}) \) such that \( \bigwedge_{y \in G_r} U(y, x_0) > r \) for any \( r \in [0, 1) \). When \( y \in G_r \), we have \( U(y, x_0) > r \) for any \( r \in [0, 1) \). Hence, \( y \in \Psi(U)[x_0] \), where \( \Psi(U)[x_0] = \{ y \in X \mid U(y, x_0) = 1 \} \). Furthermore, there is \( G_r \subseteq \Psi(U)[x_0] \). Therefore, \( \Psi(U)[x_0] \in \emptyset(\mathcal{F}) \). \( \square \)

It is easy to check the following results by Lemma 3.1 and Lemma 3.2.

**Theorem 3.4.** If \((X, \Phi(\mathcal{U}))\) is Cauchy \(1\)-complete, then \((X, \mathcal{U})\) is complete.

**Corollary 3.5.**
1. \((X, \mathcal{U})\) is complete if and only if \((X, \Phi(\mathcal{U}))\) is induced Cauchy \(1\)-complete;
2. \((X, \Psi(\mathcal{U}))\) is complete, then \((X, \mathcal{U})\) is induced Cauchy \(1\)-complete.

### 4 Relationship between the completeness of \((X, \mathcal{U})\) and \((X, \mathcal{U})\)

In this section, we will study the completeness of probabilistic quasi-uniformity \( \mathcal{U} \) and Hutton \([0, 1]\)-quasi-uniformity \( \mathcal{U} \) through the following functors \( \Lambda \) and \( \Upsilon \). Gutiérrez García in [3, 5] has studied the functors and discussed the relationship between probabilistic quasi-uniformities and Hutton \([0, 1]\)-quasi-uniformities by use of them.

**Proposition 4.1 (3, 5).** Let \((X, \mathcal{U})\) be a probabilistic quasi-uniform space and \((X, \mathcal{U})\) be a Hutton \([0, 1]\)-quasi-uniform space. Let \( \Lambda(\mathcal{U}) \) be the Hutton \([0, 1]\)-quasi-uniformity generated by \( \{ \Lambda(U) \mid U \in \mathcal{U} \} \), where \( [\Lambda(U)](a)(x) = \bigvee_{y \in X} U(x, y) * a(y) \) for each \( U \in \mathcal{U} \), \( a \in \mathcal{I}^X \) and \( x \in X \). Let \( \Upsilon(\mathcal{U}) \) be the probabilistic quasi-uniformity generated by \( \{ \Upsilon(\phi) \mid \phi \in \mathcal{U} \} \), where \( [\Upsilon(\phi)](x, y) = \bigwedge_{\alpha \in \mathcal{I}} \alpha \to [\phi(\alpha * 1_{(y)})](x) \) for each \( \phi \in \mathcal{U} \) and \( x, y \in X \). Then:
1. \( \Upsilon(\Lambda(\mathcal{U})) = \mathcal{U} \);
2. \( \Lambda(\Upsilon(\mathcal{U})) \subseteq \mathcal{U} \);
3. \( \Upsilon \) is a right adjoint of \( \Lambda \).

Since \( \alpha \) is the zero element with respect to \( * \), it follows that for each \( \alpha \in \mathcal{I} \) and \( x \in X \), \( [\Lambda(U)](\alpha * 1_{(x)}) = U(-, x) * \alpha = [\Lambda(U)](1_{(x)}) \) for each \( \phi \in \mathcal{B} \) and \( x \in X \), where \( \mathcal{B} \) is a base for \( \mathcal{U} \). First of all, we recall the completeness of Hutton \([0, 1]\)-quasi-uniform space \((X, \mathcal{U})\) in [3].

**Lemma 4.2 (3).** Let \( I \)-filter \( \mathcal{V} \) and Hutton \([0, 1]\)-quasi-uniformity \( \mathcal{U} \) on \( X \) be stratified and \( p \in X \). Then:
1. \( \mathcal{V} \) converges to \( p \) if and only if \( \forall \varphi \in \mathcal{B} \), \( \mathcal{V}(\varphi(1_{(p)})) = 1 \);
2. \( \mathcal{V} \) is a Cauchy \( I \)-filter if and only if \( \forall \varphi \in \mathcal{B} \), \( \exists p_\varphi \in X \), \( \mathcal{V}(\varphi(1_{(p_\varphi)})) = 1 \).

Where \( \mathcal{B} \) is a base for \( \mathcal{U} \).

