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System of fuzzy fractional differential equations in generalized metric space in the sense of Perov

H. V. Long[1], N. T. K. Son[2] and H. T. P. Thao[3]

[1] *Division of Computational Mathematics and Engineering, Institute for Computational Science, Ton Duc Thang University, Ho Chi Minh City, Vietnam and Faculty of Mathematics and Statistics, Ton Duc Thang University, Ho Chi Minh City, Vietnam.*
[2] *Faculty of natural sciences, Hanoi Metropolitan University, Hanoi, Vietnam*
[3] *Department of Mathematics, Hanoi National University of Education, Hanoi, Vietnam*

hoangvietlong@tdtu.edu.vn, ntkson@daihocthudo.edu.vn, hoangphuongthaohnue@gmail.com

Abstract

In this paper, we study the existence of integral solutions of fuzzy fractional differential systems with nonlocal conditions under Caputo generalized Hukuhara derivatives. These models are considered in the framework of complete generalized metric spaces in the sense of Perov. The novel feature of our approach is the combination of the convergent matrix technique with Schauder fixed point principle of vector valued operators in semilinear Banach spaces. Some computational examples are represented to demonstrate our theoretical results.

Keywords: Fuzzy fractional differential systems, Caputo gH-derivatives, vector valued metric, generalized metric space.

1 Introduction

In the theory of differential equations, nonlocal conditions (NCs) arise when we cannot measure data directly at the boundary. It was remarked by Byszewski et al. [12, 13] that the NCs can be used to depict motion phenomena more effectively than the classical Cauchy condition. The first problem modeled by NCs was the investigation of a mercury droplet in electric contact, when the droplet volume was given [36]. There exists a wide literature on DEs subject to NCs and we refer to the papers of Boucherif [11] for first order DEs, to Byszewski and Lakshmikantham [12] for semilinear evolution equations, to Octavia and Precup [28, 29] for the first order and implicit differential systems.

Recently, fractional differential equations (FDEs) with uncertainty have been attracting more and more attentions, since they are capable of modeling many real world processes. Agarwal et al. [1, 2] are the pioneers studied in fuzzy FDEs. They proposed the concept of fuzzy solutions for the FDEs with uncertainty under Riemann-Liouville differentiability. We note that the uncertainty in [1] was handled only in the initial conditions, no differentiability of fuzzy valued functions was concerned. Allahviranloo et al. [7] and Mazandarani et al. [26] introduced the Riemann-Liouville and Caputo differentiability concepts for fuzzy valued functions by using H-difference and proved the existence of solutions for some classes of fuzzy FDEs. Unfortunately, H-differentiable functions have the increasing length of their supports in time variable. Some authors overcome this limitation by considering the gH-differentiability notion, see for example in [8, 14, 15, 18, 21]. This approach can improve the set of fuzzy solutions and state further researches on the asymptotic behavior of fuzzy solutions. In the aspects of application, these concepts were applied to study some real-world problems such as: the Falling body, the Basset and the Decay problem, etc. [4, 5, 6, 31, 32, 33, 34]. Some mathematical foundations for random fuzzy fractional integral equations are considered by Malinowski [25].

The space of fuzzy valued functions is not a linear space [3, 21]. So that, we meet many troubles when using topological fixed point theorems in this abstract space. Agarwal et al. [3] proposed a fruitful alternative approach to study fuzzy DEs in the framework of semilinear Banach spaces. They generalized Schauder fixed point theorem to the

Corresponding Author: H. V. Long
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framework of semilinear Banach spaces and presented an application to the existence of solutions of fuzzy FDEs under H-differentiability.

In the present paper, we propose a new approach to deal with the existence of solution for a class of fuzzy fractional differential systems with NCs under Caputo gH-differentiability. Firstly, we will prove that the considered frameworks are the complete generalized metric space (in the sense of Perov) equipped vector valued metric (see Lemma 3.4). Secondly, we transfer the existence of solutions of problem (3)-(4) into fixed point problem of vector valued operators $T = (T_1, T_2)$ (see Lemma 3.1, Remark 3.4). To this end, we split the component T_i $(i = 1, 2)$ into two integral operators, where one is Fredholm type for the subinterval containing the points involved by the NCs and another one is Volterra type for the rest of the interval (see Remark 3.5). This approach allows us to combine the technique that uses convergent matrices and vector valued version of Schauder fixed point theorem in semilinear Banach space. Finally, the existence of four types of integral solutions was given in Theorems 3.11-3.12. The nonlinearities factors f and g satisfy the more relaxed conditions than Lipschitz condition as using in previous results [22, 23, 24], namely, the condition of at most linear growth (see Hypothesis (H1)). A computational example to illustrate our theoretical results is given in Section 4. And as usual, some conclusions and future works are discussed in Section 5.

2 Preliminaries

In this section we will briefly recall some necessary preliminaries, see [21, 22].

2.1 Generalized metric space and convergent matrices

Let X be a nonempty set, mapping $d_X : X \times X \to \mathbb{R}^n_+$ is called a vector valued metric on X if it satisfies

(i) $d_X(u, v) \geq 0, \forall u,\ v \in X$ and $d_X(u, v) = 0 \Rightarrow u = v$;

(ii) $d_X(u, v) = d_X(v, u), \forall u, v \in X$,

(iii) $d_X(u, w) \leq d_X(u, v) + d_X(v, w), \forall u, v, w \in X$.

Here, for $x = (x_1, x_2, \dots, x_n), y = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$, $x \leq y$ if and only if $x_i \leq y_i$ for all $i = 1, 2, \dots, n$.

We call the pair (X, d_X) is a generalized metric space (in the sense of Perov, see [29]).

A square matrix M with nonnegative elements is said to be convergent matrix if M^k converges to zero matrix when $k \to \infty$.

Proposition 2.1 ([27], Page 3)**.** *The property of being convergent matrix M is equivalent to each of the following conditions:*

(i) the eigenvalues of M are located inside the unit disc of the complex plane;

(ii) $I - M$ is nonsingular and $(I - M)^{-1}$ has nonnegative elements.

Lemma 2.2 ([29], Lemma 1.1)**.** *If A is a convergent matrix and all elements of the same size matrix B are small enough, then $A + B$ is also a convergent matrix.*

2.2 Complete generalized fuzzy metric space

Let E be the space of fuzzy sets on $\mathbb{R}$, that are nonempty subsets $\{(x, u(x)) : x \in \mathbb{R}\}$ in $\mathbb{R} \times [0, 1]$ of certain functions u: $\mathbb{R} \to [0, 1]$ being normal, fuzzy convex, semi-continuous and compact support. Let $u \in E$, the $\alpha-$level sets of u are defined by $[u]^\alpha = \begin{cases} \{x \in \mathbb{R} : u(x) \geq \alpha\} & \text{if } \alpha \in (0, 1], \\ \overline{\{x \in \mathbb{R} : u(x) > 0\}} & \text{if } \alpha = 0, \end{cases}$ which are nonempty compact, convex subsets of $\mathbb{R}$ for all $\alpha \in [0, 1]$. Denote $\hat{0}$ by $[\hat{0}]^\alpha = [0, 0]$ for all $\alpha \in [0, 1]$.

As a result of the Zadeh's extension principle (see [9]), the operation of addition and scalar multiplication of fuzzy numbers on E have the form $[u \oplus v]^\alpha = [u]^\alpha + [v]^\alpha$, $[\lambda \odot u]^\alpha = \lambda[u]^\alpha$, $\lambda \in \mathbb{R}$, where $[u]^\alpha + [v]^\alpha = \{a + b : a \in [u]^\alpha,\ b \in [v]^\alpha\}$ is the Minkowski sum of $[u]^\alpha$ and $[v]^\alpha$ and $\lambda[u]^\alpha = \{\lambda a : a \in [u]^\alpha\}$.

For $u \in E$, the $\alpha-$level sets of u can be presented in parametric form $[u]^\alpha = [u^-_\alpha, u^+_\alpha]$, $\alpha \in [0, 1]$ and $len[u]^\alpha = u^+_\alpha - u^-_\alpha$ is the diameter of $[u]^\alpha$.

A triangular fuzzy number denoted by $u = (a, b, c)$, $a < b < c$ is defined by $u(x) = \begin{cases} \frac{x-a}{b-a} & \text{if } a \leq x \leq b \\ \frac{c-x}{c-b} & \text{if } b \leq x \leq c \\ 0 & \text{if } x \notin [a, c]. \end{cases}$ The level set of u is represented in form $[u]^\alpha = [(b - a)\alpha + a, (b - c)\alpha + c]$, $\alpha \in [0, 1]$.

We recall that (E, d_∞) is a complete metric space with $d_\infty(u, v) = \sup\limits_{0 \leq \alpha \leq 1} d_H([u]^\alpha, [v]^\alpha)$, where d_H is the well-known Hausdorff-Pompeiu distance between compact convex sets.

If there exists $w \in E$ such that $u = v \oplus w$, we call $w = u \ominus v$ the H-difference of u and v. If $u \ominus v$ exists, it is unique and $[u \ominus v]^\alpha = [u_\alpha^- - v_\alpha^-, u_\alpha^+ - v_\alpha^+]$ for all $0 \le \alpha \le 1$.

