

First order linear fuzzy dynamic equations on time scales

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Abstract

In this paper, we study the concept of generalized differentiability for fuzzy-valued functions on time scales. Using the derivative of the product of two functions, we provide solutions to first order linear fuzzy dynamic equations. We present some examples to illustrate our results.

Keywords: Time scales, Generalized differentiability, First order, Linear fuzzy dynamic equations.

1 Introduction

Time scale calculus is a unification of the theory of difference equations with differential equations, unifying integral and differential calculus with the calculus of finite differences, offering a formalism for studying hybrid discrete-continuous dynamical systems. Many results concerning differential equations carry over quite easily to corresponding results for difference equations, while other results seem to be completely different from their continuous counterparts. The study of dynamic equations on time scales reveals such discrepancies, and helps avoid proving results twice, once for differential equations and once again for difference equations. The general idea is to prove a result for a dynamic equation where the domain of the unknown function is a so-called time scale. In this way, results apply not only to the set of real numbers or set of integers, but also to more general time scales such as a Cantor set. Time scale calculus has applications in any field that requires simultaneous modeling of discrete and continuous data, such as population models and engineering [?]. It gives a new definition of a derivative such that if one differentiates a function which acts on the real numbers, then the definition is equivalent to standard differentiation, but if one uses a function acting on the integers, then it is equivalent to the difference operator [?].

For the first time, the theory of time scales was introduced and presented by Hilger in 1988, to unify continuous and discrete analysis [?]. Dynamic equations on time scales and calculus on time scales was studied in [?]. Later, calculus on time scales was developed for multivalued functions by Hong in [?]. Recently, Lupulescu in [?] introduced the Hukuhara differentiability of interval-valued functions and interval differential equations on time scales. In [?], Vasavi et. al. studied Hukuhara delta derivative for fuzzy functions on time scales and presented sufficient conditions for existence and uniqueness of solutions to fuzzy dynamic equations on time scales. In [?], the authors studied generalized differentiability and integrability of fuzzy functions on time scales. Fard and Bidgoli in [?] presented some results on calculus of fuzzy functions on time scales. They used concept of Henstock-Kurzweil integral for fuzzy functions. In [?], Fard et. al. presented a Hukuhara approach to study the hybrid fuzzy systems on time scales and obtained some results for stability of hybrid fuzzy systems. Fuzzy dynamic equations on time scales under second Hukuhara delta derivative were studied in [?]. They presented some properties of second Hukuhara delta derivative for fuzzy functions and established the existence and uniqueness criteria for fuzzy dynamic equations on time scales, using Banach contraction mapping. Usage of fuzzy dynamic equations is a natural way to model dynamical systems under possibilistic uncertainty. The generalized differentiability of fuzzy number valued functions was introduced in [?] and studied in [?, ?, ?]. Fuzzy differential equations under generalized derivative was investigated in [?, ?, ?, ?, ?, ?]. Difference equations appear naturally as discrete analogues and as numerical solutions of differential and delay differential equations which model various diverse phenomena in biology, ecology, physiology, physics, engineering and economics [?]. Fuzzy difference

equations is a difference equation with fuzzy parameters and fuzzy initial values, and the solution is a sequence of the fuzzy numbers. Fuzzy difference equations is important due for analysis of real world phenomena. For example, it is suitable to study finance problems and population model [?, ?]. Lakshmikantham and Vatsala presented the basic theory of fuzzy difference equations in [?]. Recently, there has been an increasing investigation concerning the existence of solutions for fuzzy difference equations [?, ?, ?]. First order linear fuzzy dynamic equations are one of the simplest fuzzy dynamic equations which may appear in many applications. Bede et. al. in [?] studied first order linear fuzzy differential equations under generalized differentiability concept and presented solutions of this problem in some cases. Later in [?, ?], solutions of this problem in general cases are presented. In this paper, our main purpose is to study the linear fuzzy dynamic equations on time scales using the generalized differentiability concept and introduce the general form of their solutions.

The paper is organized as follows. We present some basic concepts of time scales and fuzzy calculus in section 2. In section 3, we study the generalized differentiability of fuzzy-valued functions on time scales. In section 4, we investigate the solutions of linear fuzzy dynamic equations on time scales. Finally, we present some examples to illustrate our results.

2 Preliminaries

In this section, we give some definitions and useful results and introduce the necessary notation which will be used throughout the paper. Most of it can be found, for example, in [?, ?, ?].