**Definition 4.3 (3).** A Hutton \([0, 1]\)-quasi-uniform space \((X, \mathcal{U})\) is said to be complete if any stratified and tight Cauchy \( I \)-filter on \( X \) is convergent.

Next, we will discuss the relationship between the completeness of probabilistic quasi-uniformities and Hutton \([0, 1]\)-quasi-uniformities.

**Proposition 4.4.** Let \((X, \mathcal{U})\) be a probabilistic quasi-uniform space, \((X, \mathcal{U})\) be a stratified Hutton \([0, 1]\)-quasi-uniform space, \( \mathcal{F} \) be a \( I \)-filter and \( \nu \) be a stratified \( I \)-filter. Then:
1. \( \mathcal{F} \) is a Cauchy \(1\)-filter in \((X, \mathcal{U})\) if and only if \( \nu \) is a stratified Cauchy \( I \)-filter in \((X, \Lambda(\mathcal{U}))\);
2. \( \nu \) is a stratified Cauchy \( I \)-filter in \((X, \mathcal{U})\) if and only if \( \mathcal{F}_\nu \) is a Cauchy \( I \)-filter in \((X, \Upsilon(\mathcal{U}))\).
Proof. (1) Necessity: Let \( U \in \mathcal{U} \) and \( \Lambda(U) \) be a base element for \((X, \Lambda(\mathcal{U}))\). Since \( F \) is a Cauchy 1-filter in \((X, \mathcal{U})\), there exists \( p \in X \) such that \( U(-, p) \in F \). Hence,
\[
v_F(\Lambda(U)(1_{(p_0)})) = \bigvee_{\beta \in [0,1]} \alpha \rightarrow \Lambda(U)(1_{(p_0)}) \in F = \bigvee_{\beta \in [0,1]} \alpha \rightarrow U(-, p) \in F \geq \{1 \in [0,1] | 1 \rightarrow U(-, p) \in F \} = 1.
\]
Therefore, \( v_F \) is a Cauchy I-filter on \((X, \Lambda(\mathcal{U}))\).

Sufficiency: Let \( U \in \mathcal{U} \). Then \( \Lambda(U) \) is a base element for \((X, \Lambda(\mathcal{U}))\). Since \( v_F \) is a stratified Cauchy I-filter in \((X, \Lambda(\mathcal{U}))\), there exists \( p \in X \) such that \( v_F(\Lambda(U)(1_{(p_0)})) = 1 \). Specifically,
\[
v_F(\Lambda(U)(1_{(p_0)})) = \bigvee_{\beta \in [0,1]} \alpha \rightarrow \Lambda(U)(1_{(p_0)}) \in F = \bigvee_{\beta \in [0,1]} \alpha \rightarrow U(-, p) \in F = 1.
\]
So for any \( \alpha \in [0,1] \), we can find \( \beta \in [0,1] \) satisfying \( \beta \rightarrow U(-, p) \in F \) such that \( \beta \geq \alpha \). Hence,
\[
\bigvee_{B \in F} S(B, U(-, p)) \geq \bigvee_{\alpha \in [0,1]} S(\beta \rightarrow U(-, p), U(-, p)) = \bigvee_{\alpha \in [0,1]} \bigvee_{x \in X} ((\beta \rightarrow U(x, p)) \rightarrow U(x, p)) \geq \bigvee_{\alpha \in [0,1]} \beta \geq \bigvee_{\alpha \in [0,1]} \alpha = 1.
\]
Therefore, \( U(-, p) \in F \).

(2) Necessity: Let \( \phi \in \mathcal{B} \) and \( \Upsilon(\phi) \in \Upsilon(\mathcal{U}) \), where \( \mathcal{B} \) is a base for \( \mathcal{U} \). Since \( v \) is a stratified Cauchy I-filter in \((X, \mathcal{U})\), there exists \( p \in X \) such that \( v(\phi(1_{(p_0)})) = 1 \). An account of \( F_v = \{ f \in I^X \mid v(f) = 1 \} \), it follows that \( \phi(1_{(p_0)}) \in F_v \). Hence,
\[
\Upsilon(\phi)(-p) = \phi(1_{(p_0)}) \in F_v.
\]
Sufficiency: Let \( \phi \in \mathcal{B} \), where \( \mathcal{B} \) is a base for \( \mathcal{U} \). Then \( \Upsilon(\phi) \in \Upsilon(\mathcal{U}) \). Since \( F_v \) is a Cauchy 1-filter in \((X, \Upsilon(\mathcal{U}))\), there exists \( p \in X \) such that \( \Upsilon(\phi)(-p) \in F_v \). Hence, \( 1 = v(\Upsilon(\phi)(-p)) = v(\phi(1_{(p_0)})) \).