From here, one can easily find that d_∞ has the following properties

(a) $d_\infty(u \ominus v, w \ominus e) \le d_\infty(u, w) + d_\infty(v, e)$, $\forall u,\ v,\ w,\ e \in E$ and H-differences $u \ominus v,\ w \ominus e$ exist,

(b) $d_\infty(u \oplus w, v \oplus w) = d_\infty(u, v)$, $\forall u,\ v,\ w \in E$,

(c) $d_\infty(\lambda \odot u, \lambda \odot v) = |\lambda| d_\infty(u, v)$, $\forall u,\ v \in E,\ \lambda \in \mathbb{R}$,

(d) $d_\infty(u \oplus e, v \oplus f) \le d_\infty(u, v) + d_\infty(e, f)$, $\forall u,\ v,\ e,\ f \in E$.

For $u, v \in E$, the gH-difference of u and v, denoted by $u \ominus_{gH} v$ is defined as the element $w \in E$ such that (i) $u = v \oplus w$ or (ii) $v = u \oplus (-1) \odot w$. Notice that if $u \ominus v$ exists, then $u \ominus_{gH} v = u \ominus v$.

Denote E_c by the space of all fuzzy numbers $u \in E$ with the property that the function $\alpha \mapsto [u]^\alpha$ is continuous with respect to the Hausdorff-Pompeiu metric d_H on $[0,1]$. On the space E_c, we recall some following concepts (see in [3]).

(1) A subset $A \subset E_c$ is said to be compact-supported if there exists a compact set $K \subseteq \mathbb{R}$ such that $[u]^0 \subseteq K$ for all $u \in A$.

(2) A subset $A \subset E_c$ is said to be level-equicontinuous at $\alpha_0 \in [0,1]$ if for all $\epsilon > 0$, there exists $\delta > 0$ such that $|\alpha - \alpha_0| < \delta$ implies $d_H([u]^\alpha, [u]^{\alpha_0}) < \epsilon$ for all $u \in A$. A is level-equicontinuous on $[0,1]$ if A is level-equicontinuous at all $\alpha \in [0,1]$.

(3) Let $J = [a, b] \subset \mathbb{R}$, a continuous function $f : J \times E_c \to E_c$ is said to be compact if for all $I \subseteq J,\ A \subseteq E_c$ is bounded, we imply that $f(I \times A)$ is relatively compact in E_c.

Lemma 2.3 ([3], Theorem 2.2)**.** *Let A be a compact-supported subset of E_c. Then the following are equivalent*

(1) A is a relatively compact subset of (E_c, d_∞).

(2) A is level-equicontinuous on $[0,1]$.

Remark 2.4. *Denote $C(J, E_c)$ by the space of all continuous functions $f : J \to E_c$. Then $C(J, E_c)$ is a semilinear Banach space having the cancelation property (see [3]). Based on Theorem 3.4 in [3], we can indicate a variant of the Schauder fixed point theorem in space $C(J, E_c)$ as follows.*

Theorem 2.5 (Schauder fixed point principle in semilinear Banach spaces)**.** *Let B be a nonempty, closed, bounded and convex subset of $C(J, E_c)$ and $P : B \to B$ be a compact operator. Then P has at least one fixed point in B.*

In space $C(J, E_c)$, we construct the supremum metric $H_J(u, \overline{u}) = \sup_{t \in J} d_\infty(u(t), \overline{u}(t))$ and the weighted metric

$$\tilde{H}_J(u, \overline{u}) = \sup_{t \in J} \left\{ e^{-\lambda(t-a)} d_\infty(u(t), \overline{u}(t)) \right\}, \text{ where } \lambda > 0.$$

For fixed $0 < \mu \le 1$, denote $H_\mu(u, \overline{u}) = \max\{H_{[0,\mu]}(u, \overline{u}), \tilde{H}_{[\mu,1]}(u, \overline{u})\}$. We can see that $(C(J, E_c), H_J)$, $(C(J, E_c), \tilde{H}_J)$ and $(C([0,1], E_c), H_\mu)$ are complete metric spaces (see in [18, 21, 22]).

Denote $\mathbb{Y} := C(J, E_c) \times C(J, E_c)$ and vector valued metric $\rho : \mathbb{Y} \times \mathbb{Y} \to \mathbb{R}_+^2$ with $\rho(\nu, \overline{\nu}) = (H_\mu(u, \overline{u}), H_\mu(v, \overline{v}))$, where $\nu = (u, v),\ \overline{\nu} = (\overline{u}, \overline{v}) \in \mathbb{Y}$. It follows from Lemma 2.3 in [22] that $(\mathbb{Y}, \rho)$ is a complete generalized metric space.

Definition 2.6 ([10], Definition 20)**.** *Let $t_0 \in J,\ h \in \mathbb{R}$ such that $t_0 + h \in J$ and $f \in C(J, E)$. If there exists $f'(t_0) \in E$ such that $f'(t_0) = \lim_{h \to 0} \frac{1}{h} \odot [f(t_0 + h) \ominus_{gH} f(t_0)]$ then we say that f is gH-differentiable at t_0. Denote $C^1(J, E)$ by the space of all continuously gH-differentiable functions on J.*

Definition 2.7 ([10], Definition 26)**.** *Suppose that $f \in C(J, E)$ is gH-differentiable at $t_0 \in J$, $[f(t)]^\alpha = [f_\alpha^-(t), f_\alpha^+(t)]$ for all $0 \le \alpha \le 1$, $t \in J$. Then we say that*

(i) f is (i)-gH differentiable at t_0 if $[f'(t_0)]^\alpha = [(f_\alpha^-)'(t_0), (f_\alpha^+)'(t_0)],\ \forall \alpha \in [0,1]$. *(1)*

(ii) f is (ii)-gH differentiable at t_0 if $[f'(t_0)]^\alpha = [(f_\alpha^+)'(t_0), (f_\alpha^-)'(t_0)],\ \forall \alpha \in [0,1]$. *(2)*

Adapting to Definition 26 in [37], we have following definition.

Definition 2.8. *We say that a point $t_0 \in J$ is a switching point for the differentiability of f, if in any neighborhood V of $t_0 \in J$ there exist points $t_1 < t_0 < t_2$ in J such that*

(type I) at t_1 (1) holds while (2) does not hold and at t_2 (2) holds while (1) does not hold, or

(type II) at t_1 (2) holds while (1) does not hold and at t_2 (1) holds while (2) does not hold.

The left-sided Riemann-Liouville fractional integrals of order $q \in (0,1]$ (see [16]) for a real function $f : \mathbb{R} \to \mathbb{R}$ is defined by ${}^{RL}I_{0^+}^q f(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s) ds,\ t \in \mathbb{R}$, provided that the integral is well-defined and Γ is the Gamma function.

Definition 2.9 ([21], Lemma 3.1)**.** *Let $q \in (0,1]$ and $u \in C(J,E)$, $[u(t)]^\alpha = [u_\alpha^-(t), u_\alpha^+(t)]$, $\alpha \in [0,1]$. The Riemann - Liouville fractional integral of order q for fuzzy valued function u, denoted by ${}^{RL}_{F}\mathcal{I}^q_{0+}u(t) = \dfrac{1}{\Gamma(q)} \odot \int_0^t (t-s)^{q-1} \odot u(s)ds,\ t \in J$, is defined by levelsetwise, for all $\alpha \in [0,1]$, $[{}^{RL}_{F}\mathcal{I}^q_{0+}u(t)]^\alpha = [{}^{RL}I^q_{0+}u_\alpha^-(t), {}^{RL}I^q_{0+}u_\alpha^+(t)],\ t \in J$*

Proposition 2.10 ([21], Proposition 3.1)**.** *Let $p, q \in (0,1]$ such that $p+q \in (0,1]$ and $u \in C(J,E)$ then $({}^{RL}_{F}\mathcal{I}^p_{0+})({}^{RL}_{F}\mathcal{I}^q_{0+})u = {}^{RL}_{F}\mathcal{I}^{p+q}_{0+}u$.*

Definition 2.11. *Let $q \in [0,1)$ and $u \in C^1(J,E)$. The Caputo fractional gH-derivative of order q is defined by ${}^{C}_{gH}\mathcal{D}^q u(t) = {}^{RL}_{F}\mathcal{I}^{1-q}_{0+}u'(t),\ t \in J$, provided that the expression on the right hand side is defined, where u' is gH-derivative of u.*

3 The nonlocal problem for fractional differential systems

We consider the following fuzzy fractional differential system of order $q \in (0,1)$

$$\begin{cases} {}^{C}_{gH}\mathcal{D}^q u(t) = f(t,u(t),v(t)) \\ {}^{C}_{gH}\mathcal{D}^q v(t) = g(t,u(t),v(t)) \end{cases} \quad t \in J := [0,1] \tag{3}$$

with NCs

$$\begin{cases} u(0) \oplus \sum\limits_{k \in J_1} a_k \odot u(t_k) = \sum\limits_{k \in J_2} a_k \odot u(t_k) \\ v(0) \oplus \sum\limits_{k \in Q_1} \tilde{a}_k \odot v(t_k) = \sum\limits_{k \in Q_2} \tilde{a}_k \odot v(t_k). \end{cases} \tag{4}$$

Here

(1) $f,\ g : J \times E_c \times E_c \to E_c$ are given mappings such that $f(.,u(.),v(.))$ and $g(.,u(.),v(.))$ belong to $L^1(J,E_c)$-the space of fuzzy valued integrable functions;

(2) $J_1 \cap J_2 = Q_1 \cap Q_2 = \emptyset$ and $J_1 \cup J_2 = Q_1 \cup Q_2 = \{1,2...,m\}$;

(3) $t_1, t_2, ..., t_m$ are given points satisfying $0 < t_1 \le t_2 \le \ldots \le t_m < 1$;

(4) $a_k, \tilde{a}_k,\ k = 1,2,...,m$ are real positive numbers such that

$$\sum_{k \in J_2} a_k t_k^q - \sum_{k \in J_1} a_k t_k^q > 0 \text{ and } \sum_{k \in Q_2} \tilde{a}_k t_k^q - \sum_{k \in Q_1} \tilde{a}_k t_k^q > 0,$$

such that

$$a = \big(1 + \sum_{k \in J_1} a_k - \sum_{k \in J_2} a_k\big)^{-1},\ \tilde{a} = \big(1 + \sum_{k \in Q_1} \tilde{a}_k - \sum_{k \in Q_2} \tilde{a}_k\big)^{-1}$$

are both positive real numbers;

(5) Both u and v do not have switching point for the gH-differentiability on J.