2.1 Time scales calculus

A time scale is an arbitrary nonempty closed subset of the real numbers. For $t \in T$, we define the forward jump operator $\sigma : T \rightarrow T$ by $\sigma(t) := \inf \{t \in T : s > t\}$ and backward jump operator $\rho : T \rightarrow T$ by $\rho(t) := \sup \{t \in T : s < t\}$. If $\sigma(t) > t$, we say that t is right-scattered, while if $\rho(t) < t$, we say that t is left-scattered. Points that are right-scattered and left-scattered at the same time are called isolated. If $\sigma(t) = t$, then t is right-dense, if $\rho(t) = t$, then t is left-dense. Points that are right-dense and left-dense at the same time are called dense. The graininess function $\mu : T \rightarrow [0, \infty)$ is defined by $\mu(t) := \sigma(t) - t$. We need the set T^k which is derived from the time scale T as follows: if T has a left-scattered maximum m , then $T^k = T - \{m\}$, otherwise $T^k = T$.

Definition 2.1. [?] *A function $f : T \rightarrow \mathbb{R}$ is called regulated if its right-sided limits exists at all right-dense points in T , and its left-sided limits exist (finite) at all left-dense points in T . A function $f : T \rightarrow \mathbb{R}$ is called rd-continuous if it is continuous at all right-dense points in T and its left-sided limits exist (finite) at all left-dense points in T .*

Definition 2.2. [?] *Assume $f : T \rightarrow \mathbb{R}$ is a function and let $t \in T$. Then, we define $f^\Delta(t)$ to be the number (provided it exists) with the property that given any $\epsilon > 0$, there is a neighborhood $U_T(t, \delta)$ ($U_T(t, \delta) := (t - \delta, t + \delta) \cap T$) of t , such that, for all $s \in U_T(t, \delta)$, $|[f(\sigma(t)) - f(s)] - f^\Delta(t)[(\sigma(t) - s)]| \leq \epsilon|\sigma(t) - s|$. We call $f^\Delta(t)$ the Δ -derivative of f at t . Moreover, we say that f is Δ -differentiable on T^k provided $f^\Delta(t)$ exists for all $t \in T^k$.*

In the previous definition, for $s = t + h \in U_T(t, \delta)$, since $\sigma(t) - t = \mu(t)$ and $h - \mu(t) > 0$, we obtain

$$|[f(\sigma(t)) - f(t + h)] - f^\Delta(t)[(\mu(t) - h)]| \leq \epsilon(h - \mu(t)).$$

Similarly, for $t - h \in U_T(t, \delta)$, we have $|[f(\sigma(t)) - f(t - h)] - f^\Delta(t)[(h + \mu(t))]| \leq \epsilon(h + \mu(t))$. Now, we recall some properties of the exponential function on time scales. For $h > 0$, we define the Hilger complex numbers as [?] $\mathbb{C}_h := \{z \in \mathbb{C} : z \neq \frac{-1}{h}\}$, and for $h > 0$, $\mathbb{Z}_h := \{z \in \mathbb{C} : \frac{-\pi}{h} < \text{Im}(z) \leq \frac{\pi}{h}\}$, and for $h = 0$, let $\mathbb{Z}_0 := \mathbb{C}$.

Definition 2.3. [?] *For $h > 0$, we define the cylinder transformation $\xi_h(z) = \mathbb{C}_h \rightarrow \mathbb{Z}_h$ by $\xi_h(z) = \frac{1}{h} \log(1 + zh)$, where log is the principal logarithm function. For $h = 0$, we define $\xi_0(z) = z$, for all $z \in \mathbb{C}$.*

Remark 2.4. [?] *We say that a function $P : T \rightarrow \mathbb{R}$ is regressive provided, for all $t \in T^k$, $1 + \mu(t)P(t) \neq 0$, holds. The set of all regressive and rd-continuous functions $f : T \rightarrow \mathbb{R}$ will be denoted by R .*

Definition 2.5. [?] *The exponential function on time scales for $P \in R$ is defined, for $s, t \in T$, by, $e_P(t, s) = \exp(\int_s^t \xi_{\mu(\tau)}(P(\tau))\Delta\tau)$, where ξ_z is introduced in Definition 2.3.*

The following properties of the exponential function are well-known [?]:

1. $e_0(t, s) = 1$, and $e_P(t, s) = 1$,
2. $e_P(\sigma(t), s) = [1 + \mu(t)P(t)]e_P(t, s)$,
3. $e_P(t, r)e_P(r, s) = e_P(t, s)$,
4. $e_P(t, s) = \frac{1}{e_P(s, t)} = e_{\ominus P}(s, t)$, where $\ominus P(t) := -\frac{P(t)}{1 + \mu(t)P(t)}$.