Proposition 4.5. Let \((X, \mathcal{U})\) be a probabilistic quasi-uniform space, \((X, \Upsilon(\mathcal{U}))\) be a stratified Hutton \([0,1]\)-quasi-uniform space, \( F \) be a 1-filter, \( v \) be a stratified I-filter and \( p \in X \). Then:

(1) \( F \) converges to \( p \) in \((X, \mathcal{U})\) if and only if \( v_F \) converges to \( p \) in \((X, \Lambda(\mathcal{U}))\);

(2) \( v \) converges to \( p \) in \((X, \mathcal{U})\) if and only if \( F_v \) converges to \( p \) in \((X, \Upsilon(\mathcal{U}))\).

Proof. (1) Necessity: Let \( U \in \mathcal{U} \) and \( \Lambda(U) \) be a base element for \((X, \Lambda(\mathcal{U}))\). Since \( F \) converges to \( p \) in \((X, \mathcal{U})\), we have \( N_{\mathcal{U}} \subseteq F \). Then for any \( U(-, p) \in N_{\mathcal{U}} \), there is \( U(-, p) \in F \). Hence,
\[
v_F(\Lambda(U)(1_{(p_0)})) = \bigvee_{\alpha \in [0,1]} \alpha \rightarrow \Lambda(U)(1_{(p_0)}) \in F = \bigvee_{\alpha \in [0,1]} \alpha \rightarrow U(-, p) \in F \geq \{1 \in [0,1] | 1 \rightarrow U(-, p) \in F \} = 1.
\]
Therefore, \( v_F \) converges to \( p \) in \((X, \Lambda(\mathcal{U}))\).

Sufficiency: Let \( U \in \mathcal{U} \) and \( U(-, p) \in N_{\mathcal{U}} \). Since \( v_F \) converges to \( p \) in \((X, \Lambda(\mathcal{U}))\), there exist \( \Lambda(U) \) being a base element for \( \Lambda(\mathcal{U}) \) such that \( v_F(\Lambda(U)(1_{(p_0)})) = 1 \). Specifically,
\[
v_F(\Lambda(U)(1_{(p_0)})) = \bigvee_{\alpha \in [0,1]} \alpha \rightarrow \Lambda(U)(1_{(p_0)}) \in F = \bigvee_{\alpha \in [0,1]} \alpha \rightarrow U(-, p) \in F = 1.
\]
So for any \( \alpha \in [0,1] \), we can find \( \beta \in [0,1] \) satisfying \( \beta \rightarrow U(-, p) \in F \) such that \( \beta \geq \alpha \). Hence,
\[
\bigvee_{B \in F} S(B, U(-, p)) \geq \bigvee_{\alpha \in [0,1]} S(\beta \rightarrow U(-, p), U(-, p)) = \bigvee_{\alpha \in [0,1]} \bigvee_{x \in X} ((\beta \rightarrow U(x, p)) \rightarrow U(x, p)) \geq \bigvee_{\alpha \in [0,1]} \beta \geq \bigvee_{\alpha \in [0,1]} \alpha = 1.
\]
Therefore, \( U(-, p) \in F \).

(2) Necessity: Let \( \phi \in \mathcal{B} \) and \( \Upsilon(\phi)(-p) \in N_{\Upsilon(\mathcal{U})} \), where \( \mathcal{B} \) is a base for \( \mathcal{U} \). Since \( v \) converges to \( p \) in \((X, \mathcal{U})\), we have \( v(\Upsilon(\phi)(-p)) = 1 \). Hence, \( \Upsilon(\phi)(-p) \in F_v \).

Sufficiency: Let \( \phi \in \mathcal{B} \), where \( \mathcal{B} \) is a base for \( \mathcal{U} \). Since \( F_v \) converges to \( p \) in \((X, \Upsilon(\mathcal{U}))\), we have \( \Upsilon(\phi)(-p) \in N_{\mathcal{U}} \subseteq F_v \). Furthermore, we have \( v(\Upsilon(\phi)(-p)) = 1 \). Therefore, \( v(\phi(1_{(p_0)})) = v(\Upsilon(\phi)(-p)) = 1 \).