Remark 3.1. *Denote $A_1 = 1 + a\sum\limits_{k=1}^{m} a_k,\ \tilde{A}_1 = 1 + \tilde{a}\sum\limits_{k=1}^{m} \tilde{a}_k$, and $I^t[h](u,v) = \dfrac{1}{\Gamma(q)} \odot \int\limits_0^t (t-s)^{q-1} \odot h(s,u(s),v(s))ds$. For simplicity in presentation, we introduce some integral operators:*

$$\begin{aligned} F_m^1(u,v) &= \sum_{k \in J_2} aa_k \odot I^{t_k}[f](u,v) \ominus \sum_{k \in J_1} aa_k \odot I^{t_k}[f](u,v), \\ G_m^1(u,v) &= \sum_{k \in Q_2} \tilde{a}\tilde{a}_k \odot I^{t_k}[g](u,v) \ominus \sum_{k \in Q_1} \tilde{a}\tilde{a}_k \odot I^{t_k}[g](u,v), \\ F_m^2(u,v) &= \sum_{k \in J_1} (-aa_k) \odot I^{t_k}[f](u,v) \ominus \sum_{k \in J_2} (-aa_k) \odot I^{t_k}[f](u,v), \\ G_m^2(u,v) &= \sum_{k \in Q_1} (-\tilde{a}\tilde{a}_k) \odot I^{t_k}[g](u,v) \ominus \sum_{k \in Q_2} (-\tilde{a}\tilde{a}_k) \odot I^{t_k}[g](u,v). \end{aligned}$$

Lemma 3.2. *Suppose that $u,\ v \in C^1(J,E_c)$ satisfy the nonlocal problem (3)-(4).*

1) If u,v are both $(i)-gH$ differentiable on J then (u,v) is a solution of following fuzzy integral system

$$\begin{cases} u(t) = F_m^1(u,v) \oplus I^t[f](u,v) \\ v(t) = G_m^1(u,v) \oplus I^t[g](u,v) \end{cases}$$

$$(5)$$

2) If u, v are both $(ii)-gH$ differentiable on J then (u, v) is a solution of a following fuzzy integral system

$$\begin{cases} u(t) = F_m^2(u,v) \ominus (-1) \odot I^t[f](u,v) \\ v(t) = G_m^2(u,v) \ominus (-1) \odot I^t[g](u,v) \end{cases} \tag{6}$$

3) If u is $(i)-gH$ differentiable on J, v is $(ii)-gH$ differentiable on J then (u, v) is a solution of a following fuzzy integral system

$$\begin{cases} u(t) = F_m^1(u,v) \oplus I^t[f](u,v) \\ v(t) = G_m^2(u,v) \ominus (-1) \odot I^t[g](u,v) \end{cases} \tag{7}$$

4) If u is $(ii)-gH$ differentiable on J, v is $(i)-gH$ differentiable on J then (u, v) is a solution of a following fuzzy integral system

$$\begin{cases} u(t) = F_m^2(u,v) \ominus (-1) \odot I^t[f](u,v) \\ v(t) = G_m^1(u,v) \oplus I^t[g](u,v) \end{cases} \tag{8}$$

Proof. If $u \in C^1(J, E_c)$ and u is (i)-gH differentiable, by taking integral operator ${}_F^{RL}\mathcal{I}_{0+}^q$ both sides of equation ${}_{gH}^C\mathcal{D}^q u(t) = f(t, u(t), v(t))$, we have $\int\limits_0^t u'(s)ds = {}_F^{RL}\mathcal{I}_{0+}^q f(t, u(t), v(t))$. This implies

$$u(t) \ominus u(0) = \frac{1}{\Gamma(q)} \odot \int\limits_0^t (t-s)^{q-1} \odot f(s, u(s), v(s))ds.$$

Thus, $u(t) = u(0) \oplus \dfrac{1}{\Gamma(q)} \odot \int\limits_0^t (t-s)^{q-1} \odot f(s, u(s), v(s))ds = u(0) \oplus I^t[f](u,v).$ (9)

For each t_k $(k = 1, ..., m)$, we have $u(t_k) = u(0) \oplus I^{t_k}[f](u, v)$. Substitute these equations into (3), we get

$$u(0) \oplus \sum_{k\in J_1} a_k \odot (u(0) \oplus I^{t_k}[f](u,v)) = \sum_{k\in J_2} a_k \odot (u(0) \oplus I^{t_k}[f](u,v)).$$

Since $1 + \sum\limits_{k\in J_1} a_k - \sum\limits_{k\in J_2} a_k > 0$, we have $\big(1 + \sum\limits_{k\in J_1} a_k - \sum\limits_{k\in J_2} a_k\big) \odot u(0) = \sum\limits_{k\in J_2} a_k \odot I^{t_k}[f](u,v) \ominus \sum\limits_{k\in J_1} a_k \odot I^{t_k}[f](u,v)$. It implies

$$u(0) = a\Big(\sum_{k\in J_2} a_k \odot I^{t_k}[f](u,v) \ominus \sum_{k\in J_1} a_k \odot I^{t_k}[f](u,v)\Big) = F_m^1(u,v). \tag{10}$$

By combining (9) and (10) we obtain the first equation in system (5).

If u is (ii)-gH differentiable, then by taking integral both sides of the formula ${}_{gH}^C\mathcal{D}^q u(t) = f(t, u(t), v(t)$, we have

$$(-1) \odot u(0) \ominus (-1) \odot u(t) = \frac{1}{\Gamma(q)} \odot \int\limits_0^t (t-s)^{q-1} \odot f(s, u(s), v(s))ds.$$

Therefore, $u(t) = u(0) \ominus (-1) \odot I^t[f](u, v)$ for all $t \in J$. (11)

This implies $u(t_k) = u(0) \ominus (-1)I^{t_k}[f](u, v),\ k = 1, ..., m$. By substituting these equations into (4) we obtain

$$u(0) = \sum_{k\in J_1} (-aa_k) \odot I^{t_k}[f](u,v) \ominus \sum_{k\in J_2} (-aa_k) \odot I^{t_k}[f](u,v) = F_m^2(u,v). \tag{12}$$

From (11) and (12), we obtain the first equation of system (6).

Now by combining the equation of u and v for both cases of $(i)-$gH differentiability and $(ii)-$gH differentiability we receive four fuzzy integral systems (5)-(8). □

Example 3.3. *We consider following fuzzy fractional differential equations*

$$\begin{cases} {}^{C}_{gH}\mathcal{D}^{1/3}u(t) = t^2 \odot K \\ {}^{C}_{gH}\mathcal{D}^{1/3}v(t) = t^3 \odot K \end{cases} \tag{13}$$

for $t \in [0,1]$, $K = (1,2,3)$ *with NCs*

$$\begin{cases} u(0) + \dfrac{1}{4\sqrt[3]{2}}u(1) = u\left(\dfrac{1}{2}\right) \\ v(0) + \dfrac{1}{8\sqrt[3]{2}}v(1) = v\left(\dfrac{1}{2}\right). \end{cases} \tag{14}$$

Let $f(t,u(t),v(t)) = t^2 \odot K$, $g(t,u(t),v(t)) = t^3 \odot K$ *and* $I^t[f](u,v) = {}^{RL}_{F}\mathcal{I}^{\frac{1}{3}}_{0+}(t^2 \odot K)$. *Applying formula* ${}^{RL}I^{\alpha}_{0+}(x-a)^{\beta-1} = \dfrac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}(x-a)^{\alpha+\beta-1}$ *(see [35], page 173), we have* $I^t[f](u,v) = C_1 t^{\frac{7}{3}} \odot K$ *with* $C_1 = \dfrac{\Gamma(3)}{\Gamma(\frac{10}{3})}$.