In particular, if $P \in \mathbb{R}$ is such that $1 + \mu(t)P > 0$ for all $t \in T$, we have $e_P(t, 0) = e^{Pt}$ if $T = \mathbb{R}$, and $e_P(t, 0) = (1 + Ph)^{\frac{t}{h}}$ if $T = h\mathbb{Z}$, with $h > 0$.

In the following, we recall the definition of integration on time scales and some useful properties. It is known that for every $\delta > 0$, there exists at least one partition $P : a = t_0 < t_1 < \dots < t_n = b$ of $[a, b)_T = \{t \in T : a \leq t < b\}$ such that for each $i \in \{1, 2, \dots, n\}$, either, $t_i - t_{i-1} \leq \delta$ or $t_i - t_{i-1} > \delta$ and $\rho(t_i) = t_{i-1}$ [?]. For given $\delta > 0$, we denote $\rho([a, b)_T, \delta)$, the set of all partitions $P : a = t_0 < t_1 < \dots < t_n = b$ that possess the above property. Let $f : T \rightarrow \mathbb{R}$ be a bounded function on $[a, b)_T$, and, let $P : a = t_0 < t_1 < \dots < t_n = b$ be a partition of $[a, b)_T$. In each interval $[t_{i-1}, t_i)_T$, choose an arbitrary point ξ_i and form the sum $S = \sum_{i=1}^n (t_i - t_{i-1})f(\xi_i)$. We call S a Riemann Δ -sum of f corresponding to the partition P .

Definition 2.6. [?] We say that f is Riemann Δ -integrable from a to b if there exists a number I with the following property: for each $\epsilon > 0$, there exists $\delta > 0$ such that $|S - I| < \epsilon$, for every Riemann Δ -sum S of f , corresponding to a partition $P \in \rho([a, b)_T, \delta)$ independent of the way in which we choose $\xi_i \in [t_{i-1}, t_i)$, $i = 1, \dots, n$. It is easily seen that such a number I is unique. The number I is the Riemann Δ -integral of f from a to b and we will denote it by $\int_a^b f(t)\Delta t$.

Proposition 2.7. [?] Assume that $a, b \in T$, $a < b$ and $f : T \rightarrow \mathbb{R}$ is rd-continuous. Then, the integral has the following properties:

1. If $T = \mathbb{R}$, then $\int_a^b f(t)\Delta t = \int_a^b f(t)dt$, where the integral on the right-hand side is the Riemann integral.
2. If T consists of isolated points, then $\int_a^b f(t)\Delta t = \sum_{t \in [a, b)_T} \mu(t)f(t)$.

2.2 Fuzzy calculus

Let X be a nonempty set. A fuzzy set u in X is characterized by its membership function $u : X \rightarrow [0, 1]$, where $u(x)$ is interpreted as the degree of membership of an element x in the fuzzy set u for each $x \in X$. In the following, we denote the space of fuzzy numbers by \mathbb{R}_F . Given a fuzzy number $u \in \mathbb{R}_F$ and $0 < \alpha \leq 1$, we obtain the α -level set of u by $[u]^\alpha = \{s \in \mathbb{R} \mid u(s) \geq \alpha\}$ and the support of u as $[u]^0 = cl\{s \in \mathbb{R} \mid u(s) > 0\}$. For any $\alpha \in [0, 1]$, due to the properties imposed on the set of fuzzy numbers, we have that $[u]^\alpha$ is a bounded closed interval. The notation $[u]^\alpha = [\underline{u}^\alpha, \bar{u}^\alpha]$, denotes explicitly the α -level set of u .

For given $u, v \in \mathbb{R}_F$ and $\lambda \in \mathbb{R}$, we define the sum $u + v$ and the product λu by the standard level-set operations $[u + v]^\alpha = [u]^\alpha + [v]^\alpha$, $[\lambda u]^\alpha = \lambda[u]^\alpha$, $\forall \alpha \in [0, 1]$, where $[u]^\alpha + [v]^\alpha$ means the usual addition of two intervals (subsets) of \mathbb{R} and $\lambda[u]^\alpha$ means the usual product between a scalar and a subset of \mathbb{R} . The metric structure is given by the Hausdorff distance $D : \mathbb{R}_F \times \mathbb{R}_F \rightarrow \mathbb{R}_+ \cup \{0\}$, and, for all $u, v \in \mathbb{R}_F$, $D[u, v] = \sup_{\alpha \in [0, 1]} \max\{|\underline{u}^\alpha - \underline{v}^\alpha|, |\bar{u}^\alpha - \bar{v}^\alpha|\}$. For $u \in \mathbb{R}_F$, we set $\|u\| = D[u, \hat{0}]$ where $\hat{0} = \chi_{\{0\}}$, and $\hat{0} \in \mathbb{R}_F$ is the neutral element with respect to $+$.