Theorem 4.6. Let \((X, \mathcal{U})\) be a probabilistic quasi-uniform space and \( \Lambda(\mathcal{U}) \) be Hutton \([0,1]\)-quasi-uniformity induced by the mapping \( \Lambda \). Then, \((X, \mathcal{U})\) is Cauchy 1-complete if and only if \((X, \Lambda(\mathcal{U}))\) is complete.

Proof. Sufficiency: Suppose that \((X, \Lambda(\mathcal{U}))\) is complete, then \((X, \mathcal{U})\) is Cauchy 1-complete by Proposition 4.5(1) and Proposition 4.3(1).

Necessity: Let \( v \) be a Cauchy I-filter on \((X, \Lambda(\mathcal{U}))\). Since \( \Upsilon(\Lambda(\mathcal{U})) = \mathcal{U} \), \( F_v \) is a Cauchy filter on \((X, \Upsilon(\Lambda(\mathcal{U})))\) by Proposition 4.3(2). Since \((X, \Upsilon(\Lambda(\mathcal{U})))\) is Cauchy 1-complete, then \( F_v \) converges to \( p \) in \((X, \Upsilon(\Lambda(\mathcal{U})))\). By Proposition 4.3(2), we have \( v_{F_v} \) converges to \( p \) in \((X, \Lambda(\mathcal{U})))\). On account of \( v_{F_v} \leq v \), it follows that \( v \) converges to \( p \) in \((X, \Lambda(\mathcal{U})))\). Hence, \((X, \Lambda(\mathcal{U})))\) is complete.

Theorem 4.7. Let \((X, \mathcal{U})\) be a Hutton \([0,1]\)-quasi-uniform space and \( \Upsilon(\mathcal{U}) \) be probabilistic quasi-uniformity induced by the mapping \( \Upsilon \). Then, if \((X, \Upsilon(\mathcal{U}))\) is Cauchy 1-complete, then \((X, \mathcal{U})\) is complete.

Proof. It is easy to check the result by Proposition 4.3(2) and Proposition 4.4(2).
5 Completeness in fuzzy quasi-metric spaces

In this section, we will discuss the relationship between the completeness of induced probabilistic quasi-uniform spaces and induced classical quasi-uniform spaces and the relationship between the completeness of induced probabilistic quasi-uniform spaces and induced Hutton \([0,1]\)-quasi-uniform space in fuzzy quasi-metric spaces. Let \((X, M, \ast)\) be a fuzzy quasi-metric space. From [2], we can associate \((X, M, \ast)\) with a classical quasi-uniformity \(U_M\) induced by the base \(\{U_{\varepsilon,t} | \varepsilon \in (0, 1), t > 0\}\), where \(U_{\varepsilon,t} = \{(x, y) | M(x, y, t) > 1 - \varepsilon\}\). We can also associate \((X, M, \ast)\) with a probabilistic quasi-uniformity \(U_M\) induced by the base \(\{M(\ast, - , t) | t > 0\}\), i.e., \(U_M = \{U \in [0, 1]^{X \times X} | \bigvee_{t>0} S(M(\ast, - , t), U) = 1\}\).

From [3], we can associate \((X, M, \ast)\) with a Hutton \([0,1]\)-quasi-uniformity \(U_M\) induced by the base \(\{\phi^M_{\varepsilon,t} | \varepsilon \in (0, 1), t > 0\}\), where the mapping \(\phi^M_{\varepsilon,t} : [0, 1]^{X} \rightarrow [0, 1]^{X}\) is defined by \(\phi^M_{\varepsilon,t}(\alpha \ast 1_{\{x\}})(y) = \alpha \ast ((1 - \varepsilon) \rightarrow M(x, y, t))\) for each \(x, y \in X\) and \(\alpha \in [0, 1]\).

**Lemma 5.1.** Let \((X, M, \ast)\) be a fuzzy quasi-metric space, \(F\) be a classical filter on \(X\) and \(\mathcal{F}\) be a 1-filter. Then:

1. \(\mathcal{F}\) is a Cauchy filter in \((X, U_M)\) if and only if \(\omega(\mathcal{F})\) is a Cauchy \(\ast\)-filter in \((X, U_M)\);
2. If \(\mathcal{F}\) is a Cauchy \(\ast\)-filter in \((X, U_M)\), then \(\omega(\mathcal{F})\) is a Cauchy filter in \((X, U_M)\).