Doing the same way, we receive $I^t[g](u,v) = {}^{RL}_{F}\mathcal{I}^{\frac{1}{3}}_{0+}(t^3 \odot K) = C_2 t^{\frac{10}{3}} \odot K$ *with* $C_2 = \dfrac{\Gamma(4)}{\Gamma(\frac{13}{3})}$. *With* $m = 2$, $t_1 = 1, t_2 = \dfrac{1}{2}, a_1 = \dfrac{1}{4\sqrt[3]{2}} a_2 = 1, \tilde{a}_1 = a_1 = \dfrac{1}{8\sqrt[3]{2}}, \tilde{a}_2 = 1$, $a = 4\sqrt[3]{2}, \tilde{a} = 8\sqrt[8]{2}$, *we have*

$$F^1_2(u,v) = \sum_{k=1}^{2} aa_k \odot I^{t_k}[f](u,v) = 4\sqrt[3]{2}\left(\frac{C_1}{\sqrt[3]{2^7}} - \frac{C_1}{4\sqrt[3]{2}}\right) \odot K = \hat{0}$$

$$G^1_2(u,v) = \sum_{k=1}^{2} \tilde{a}\tilde{a}_k \odot I^{t_k}[g](u,v) = 8\sqrt[3]{2}\left(\frac{C_2}{\sqrt[3]{2^{10}}} - \frac{C_2}{8\sqrt[3]{2}}\right) \odot K = \hat{0}.$$

Denote

$$\begin{cases} \hat{u}(t) = F^1_2(u,v) \oplus I^t[f](u,v) = C_1 t^{\frac{7}{3}} \odot K \\ \hat{v}(t) = G^1_2(u,v) \oplus I^t[g](u,v) = C_2 t^{\frac{10}{3}} \odot K. \end{cases}$$

Thus, $(\hat{u}(t), \hat{v}(t))$ *is a solutions of integral system (5). Let* $\hat{u}^-_\alpha = C_1 t^{\frac{7}{3}}(\alpha+1)$, $\hat{u}^+_\alpha = C_1 t^{\frac{7}{3}}(-\alpha+3)$. *Then* $d(\hat{u}^-_\alpha(t))/dt = \frac{7}{3}C_1 t^{\frac{4}{3}}(\alpha+1)$ *is an increasing function with respect to* α, $d(\hat{u}^+_\alpha(t))/dt = \frac{7}{3}C_1 t^{\frac{4}{3}}(-\alpha+3)$ *is a decreasing function with respect to* α *for all* $t \in [0,1]$. *So from Theorem 24 in [10], we imply that* $\hat{u}(t)$ *is (i)-gH differentiable in* $[0,1]$. *Using similar arguments,* $\hat{v}(t)$ *is (i)-gH differentiable in* $[0,1]$. *Moreover,* $(\hat{u},\hat{v})$ *satisfies system (13) with NCs (14). Hence,* $(\hat{u},\hat{v})$ *is one classical solution of problem (13) - (14).*

Remark 3.4. *If* (u,v) *is a classical solution of problem (3)-(4), i.e.,* u,v *are gH-differentiable on* J *and satisfy (3) and (4). From Lemma 3.2, the classical solution* (u,v) *must be a solution of one of four integral systems (5)-(8).*

Example 3.3 shows a concrete case when u,v *are (i)-gH-differentiable in* $[0,1]$, *satisfy integral system (5) and* (u,v) *is a solution of nonlocal problem (3)-(4).*

However, in the converse direction, if (u,v) *satisfies one of four integral systems (5)-(8), we do not require the gH-differentiability of* u,v. *All conditions considered are the continuity of* u,v *on* J *and satisfing integral systems (5) or (6) or (7) or (8). Then we can not infer that* (u,v) *is a classical solution of problem (3)-(4). The following counter-example will clarify this statement.*

Example 3.5.

$$\begin{cases} {}^{C}_{gH}\mathcal{D}^{1/2}u(t) = (t^{\frac{1}{2}}, t^{-\frac{1}{2}} + \frac{1}{2}, t^{-\frac{1}{2}} + 1) \\ {}^{C}_{gH}\mathcal{D}^{1/2}v(t) = \frac{2}{\sqrt{\pi}}(t^{\frac{3}{2}}, t^{\frac{1}{2}} + \frac{1}{2}, t^{\frac{1}{2}} + 1) \end{cases} \quad t \in J = (0,1) \tag{15}$$

with NCs

$$\begin{cases} u(0) = \frac{1}{2}u\left(\frac{1}{\sqrt{\pi}}\right) \\ v(0) = \frac{1}{2}v\left(\frac{1}{2}\right). \end{cases} \tag{16}$$

The NCs can be rewritten in the form $\begin{cases} u(0) = \sum\limits_{k=1}^{2} a_k \odot u(t_k) \\ v(0) = \sum\limits_{k=1}^{2} \tilde{a}_k \odot v(t_k) \end{cases}$ *with* $t_1 = \frac{1}{2},\ t_2 = \frac{1}{\sqrt{\pi}}, a_1 = 0, a_2 = \frac{1}{2}, \tilde{a}_1 = \frac{1}{2}, \tilde{a}_2 = 0,$ $a = \tilde{a} = 2$. *Let* $f(t,u,v) = (t^{\frac{1}{2}}, t^{-\frac{1}{2}} + \frac{1}{2}, t^{-\frac{1}{2}} + 1)$, $g(t,u,v) = \frac{2}{\sqrt{\pi}}(t^{\frac{3}{2}}, t^{\frac{1}{2}} + \frac{1}{2}, t^{\frac{1}{2}} + 1)$. *Then*

$$I^t[f](u,v) = \frac{1}{\Gamma(\frac{1}{2})} \odot \int_0^t (t-s)^{-\frac{1}{2}} \odot f(s,u(s),v(s))ds = \left(\frac{\sqrt{\pi}}{2}t, \sqrt{\pi} + \frac{\sqrt{t}}{\sqrt{\pi}}, \sqrt{\pi} + \frac{2\sqrt{t}}{\sqrt{\pi}}\right),$$

$$I^t[g](u,v) = \frac{1}{\Gamma(\frac{1}{2})} \odot \int_0^t (t-s)^{-\frac{1}{2}} \odot g(s,u(s),v(s))ds = (\frac{3}{4}t^2, t + \frac{2\sqrt{t}}{\pi}, t + \frac{4\sqrt{t}}{\pi}),$$

$$F_2^1(u,v) = \sum_{k=1}^{2} aa_k \odot I^{t_k}[f](u,v) = \left(\frac{1}{2}, \sqrt{\pi} + \frac{1}{\sqrt[4]{\pi^3}}, \sqrt{\pi} + \frac{2}{\sqrt[4]{\pi^3}}\right),$$

$$G_2^1(u,v) = \sum_{k=1}^{2} \tilde{a}\tilde{a}_k \odot I^{t_k}[g](u,v) = (\frac{3}{16}, \frac{1}{2} + \frac{\sqrt{2}}{\pi}, \frac{1}{2} + \frac{2\sqrt{2}}{\pi}).$$

Now for $t \in J$, *we consider the following functions*

$$\begin{cases} u(t) = F_2^1 \oplus I^t[f] = \left(\frac{1}{2} + \frac{t\sqrt{\pi}}{2}, 2\sqrt{\pi} + \frac{1}{\sqrt[4]{\pi^3}} + \frac{\sqrt{t}}{\sqrt{\pi}}, 2\sqrt{\pi} + \frac{2}{\sqrt[4]{\pi^3}} + \frac{2\sqrt{t}}{\sqrt{\pi}}\right) \\ v(t) = G_2^1 \oplus I^t[g] = (\frac{3}{4}t^2 + \frac{3}{16}, t + \frac{2\sqrt{t}}{\pi} + \frac{1}{2} + \frac{\sqrt{2}}{\pi}, t + \frac{4\sqrt{t}}{\pi} + \frac{1}{2} + \frac{2\sqrt{2}}{\pi})). \end{cases} \tag{17}$$

It is easy to see that u, v *satisfy integral system (5). However, they do not satisfy nonlocal problem (15)-(16). Indeed, we have*

$$u_\alpha^- = \left[\left(2\sqrt{\pi} + \frac{1}{\sqrt[4]{\pi^3}} - \frac{1}{2}\right) + \frac{\sqrt{t}}{\sqrt{\pi}} - \frac{\sqrt{\pi}t}{2}\right]\alpha + \frac{1}{2} + \frac{\sqrt{\pi}t}{2}, \text{ and } u_\alpha^+ = -\left(\frac{1}{\sqrt[4]{\pi^3}} + \frac{\sqrt{t}}{\sqrt{\pi}}\right)\alpha + 2\sqrt{\pi} + \frac{2}{\sqrt[4]{\pi^3}} + \frac{2\sqrt{t}}{\sqrt{\pi}}.$$

Then $\frac{du_\alpha^-}{dt} = \left(\frac{1}{2\sqrt{\pi t}} - \frac{\sqrt{\pi}}{2}\right)\alpha + \frac{\sqrt{\pi}}{2}$ *is a increasing with respect to* α *when* $0 < t < \dfrac{1}{\pi^2}$ *and it is a nonincreasing function of* α *when* $\dfrac{1}{\pi^2} < t < 1$. *So from Theorem 24 in [10], we imply that* $u(t)$ *is not (i)-gH differentiable or (ii)-gH differentiable in* $(0,1)$ *(here* $t_0 = \dfrac{1}{\pi^2}$ *is a switching point for the differentiability of* u*). Therefore,* (u,v) *defined in (17) is not a classical solution of (15)-(16).*

This counter-example implies that a solution of the fractional fuzzy integral systems in Lemma 3.2, in general, is not a classical solution of the fractional fuzzy differential system (3)-(4). Up to now, we do not know in what conditions this inverse happens? So it is reasonable to call (u,v) *satisfied integral systems in Lemma 3.2 by integral solutions of problem (3)-(4).*

Definition 3.6. *A pair of functions* $(u,v) \in \mathbb{Y}$ *is called*

(1) an integral solution in type 1 of the problem (3)-(4) if it satisfies (5);

(2) an integral solution in type 2 of the problem (3)-(4) if it satisfies (6);

(3) an integral solution in type 3 of the problem (3)-(4) if it satisfies (7);

(4) an integral solution in type 4 of the problem (3)-(4) if it satisfies (8).