Remark 2.8. The following properties are well-known [?]:

1. (\mathbb{R}_F, D) is a complete metric space.
2. $D[u + w, v + w] = D[u, v]$, $\forall u, v, w \in \mathbb{R}_F$,
3. $D[ku, kv] = |k|D[u, v]$, $\forall k \in \mathbb{R}$,
4. $D[u + v, w + e] \leq D[u, w] + D[v, e]$, $\forall u, v, w, e \in \mathbb{R}_F$,
5. For any $\lambda, \mu \in \mathbb{R}$ and any $u \in \mathbb{R}_F$, we have $\lambda(\mu u) = (\lambda\mu)u$,
6. For any $a, b \in \mathbb{R}$ with $ab \geq 0$ and any $u \in \mathbb{R}_F$, we have $(a + b)u = au + bu$. For general $a, b \in \mathbb{R}$, the above property does not hold,

7. For any $\lambda \in \mathbb{R}$ and any $u, v \in \mathbb{R}_F$, we have $\lambda(u + v) = \lambda u + \lambda v$,

8. For any $a, b \in \mathbb{R}$; $ab \geq 0$ and any $u \in \mathbb{R}_F$ we have $D(au, bu) = |a - b|D(u, \hat{0})$.

For $u, v \in \mathbb{R}_F$, if there exists $w \in \mathbb{R}_F$ such that $u = v + w$, then w is called the H-difference of u, v and it is denoted $u \ominus v$. We use this notation $u \ominus v$ to represent the H-difference of u and v , which is different, in general, from $u - v = u + (-1)v$. In the following, we present some lemmas that we need in our main results.

Lemma 2.9. *Let $a, b, c, d, w, r \in \mathbb{R}_F$ and the H-differences $c \ominus d$, $w \ominus d$ exist. Then, we have*

$$D[c \ominus d, a + b] \leq D[w \ominus d, a] + D[c, w + r] + D[b, r].$$

Proof. Using Remark ??, we have

$$\begin{aligned} D[c \ominus d, a + b] &= D[c \ominus d + w, a + b + w] \leq D[w \ominus d, a] + D[c, b + w] = D[w \ominus d, a] + D[c + r, b + w + r] \\ &\leq D[w \ominus d, a] + D[c, w + r] + D[r, b]. \end{aligned}$$

□

Lemma 2.10. *Let $a, b, c, d, w, r \in \mathbb{R}_F$ and the H-differences $c \ominus d$, $c \ominus w$ exist. Then, we have*

$$D[c \ominus d, a + b] \leq D[c \ominus w, a] + D[w, d + r] + D[b, r].$$

Proof. By Remark ??, we have

$$\begin{aligned} D[c \ominus d, a + b] &= D[c + w, a + b + d + w] \leq D[c, w + a] + D[w, b + d] \\ &= D[c, w + a] + D[w + r, b + d + r] \leq D[c \ominus w, a] + D[w, d + r] + D[r, b]. \end{aligned}$$

□

Lemma 2.11. *Let $a, b, c, d, e, w, r \in \mathbb{R}_F$ and H-differences $c \ominus d$, $a \ominus b$, $r \ominus c$ exist, then*

$$D[c \ominus d, a \ominus b] \leq D[r, d + w] + D[a, w] + D[e, r \ominus c] + D[e, b].$$

Proof. Using Remark ??, we have

$$\begin{aligned} D[c \ominus d, a \ominus b] &= D[c \ominus d + r, a \ominus b + r] = D[b + r + c, a + d + r] = D[r + b, a + d + r \ominus c] \leq D[r, a + d] + D[b, r \ominus c] \\ &= D[r + w, a + d + w] + D[b + e, r \ominus c + e] \leq D[r, d + w] + D[w, a] + D[e, r \ominus c] + D[b, e]. \end{aligned}$$

□

Lemma 2.12. *Let $a, b, c, d, w, r \in \mathbb{R}_F$ and $c \ominus d$, $a \ominus b$, $c \ominus w$ exist. Then, we have*

$$D[c \ominus d, a \ominus b] \leq D[c \ominus w, a] + D[w + r, d] + D[r, b].$$

Proof. Using Remark ??, we have

$$\begin{aligned} D[c \ominus d, a \ominus b] &= D[c + b, a + d] = D[c + b + w, a + d + w] \leq D[c, a + w] + D[b + w, d] = D[c \ominus w, a] + D[b + w + r, d + r] \\ &\leq D[c \ominus w, a] + D[w + r, d] + D[r, b]. \end{aligned}$$