**Lemma 5.2.** Let \((X, M, \ast)\) be a fuzzy quasi-metric space, \(F\) be a classical filter on \(X\) and \(\mathcal{F}\) be a 1-filter and \(x_0 \in X\). Then:

1. \(F\) converges to \(x_0\) in \((X, U_M)\) if and only if \(\omega(F)\) converges to \(x_0\) in \((X, U_M)\);
2. If \(\mathcal{F}\) converges to \(x_0\) in \((X, U_M)\), then \(\omega(\mathcal{F})\) converges to \(x_0\) in \((X, U_M)\).

**Corollary 5.3.** Let \((X, M, \ast)\) be a fuzzy quasi-metric space and \(x_0 \in X\). If \(\mathcal{F}\) is an induced 1-filter, Then:

1. \(\mathcal{F}\) is a Cauchy \(\ast\)-filter in \((X, U_M)\) if and only if \(\omega(\mathcal{F})\) is a Cauchy filter in \((X, U_M)\);
2. \(\mathcal{F}\) converges to \(x_0\) in \((X, U_M)\) if and only if \(\omega(\mathcal{F})\) converges to \(x_0\) in \((X, U_M)\).

The lemmas above can be similarly proved according to [30]. Through the lemmas above, we obtain the relationship between the completeness of induced probabilistic quasi-uniform spaces and induced classical quasi-uniform spaces in fuzzy quasi-metric spaces.

**Theorem 5.4.** If \((X, U_M)\) is Cauchy 1-complete, then \((X, U_M)\) is complete.

**Corollary 5.5.** \((X, U_M)\) is induced Cauchy 1-complete if and only if \((X, U_M)\) is complete.

By the following proposition, we have the relationship between the completeness of induced probabilistic quasi-uniform spaces and induced Hutton \([0,1]\)-quasi-uniform spaces. According to the idea of Gutiérrez García [11] Corollary 16, we have the following results.

**Proposition 5.6.** Let \((X, M, \ast)\) be a fuzzy quasi-metric space. Then:

1. \(U_M = \Lambda(U_M)\);
2. \(U_M = \Upsilon(U_M)\).

**Proof.** (1) We know that the collection \(\Lambda((1 - \varepsilon) \rightarrow M(\ast, - , t)) | \varepsilon \in (0, 1), t > 0\) is a basis for \(\Lambda(U_M)\). According to the definition of \(\Lambda\), we have that \(\Lambda((1 - \varepsilon) \rightarrow M(\ast, - , t)) \ast 1_{\{y\}}(x) = (1 - \varepsilon) \rightarrow M(x, y, t)\) as desired.

(2) We know that the collection \(\Upsilon(\phi^M_{\varepsilon,t}) | \varepsilon \in (0, 1), t > 0\) is a basis for \(\Upsilon(U_M)\). According to the definition of \(\Upsilon\), we have that \(\Upsilon(\phi^M_{\varepsilon,t})(x, y) = \phi^M_{\varepsilon,t}(1_{\{y\}})(x) = (1 - \varepsilon) \rightarrow M(x, y, t)\) for each \(\varepsilon \in (0, 1)\) and \(t > 0\) as desired. \(\Box\)

**Theorem 5.7.** Let \((X, M, \ast)\) be a fuzzy quasi-metric space, \(U_M\) be an induced probabilistic quasi-uniformity and \(U_M\) be an induced Hutton \([0,1]\)-quasi-uniformity. Then, \((X, U_M)\) is Cauchy 1-complete if and only if \((X, U_M)\) is complete.

6 Conclusions

In this paper, we use 1-filters to give a kind of Cauchy 1-completeness of probabilistic quasi-uniform spaces. We have established close relationships between the completeness of probabilistic quasi-uniform spaces and both completeness of classical quasi-uniform spaces and Hutton \([0,1]\)-quasi-uniform spaces. In the framework of fuzzy quasi-metric spaces, we establish the relationship between completeness of induced classical quasi-uniform spaces and induced probabilistic quasi-uniform spaces and the relationship between the completeness of induced probabilistic quasi-uniform spaces and induced Hutton \([0,1]\)-quasi-uniform spaces. These results illustrate the reasonableness of 1-filters in discussing the problem of completeness in probabilistic quasi-uniform spaces. Consequently, we have reasons to believe 1-filters are a good tool to study probabilistic quasi-uniform convergence spaces. In the future, we will investigate the relationship between the completeness of probabilistic quasi-uniform spaces and probabilistic quasi-uniform convergence spaces.
Acknowledgement

This work was supported by NSFC (11471297,11401547), Natural Science foundation of Shandong Province (ZR2017MA017).

References