We denote the zero fuzzy valued process by $\tilde{0} : [0,1] \to E_c$, where $\tilde{0}(t)$ is zero fuzzy number for all $t \in [0,1]$. For simplicity in presentation, sometimes we also use the notion $\tilde{0}$ to express a zero fuzzy number. We need the following hypothesis.

Hypothesis (H1). Functions $f,\ g \in C(J \times E_c \times E_c, E_c)$ satisfy conditions of at most linear growth:

$$d_\infty(f(t,u(t),v(t)),\tilde{0}(t)) \leq \begin{cases} b_1 d_\infty(u(t),\tilde{0}(t)) + \tilde{b}_1 d_\infty(v(t),\tilde{0}(t)) + d_1 \text{ if } 0 \leq t \leq t_m \\ c_1 d_\infty(u(t),\tilde{0}(t)) + \tilde{c}_1 d_\infty(v(t),\tilde{0}(t)) + d_2 \text{ if } t_m \leq t \leq 1, \end{cases}$$

$$d_\infty(g(t,u(t),v(t)),\tilde{0}(t)) \le \begin{cases} B_1 d_\infty(u(t),\tilde{0}(t)) + \tilde{B}_1 d_\infty(v(t),\tilde{0}(t)) + D_1 \text{ if } 0 \le t \le t_m \\ C_1 d_\infty(u(t),\tilde{0}(t)) + \tilde{C}_1 d_\infty(v(t),\tilde{0}(t)) + D_2 \text{ if } t_m \le t \le 1, \end{cases}$$

where $b_1, \tilde{b}_1, c_1, \tilde{c_1}, d_1, d_2, B_1, \tilde{B}_1, C_1, \tilde{C}_1, D_1, D_2$ are positive real numbers satisfying

$$\begin{cases} (b_1 A_1 + \tilde{B}_1 \tilde{A}_1) t_m \le 2\Gamma(q+1) \\ (b_1 A_1 + \tilde{B}_1 \tilde{A}_1) t_m \le ((b_1 \tilde{B}_1 - \tilde{b}_1 B_1) A_1 \tilde{A}_1 t_m^2 + 1)\Gamma(q+1). \end{cases}$$

Remark 3.7. *Assume that f satisfies Hypothesis (H1). For all $t \in [0, t_m]$ we have*

$$\begin{aligned} d_\infty\left(I^t[f](u,v),\tilde{0}(t)\right) &\le \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} d_\infty(f(s,u(s),v(s)),\tilde{0}(s))ds \\ &\le \frac{1}{\Gamma(q)} \left[b_1 H_{[0,t_m]}(u,\tilde{0}) + \tilde{b}_1 H_{[0,t_m]}(v,\tilde{0}) + d_1\right] \int_0^t (t-s)^{q-1} ds \\ &\le \frac{1}{\Gamma(q)} \left[b_1 H_{[0,t_m]}(u,\tilde{0}) + \tilde{b}_1 H_{[0,t_m]}(v,\tilde{0}) + d_1\right] B(1,q) \\ &\le \frac{1}{\Gamma(q+1)} \left[b_1 H_{[0,t_m]}(u,\tilde{0}) + \tilde{b}_1 H_{[0,t_m]}(v,\tilde{0}) + d_1\right], \end{aligned}$$

where $B(x,y)$ denotes for Beta function. Thus, we have

$$\begin{aligned} d_\infty(F_m^1(u,v),\tilde{0}) &\le \sum_{k\in J_2} aa_k d_\infty(I^{t_k}[f](u,v),\tilde{0}) + \sum_{k\in J_1} aa_k d_\infty(I^{t_k}[f](u,v),\tilde{0}) = \sum_{k=1}^m aa_k d_\infty(I^{t_k}[f](u,v),\tilde{0}) \\ &\le \frac{a}{\Gamma(q+1)} \sum_{k=1}^m a_k (b_1 H_{[0,t_m]}(u,\tilde{0}) + \tilde{b}_1 H_{[0,t_m]}(v,\tilde{0}) + d_1). \end{aligned}$$

Remark 3.8. *We define a vector valued integral operator $T = (T_1, T_2)$, where*

$$T_1(u,v)(t) = F_m^1(u,v) \oplus I^t[f](u,v), \; T_2(u,v)(t) = G_m^1(u,v) \oplus I^t[g](u,v),$$

where $(u,v) \in \mathbb{Y}$ and $t \in J$.

Next, we will transfer the existence of integral solutions of nonlocal problem (3)-(4) into fixed point problem of operator T.

Lemma 3.9. *Under Hypothesis (H1) the following inequalities*

$$H_{[0,t_m]}(T_1(u,v),\tilde{0}) \le \frac{A_1}{\Gamma(q+1)} (b_1 H_{[0,t_m]}(u,\tilde{0}) + \tilde{b}_1 H_{[0,t_m]}(v,\tilde{0}) + d_1) \tag{18}$$

and

$$\tilde{H}_{[t_m,1]}(T_1(u,v),\tilde{0}) \le \frac{1}{\Gamma(q+1)} \left[A_1\big(b_1 H_{[0,t_m]}(u,\tilde{0}) + \tilde{b}_1 H_{[0,t_m]}(v,\tilde{0})\big) + K(\lambda,q)\big(c_1 \tilde{H}_{[t_m,1]}\left(u,\tilde{0}\right) + \tilde{c}_1 \tilde{H}_{[t_m,1]}\left(v,\tilde{0}\right)\big) + c_0\right] \tag{19}$$

hold for all $(u,v) \in \mathbb{Y}$, where $c_0 = A_1 d_1 + d_2$ and $K(\lambda,q)$ are defined the same as $G(\lambda,q)$ in Lemma 2.3 in [19].

Proof. For all $t \in [0,t_m]$, from Remark 3.7 we have

$$\begin{aligned} d_\infty(T_1(u,v)(t),\tilde{0}(t)) = d_\infty\Big(F_m^1(u,v) \oplus I^t[f](u,v),\tilde{0}(t)\Big) &\le d_\infty\left(F_m^1(u,v),\tilde{0}\right) + d_\infty\left(I^t[f](u,v),\tilde{0}(t)\right) \\ &\le \frac{1}{\Gamma(q+1)} (a\sum_{k=1}^m a_k + 1)\Big(b_1 H_{[0,t_m]}(u,\tilde{0}) + \tilde{b}_1 H_{[0,t_m]}(v,\tilde{0}) + d_1\Big) \\ &\le \frac{A_1}{\Gamma(q+1)}\Big(b_1 H_{[0,t_m]}(u,\tilde{0}) + \tilde{b}_1 H_{[0,t_m]}(v,\tilde{0}) + d_1\Big). \end{aligned}$$

Taking supremum when $t \in [0, t_m]$, we obtain (18). For $t \in [t_m, 1]$, we have

$$d_\infty(T_1(u,v)(t), \tilde{0}(t)) \leq a \sum_{k=1}^{m} a_k d_\infty\left(I^{t_k}[f](u,v), \tilde{0}\right) + d_\infty\left(I^{t_m}[f](u,v), \tilde{0}\right) + \frac{1}{\Gamma(q)} \int\limits_{t_m}^{t} (t-s)^{q-1} d_\infty\left(f(s,u(s),v(s)), \tilde{0}(s)\right) ds.$$

It implies from Remark 3.7 and Lemma 2.3 in [19] that

$$\begin{aligned} d_\infty(T_1(u,v)(t), \tilde{0}(t)) &\leq \frac{A_1}{\Gamma(q+1)}(b_1 H_{[0,t_m]}(u,\tilde{0}) + \tilde{b}_1 H_{[0,t_m]}(v,\tilde{0}) + d_1) \\ &+ \left[c_1 \tilde{H}_{[t_m,1]}\left(u,\tilde{0}\right) + \tilde{c}_1 \tilde{H}_{[t_m,1]}\left(v,\tilde{0}\right)\right] K(q,\lambda) \frac{e^{\lambda t}}{\Gamma(q+1)} + \frac{d_2}{\Gamma(q+1)}. \end{aligned}$$

By dividing both sides of this inequality by $e^{\lambda t}$ and then taking the supremum in $[t_m, 1]$ with respect to t we obtain (19). □

Lemma 3.10. *Assume that $f : [0,1] \times B \longrightarrow E_c$ is a compact mapping, where B is nonempty, closed, convex, totally bounded subset of $\mathbb{Y}$. Then we have*
(1) $T_1(B) \subset C(J, E_c)$ is equicontinuous, and
(2) $T_1(B)(t) \subset E_c$ is level-equicontinuous on $[0,1]$ for all $t \in J$.

Proof.
Step 1: We prove that $T_1(B)$ is equicontinuous. In deed, let $t, t' \in J$, $t < t'$ and $(u,v) \in B$. We have

$$\begin{aligned} d_\infty(T_1(u,v)(t), T_1(u,v)(t')) &\leq d_\infty\left(I^t[f](u,v), I^{t'}[f](u,v)\right) \\ &\leq d_\infty\left(I^t[f](u,v), I^t[f](u,v) \oplus \frac{1}{\Gamma(q)} \int\limits_t^{t'} (t-s)^{q-1} \odot f(s,u,v) ds\right) \\ &\leq \frac{1}{\Gamma(q)} d_\infty\left(\int\limits_t^{t'} (t-s)^{q-1} \odot f(s,u,v) ds, \tilde{0}(t)\right) \leq \frac{1}{\Gamma(q)} \int\limits_t^{t'} (t-s)^{q-1} d_\infty(f(s,u,v), \tilde{0}(s)) ds. \end{aligned}$$

Since $f : [0,1] \times B \longrightarrow E_c$ is a compact mapping, we have f is bounded. Set $M_0 = \sup\limits_{(t,u,v) \in [0,1] \times B} d_\infty\left(f(t,u(t),v(t)), \tilde{0}(t)\right)$.
It follows, $d_\infty(T_1(u,v)(t), T_1(u,v)(t')) \leq \dfrac{M_0}{\Gamma(q+1)} |t'-t|^q$, $\forall (u,v) \in B$. Thus, $d_\infty(T_1(u,v)(t), T_1(u,v)(t'))$ tends to zero for all $(u,v) \in B$ when t tends to t'. This implies that $T_1(B)$ is equicontinuous.

Step 2: We prove $T_1(B)(t)$ is level-equicontinuous on $[0,1]$ for all $t \in J$. Indeed, f is a compact mapping, then $f(J \times B)$ is relatively compact in E_c. Thus $f(J \times B)$ is level-equicontinuous on $[0,1]$. Then for each $(t,u,v) \in J \times B$ and $\forall \epsilon > 0$, there exists $\delta > 0$ such that from $|\beta - \gamma| < \delta$, one gets $d_H([f(t,u,v)]^\beta, [f(t,u,v)]^\gamma) < \dfrac{\epsilon \Gamma(q+1)}{a \sum\limits_{k=1}^{m} a_k + 1}$. Hence, when $|\beta - \gamma| < \delta$, we have

$$\begin{aligned} d_H([T_1(u,v)(t)]^\beta, [T_1(u,v)(t)]^\gamma) &\leq \frac{a}{\Gamma(q)} \sum_{k=1}^{m} a_k d_H\left([f(\delta,u,v)]^\beta, [f(\delta,u,v)]^\gamma\right) \int\limits_0^{t_k} (t_k - s)^{q-1} ds \\ &+ \frac{1}{\Gamma(q)} d_H\left([f(t,u,v)]^\beta, [f(t,u,v)]^\gamma\right) \int\limits_0^{t} (t-s)^{q-1} ds \leq \frac{1}{\Gamma(q+1)} \left(a \sum_{k=1}^{m} a_k + 1\right) d_H\left([f(\delta,u,v)]^\beta, [f(\delta,u,v)]^\gamma\right) < \varepsilon. \end{aligned}$$

Therefore, $T_1(B)(t)$ is level-equicontinuous on $[0,1]$ for all $t \in J$. □

Remark 3.11 ([22], Remark 4.2)**.** *We can see that operator T is the sum of two kinds of operators, where one is the linear combination of Fredholm operators type of fractional order q, $T_F^k = \dfrac{1}{\Gamma(q)} \odot \int\limits_0^{t_k} (t-s)^{q-1} \odot f(s,u(s),v(s)) ds$, $k =$*

$1, 2, ..., m$, with their values are restricted on the interval $[0, t_k]$ and the other is the linear combination of Volterra operators type of fractional order q in the form $T_V = \frac{1}{\Gamma(q)} \odot \int_{t_m}^{t} (t-s)^{q-1} \odot f(s, u(s), v(s))ds$ with values depend only on the restrictions of functions to the interval $[t_m, t]$. This technique allows us to split the growth condition on the nonlinear terms f, g into two subintervals $[0, t_m]$ and $[t_m, 1]$. With the help of Lemma 3.9, Lemma 3.10, we can combine convergent matrix technique with Schauder fixed point theorem to prove the existence of integral solutions of the problem (3)-(4) as follows.

Theorem 3.12. *Suppose that g, f are compact mappings and Hypothesis (H1) is satisfied. Then problem (3)-(4) has at least one integral solution in type 1.*

Proof. In the space $\mathbb{Y}$, from Lemma 3.2 we have

$$H_{[0,t_m]}(T_1(u,v),\tilde{0}) \le \frac{A_1}{\Gamma(q+1)}(b_1 H_{[0,t_m]}(u,\tilde{0}) + \tilde{b}_1 H_{[0,t_m]}(v,\tilde{0}) + d_1).$$
$$\tilde{H}_{[t_m,1]}(T_1(u,v),\tilde{0}) \le \frac{A_1}{\Gamma(q+1)}(b_1 H_{[0,t_m]}(u,\tilde{0}) + \tilde{b}_1 H_{[0,t_m]}(v,\tilde{0}))$$
$$+ c_1 \tilde{H}_{[t_m,1]}\left(u,\tilde{0}\right) \frac{K(\lambda,q)}{\Gamma(q+1)} + \frac{K(\lambda,q)}{\Gamma(q+1)} \tilde{c}_1 \tilde{H}_{[t_m,1]}\left(v,\tilde{0}\right) + \frac{c_0}{\Gamma(q+1)},$$

where $c_0 = A_1 d_1 + d_2$. It follows,

$$H_{t_m}(T_1(u,v),\tilde{0}) \le \frac{1}{\Gamma(q+1)}\left[(b_1A_1 + c_1K(\lambda,q))H_{t_m}(u,\tilde{0})\right] + \frac{1}{\Gamma(q+1)}\left[(\tilde{b}_1A_1 + \tilde{c}_1K(\lambda,q))H_{t_m}(v,\tilde{0}) + \tilde{c}_0\right].$$

where $\tilde{c}_0 = \max\{d_1A_1, c_0\}$. By doing the same argument, we obtain,

$$H_{t_m}(T_2(u,v),\tilde{0}) \le \frac{1}{\Gamma(q+1)}\left[(B_1\tilde{A}_1 + C_1K(\lambda,q))H_{t_m}(u,\tilde{0})\right] + \quad \frac{1}{\Gamma(q+1)}\left[(\tilde{B}_1\tilde{A}_1 + \tilde{C}_1K(\lambda,q))H_{t_m}(v,\tilde{0}) + \tilde{C}_0\right]$$

, where $\tilde{C}_0 = \max\left\{D_1\tilde{A}_1, C_0\right\}$ and $C_0 = D_1\tilde{A}_1 + D_2$. Therefore,

$$\begin{pmatrix} H_{t_m}(T_1(u,v),\tilde{0}) \\ H_{t_m}(T_2(u,v),\tilde{0}) \end{pmatrix} \le M_\lambda \begin{pmatrix} H_{t_m}(u,\tilde{0}) \\ H_{t_m}(v,\tilde{0}) \end{pmatrix} + \frac{1}{\Gamma(q+1)} \begin{pmatrix} \tilde{c}_0 \\ \tilde{C}_0 \end{pmatrix}, \tag{20}$$

where $M_\lambda = \frac{1}{\Gamma(q+1)} \begin{pmatrix} b_1A_1 + c_1K(q,\lambda) & \tilde{b}_1A_1 + \tilde{c}_1K(\lambda,q) \\ B_1\tilde{A}_1 + C_1K(\lambda,q) & \tilde{B}_1\tilde{A}_1 + \tilde{C}_1K(q,\lambda) \end{pmatrix} = M_0 + M_1$, in which $M_0 = \frac{1}{\Gamma(q+1)} \begin{pmatrix} b_1A_1 & \tilde{b}_1A_1 \\ B_1\tilde{A}_1 & \tilde{B}_1\tilde{A}_1 \end{pmatrix}$; $M_1 = \frac{1}{\Gamma(q+1)} \begin{pmatrix} c_1K(q,\lambda) & \tilde{c}_1K(\lambda,q) \\ C_1K(\lambda,q) & \tilde{C}_1K(q,\lambda) \end{pmatrix}$. For all $(u,v) \in \mathbb{Y}$, from the Hypothesis (H1), matrix M_0 converges to zero and matrix M_1 has all small elements when λ is large enough (see Lemma 5.2). So, matrix M_λ converges to zero for large enough λ. Thus $I - M_\lambda$ is invertible and its inverse $(I - M_\lambda)^{-1}$ has nonnegative elements. Therefore, there exist two positive numbers R_1, R_2 such that

$$(I - M_\lambda)^{-1} \frac{1}{\Gamma(q+1)} \begin{pmatrix} \tilde{c}_0 \\ \tilde{C}_0 \end{pmatrix} \le \begin{pmatrix} R_1 \\ R_2 \end{pmatrix}. \tag{21}$$

This implies,

$$\frac{1}{\Gamma(q+1)} \begin{pmatrix} \tilde{c}_0 \\ \tilde{C}_0 \end{pmatrix} \le (I - M_\lambda) \begin{pmatrix} R_1 \\ R_2 \end{pmatrix} \Leftrightarrow M_\lambda \begin{pmatrix} R_1 \\ R_2 \end{pmatrix} + \frac{1}{\Gamma(q+1)} \begin{pmatrix} \tilde{c}_0 \\ \tilde{C}_0 \end{pmatrix} \le \begin{pmatrix} R_1 \\ R_2 \end{pmatrix}. \tag{22}$$

Set $B = \left\{(u,v) \in \mathbb{Y} : H_{t_m}(u,\tilde{0}) \le R_1, H_{t_m}(v,\tilde{0}) \le R_2\right\}$. For $(u,v) \in B$, from (20) and (22) we receive $T(u,v) \in B$. Hence, T is well-defined. Now, we will prove that $T_1(B)(t)$ is compact-supported in E_c for all $t \in J$. Indeed, for any $z \in T_1(B)$, there exists $(u,v) \in B$ such that $z(t) = T_1(u,v)(t)$ for all $t \in J$. Then

$$[z(t)]^0 = \left[\sum_{k\in J_2} aa_k I^{t_k}[f](u,v) \ominus \sum_{k\in J_1} aa_k I^{t_k}[f](u,v) \oplus I^t[f](u,v)\right]^0 = \sum_{k\in J2} aa_k [I^{t_k}[f](u,v)]^0 \ominus \sum_{k\in J_1} aa_k [I^{t_k}[f](u,v)]^0 + [I^t[f](u,v)]^0.$$

Because $f(J\times B)$ is relatively compact, $f(J\times B)$ is compact-supported and level-equicontinuous on $[0,1]$. Hence, there exists a compact subset $K\subset\mathbb{R}$ such that $[f(t,u(t),v(t))]^0\subseteq K$ for all $(t,u,v)\in J\times B$. It implies

$$[I^t[f](u,v)]^0=\frac{1}{\Gamma(q)}(\int_0^t (t-s)^{q-1}ds)[f(s,u(s),v(s))]^0\subseteq\Big(\frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}ds\Big)K.$$

Thus,

$$\begin{aligned}[z(t)]^0\subseteq&\frac{a}{\Gamma(q)}\Big(\sum_{k\in J_2}a_k(\int_0^{t_k}(t_k-s)^{q-1}ds)[f(s,u(s),v(s))]^0\Big)\ominus\frac{a}{\Gamma(q)}\Big(\sum_{k\in J_1}a_k(\int_0^{t_k}(t_k-s)^{q-1}ds)[f(s,u(s),v(s))]^0\Big)\\&+\frac{1}{\Gamma(q)}(\int_0^t(t-s)^{q-1}ds)[f(s,u(s),v(s))]^0\subseteq\frac{a}{\Gamma(q+1)}\Big(\sum_{k\in J_2}a_kt_k^q-\sum_{k\in J_1}a_kt_k^q\Big)K+\frac{1}{\Gamma(q+1)}K.\end{aligned}$$

Therefore, there exists a compact set $K_0\subseteq\mathbb{R}$ such that $[z(t)]^0\subseteq K_0,\ \forall z\in T_1(B)$. Which proves that $T_1(B)(t)$ is compact-supported for all $t\in J$. Hence, from part 2) in Lemma 3.10 we imply that $T_1(B)(t)$ is relatively compact in E_c.

From part 1) in Lemma 3.10 and Ascoli-Arzelá (Theorem 4.1 in [30]) we imply that $T_1(B)$ is relative compact in $C(J,E_c)$. Similarly, $T_2(B)$ is relatively compact on E_c. So that operator $T(B)=(T_1(B),T_2(B))$ is relatively compact on $\mathbb{Y}$. By applying Theorem 2.5, T has at least one fixed point $(u^*,v^*)\in B$, that is the integral solution in type 1 of problem (3)-(4). The proof is complete. □

The integral solution in type 2 is a fixed point of $\tilde{T}=(\tilde{T}_1,\tilde{T}_2)$ defined as follows

$$\tilde{T}_1(u,v)(t)=F_m^2(u,v)\ominus(-1)\odot I^t[f](u,v). \tag{23}$$

$$\tilde{T}_2(u,v)(t)=G_m^2(u,v)\ominus(-1)\odot I^t[g](u,v). \tag{24}$$

Denote $\mathcal{C}_{fg}(J,E_c)=\{(u,v)\in\mathbb{Y}:\ \text{H-differences in (23) and (24) exist }\forall t\in J\}$.

Lemma 3.13 (Sketch to Lemma 5.2 in [18])**.** *If f and g are continuous with respect to the second and the third variables then $(\mathcal{C}_{fg}(J,E_c),\rho)$ is a generalized semilinear Banach space.*

Theorem 3.14. *Assume that functions $f,\ g:\ J\times\mathbb{Y}\to E_c$ are compact, continuous with respect to the second and the third variables and satisfy the hypothesis (H1) for all $t\in J$. Moreover following assumptions are satisfied:*
(H$_2$) *$\mathcal{C}_{fg}(J,E_c)\neq\emptyset$;*
(H$_3$) *For $(u,v)\in\mathcal{C}_{fg}(J,E_c)$ satisfies system (23), there exist H-differences*

$$\begin{cases}F_m^2(\tilde{T}_1(u,v),\tilde{T}_2(u,v))\ominus(-1)\odot I^t[f](\tilde{T}_1(u,v),\tilde{T}_2(u,v))\\ G_m^2(\tilde{T}_1(u,v),\tilde{T}_2(u,v))\ominus(-1)\odot I^t[g](\tilde{T}_1(u,v),\tilde{T}_2(u,v))\end{cases}$$

for all $t\in J$. Then the problem (3)-(4) has at least one integral solution in type 2 in $\mathcal{C}_{fg}(J,E_c)$.

Proof. Consider the operator $\tilde{T}$ defined by $\tilde{T}(u,v)(t)=(\tilde{T}_1(u,v)(t),\tilde{T}_2(u,v)(t)),\ t\in J$. By Hypotheses $(H2)-(H3)$, $\tilde{T}$ is a closed operator from $\mathcal{C}_{fg}(J,E_c)$ to $\mathcal{C}_{fg}(J,E_c)$. We have

$$\begin{aligned}d_\infty(\tilde{T}_1(u,v)(t),\tilde{0}(t))&=d_\infty\Bigg(\sum_{k\in J_1}(-aa_k)\odot I^{t_k}[f](u,v)\ominus\sum_{k\in J_2}(-aa_k)\odot I^{t_k}[f](u,v)\ominus(-1)\odot I^t[f](u,v),\tilde{0}(t)\Bigg)\\&\le a\sum_{k=1}^m a_kd_\infty\left(I^{t_k}[f](u,v),\tilde{0}(t)\right)+d_\infty\left(I^t[f](u,v),\tilde{0}(t)\right).\end{aligned}$$

Applying analogous method in the proof of Theorem 3.12, we receive

$$\begin{pmatrix}H_{t_m}(\tilde{T}_1(u,v),\tilde{0})\\ H_{t_m}(\tilde{T}_2(u,v),\tilde{0})\end{pmatrix}\le M_\lambda\begin{pmatrix}H_{t_m}(u,\tilde{0})\\ H_{t_m}(v,\tilde{0})\end{pmatrix}+\frac{1}{\Gamma(q+1)}\begin{pmatrix}\tilde{c}_0\\ \tilde{C}_0\end{pmatrix}.$$

Let R_1,R_2 be defined in (21) and $\tilde{B}=\left\{(u,v)\in\mathcal{C}_{fg}(J,E_c)\times\mathcal{C}_{fg}(J,E_c):H_{t_m}(u,\tilde{0})\le R_1,H_{t_m}(v,\tilde{0})\le R_2\right\}$. We have $\tilde{T}(\tilde{B})\subset\tilde{B}$. Applying analogous method in the proof of Theorem 3.12, we can prove that $\tilde{T}(\tilde{B})$ is a relatively compact subset of $\mathcal{C}_{fg}(J,E_c)\times\mathcal{C}_{fg}(J,E_c)$. From Theorem 2.5, $\tilde{T}$ has at least one fixed point $(u^*,v^*)\in\tilde{B}$. This is the integral solution in type 2 of the problem (3)-(4). The theorem is proved completely. □

We define two new spaces $\mathcal{C}_f(J,E_c)=\{(u,v)\in\mathbb{Y}:\ \text{H-differences in (23) exist for all } t\in J\}$ and $\mathcal{C}_g(J,E_c)=\{(u,v)\in\mathbb{Y}:\ \text{H-differences in (24) exist for all } t\in J\}$. Then, by using similar method we receive following results:

Theorem 3.15. *Assume that functions $f,\ g:\ J\times\mathbb{Y}\to E_c$ are compact, continuous with respect to the second and the third variables and satisfy the hypothesis (H1) for all $t\in J$. Moreover following assumptions are satisfied.*
($\mathbf{H_4}$) $\mathcal{C}_f(J,E_c)\neq\emptyset$;
($\mathbf{H_5}$) *If $(u,v)\in\mathcal{C}_f(J,E_c)$ satisfy system (23), then there exists H-differences*

$$F^2_m(\tilde{T}_1(u,v),\tilde{T}_2(u,v))\ominus(-1)\odot I^t[f](\tilde{T}_1(u,v),\tilde{T}_2(u,v))$$

for all $t\in J$. Then the problem (3)-(4) has at least one integral solution in type 3.

Theorem 3.16. *Assume that functions $f,\ g:\ J\times\mathbb{Y}\to E_c$ are compact, continuous with respect to the second and the third variables and satisfy the hypothesis (H1) for all $t\in J$. Moreover following assumptions are satisfied.*
($\mathbf{H_6}$) $\mathcal{C}_g(J,E_c)\neq\emptyset$;
($\mathbf{H_7}$) *If $(u,v)\in\mathcal{C}_g(J,E_c)$ satisfy system (24), then there exists H-differences*

$$G^2_m(\tilde{T}_1(u,v),\tilde{T}_2(u,v))\ominus(-1)\odot I^t[g](\tilde{T}_1(u,v),\tilde{T}_2(u,v))$$

for all $t\in J$. Then the problem (3)-(4) has at least one integral solution in type 4.

4 Application Example

Example 4.1. *We consider the the following model*

$$\begin{cases} {}^{C}_{gH}\mathcal{D}^{1/2}u(t)=t^2\odot K\\ {}^{C}_{gH}\mathcal{D}^{1/2}v(t)=\dfrac{\sqrt{\pi}}{5}t\odot u(t)\end{cases}\tag{25}$$

$t\in(0,1)$ with NCs

$$\begin{cases} u(0)\oplus 3\sqrt{3}\odot u\left(\dfrac{1}{3}\right)=\pi\sqrt{\pi}\odot u\left(\dfrac{1}{\pi}\right)\\ v(0)\oplus 9\odot v\left(\dfrac{1}{3}\right)=\pi^2\odot v\left(\dfrac{1}{\pi}\right)\end{cases}\tag{26}$$

where K is a triangular fuzzy number (see in [17]), $[K]^\alpha=[\alpha,1-\alpha],\ \alpha\in[0,1]$. We have $f(t,u(t),v(t))=t^2\odot K$, $g(t,u(t),v(t))=\dfrac{\sqrt{\pi}}{5}t\odot u(t)$ and

$$d_\infty(f(t,u(t),v(t)),\tilde{0})\le\begin{cases}\dfrac{1}{81}d_\infty(u(t),\tilde{0}(t))+\dfrac{1}{450}d_\infty(v(t),\tilde{0}(t))+\dfrac{2}{9} & \text{if } t\in[0,\frac{1}{3}]\\ \dfrac{1}{81}d_\infty(u(t),\tilde{0}(t))+\dfrac{1}{450}d_\infty(v(t),\tilde{0}(t))+2 & \text{if } t\in[\frac{1}{3},1]\end{cases}$$

$$d_\infty(g(t,u(t),v(t)),\tilde{0})\le\begin{cases}\dfrac{\sqrt{\pi}}{5}d_\infty(u(t),\tilde{0}(t))+\dfrac{1}{450}d_\infty(v(t),\tilde{0}(t))+\dfrac{16\sqrt{3}}{405} & \text{if } t\in[0,\frac{1}{3}]\\ \dfrac{\sqrt{\pi}}{5}d_\infty(u(t),\tilde{0}(t))+\dfrac{1}{450}d_\infty(v(t),\tilde{0}(t))+\dfrac{16}{15} & \text{if } t\in[\frac{1}{3},1].\end{cases}$$

It is easy to see that

$$b_1=c_1=\frac{1}{81};B_1=C_1=\frac{\sqrt{\pi}}{5};\tilde{B_1}=\tilde{C_1}=\tilde{b_1}=\tilde{c_1}=\frac{1}{450},\ \sum_{k\in J_1}a_k=3\sqrt{3};\sum_{k\in J_2}a_k=\pi\sqrt{\pi};\sum_{k\in Q_1}\tilde{a}_k=9,\ \sum_{k\in Q_2}\tilde{a}_k=\pi^2$$

$$a=(1+\sum_{k\in J_1}a_k-\sum_{k\in J_2}a_k)^{-1}\approx 1,59;\tilde{a}=(1+\sum_{k\in Q_1}\tilde{a}_k-\sum_{k\in Q_2}\tilde{a}_k)^{-1}\approx 7,69$$

$$A_1=1+a\sum_{k=1}^m a_k\approx 18,15,\ \ A_2=1+\tilde{a}\sum_{k=1}^m\tilde{a}_k\approx 145,71.$$

Furthermore

$$(b_1A_1+\tilde{B}_{11}A_2)t_m=\left(18,11\frac{1}{81}+\frac{1}{450}145,71\right)\frac{1}{3}\approx 0,182\le 2\Gamma\left(\frac{3}{2}\right)\approx 1,772.$$

$$(b_1A_1+\tilde{B}_1A_2)t_m\approx 0,182<((b_1\tilde{B}_1-\tilde{b}_1B_1)A_1A_2t_m^2+1)\Gamma\left(\frac{3}{2}\right)\approx 0,688.$$

By applying Theorem 3.12, the problem (25)-(26) has at least one integral solution in type 1.

Suppose the parametric form of u,v are $[u]^\alpha=[u_\alpha^-,u_\alpha^+]$ and $[v]^\alpha=[v_\alpha^-,v_\alpha^+]$, respectively. From system (25)-(26), we receive the following interval differential system:

$$\begin{cases} {}_{gH}^{C}\mathcal{D}^{1/2}[u_\alpha^-(t),u_\alpha^+(t)]=t^2[K_\alpha^-,K_\alpha^+]\\ {}_{gH}^{C}\mathcal{D}^{1/2}[v_\alpha^-(t),v_\alpha^+(t)]=\dfrac{\sqrt{\pi}}{5}t^2[u_\alpha^-(t),u_\alpha^+(t)]\end{cases}$$

with $t\in(0,1)$. And the nonlocal condition (26) becomes

$$\begin{cases}[u_\alpha^-(0),u_\alpha^+(0)]\oplus 3\sqrt{3}\left[u_\alpha^-\left(\dfrac{1}{3}\right),u_\alpha^+\left(\dfrac{1}{3}\right)\right]=\pi\sqrt{\pi}\left[u_\alpha^-\left(\dfrac{1}{\pi}\right),u_\alpha^+\left(\dfrac{1}{\pi}\right)\right]\\ [v_\alpha^-(0),v_\alpha^+(0)]\oplus 9\left[v_\alpha^-\left(\dfrac{1}{3}\right),v_\alpha^+\left(\dfrac{1}{3}\right)\right]=\pi^2\left[v_\alpha^-\left(\dfrac{1}{\pi}\right),v_\alpha^+\left(\dfrac{1}{\pi}\right)\right]\end{cases}$$

This system is equivalent to the following systems

$$\begin{cases}{}_{gH}^{C}\mathcal{D}^{1/2}u_\alpha^-(t)=t^2K_\alpha^-\\ {}_{gH}^{C}\mathcal{D}^{1/2}v_\alpha^-(t)=\dfrac{\sqrt{\pi}}{5}tu_\alpha^-(t)\\ u_\alpha^-(0)+3\sqrt{3}u_\alpha^-\left(\dfrac{1}{3}\right)=\pi\sqrt{\pi}u_\alpha^-\left(\dfrac{1}{\pi}\right)\\ v_\alpha^-(0)+9v_\alpha^-\left(\dfrac{1}{3}\right)=\pi^2v_\alpha^-\left(\dfrac{1}{\pi}\right)\end{cases}\quad \text{and}\quad \begin{cases}{}_{gH}^{C}\mathcal{D}^{1/2}u_\alpha^+(t)=t^2K_\alpha^+\\ {}_{gH}^{C}\mathcal{D}^{1/2}v_\alpha^+(t)=\dfrac{\sqrt{\pi}}{5}tu_\alpha^+(t)\\ u_\alpha^+(0)+3\sqrt{3}u_\alpha^+\left(\dfrac{1}{3}\right)=\pi\sqrt{\pi}u_\alpha^+\left(\dfrac{1}{\pi}\right)\\ v_\alpha^+(0)+9v_\alpha^+\left(\dfrac{1}{3}\right)=\pi^2v_\alpha^+\left(\dfrac{1}{\pi}\right).\end{cases}$$

Solve these systems, we receive

$$u_\alpha^-(t)=\frac{8K_\alpha^-}{3\sqrt{\pi}}t^{3/2},\ v_\alpha^-(t)=\frac{\sqrt{\pi}}{2}t^2K_\alpha^-\quad\text{and}\quad u_\alpha^+(t)=\frac{8K_\alpha^+}{3\sqrt{\pi}}t^{3/2},\ v_\alpha^+(t)=\frac{\sqrt{\pi}}{2}t^2K_\alpha^+$$

Finally, by applying Stacking Lemma, we have integral solution in type 1 of problem (25)-(26) is $\begin{cases}u(t)=\dfrac{8}{3\sqrt{\pi}}t^{3/2}\odot K\\ v(t)=\dfrac{\sqrt{\pi}}{2}t^2\odot K\end{cases}$.

5 Conclusions

We have developed a novel approach for studying the existence of solutions for a class of fuzzy fractional differential systems with NCs. The main tool is Schauder fixed point principle for vector valued operator in complete generalized metric space. For our future research, we will develop the proposed approach for studying different types of fuzzy DEs, fuzzy random DEs in ordered generalized metric space [20] with the use of Perov, Leray-Schauder and Krasnoselskii fixed point principles under some conditions which are weaker than Lipschitz and linear growth conditions.

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