□

3 Differentiation of fuzzy-valued functions on time scales

In this section, we present some properties of the delta generalized Hukuhara derivative for fuzzy-valued functions on time scales. First, we recall some basic definitions for fuzzy number valued function on time scales [?]. Let $f : T \rightarrow \mathbb{R}_F$. We say that f has a T-lim, $u \in \mathbb{R}_F$ at $t_0 \in T$ if for every $\epsilon > 0$, there exists $\delta > 0$ such that $D(f(t), u) < \epsilon$, $\forall t \in U_T(t_0, \delta)$. The T-lim is unique and it is denoted by $T\text{-}\lim_{t \rightarrow t_0} f(t) = u$. A fuzzy-valued function f is said to be continuous at $t_0 \in T$ if $T\text{-}\lim_{t \rightarrow t_0} f(t)$ exists and equals to $f(t_0)$, i.e. $T\text{-}\lim_{t \rightarrow t_0} f(t) = f(t_0)$, $\forall t \in U_T(t_0, \delta)$. We say that f is regulated if its right-sided T-lims exist (finite) at all right-dense points in T and its left-sided T-lims exist (finite) at all left-dense points in T . A fuzzy function f is called rd-continuous provided it is continuous at all right-dense points in T and its left-sided T-lims exist (finite) at all left-dense points in T .

Definition 3.1. [?] Let $f : T \rightarrow \mathbb{R}_F$ and $t \in T$. We say that f is delta generalized Hukuhara differentiable (Δ_{GH} -differentiable) at t if there exists an element $f^{\Delta_{GH}}(t) \in \mathbb{R}_F$, such that for any $\epsilon > 0$, there exists a neighborhood $U_T(t, \delta)$ such that

1) for all $0 < h < \delta$, $\exists f(t+h) \ominus f(\sigma(t)), f(\sigma(t)) \ominus f(t-h)$ and for all $t-h, t+h \in U_T(t, \delta)$ we have

$$D [f(t+h) \ominus f(\sigma(t)), f^{\Delta_{GH}}(t)(h - \mu(t))] \leq \epsilon(h - \mu(t)), \quad (1)$$

$$D [f(\sigma(t)) \ominus f(t-h), f^{\Delta_{GH}}(t)(h + \mu(t))] \leq \epsilon(h + \mu(t)), \quad (2)$$

or

2) for all $0 < h < \delta$, $\exists f(\sigma(t)) \ominus f(t+h), f(t-h) \ominus f(\sigma(t))$ and for all $t-h, t+h \in U_T(t, \delta)$ we have

$$D [f(\sigma(t)) \ominus f(t+h), f^{\Delta_{GH}}(t)(\mu(t) - h)] \leq \epsilon(h - \mu(t)), \quad (3)$$

$$D [f(t-h) \ominus f(\sigma(t)), f^{\Delta_{GH}}(t)(-(h + \mu(t)))] \leq \epsilon(h + \mu(t)). \quad (4)$$

We say that f is Δ_1 -differentiable if it is differentiable as case (1) in Definition ?? and f is Δ_2 -differentiable if it is differentiable as case (2) in Definition ??.

Theorem 3.2. [?] Let $f : T \rightarrow \mathbb{R}_F$ be a fuzzy-valued function and $t \in T^k$. Then, if $f : T \rightarrow \mathbb{R}_F$ is Δ_{GH} -differentiable at $t \in T^k$, then it is continuous at t .

Theorem 3.3. 1) Let $f, g : T \rightarrow \mathbb{R}_F$ be Δ_{GH} -differentiable at $t \in T^k$ in the same case of differentiability, then $f + g : T \rightarrow \mathbb{R}_F$ is Δ_{GH} -differentiable at t and we have $(f + g)^{\Delta_{GH}}(t) = f^{\Delta_{GH}}(t) + g^{\Delta_{GH}}(t)$.

2) Let $\lambda \in \mathbb{R}$ and f be Δ_1 -differentiable (Δ_2 -differentiable) then $\lambda f : T \rightarrow \mathbb{R}_F$ is Δ_1 -differentiable (Δ_2 -differentiable) at t and we have $(\lambda f)^{\Delta_1}(t) = \lambda f^{\Delta_1}(t)$ and $(\lambda f)^{\Delta_2}(t) = \lambda f^{\Delta_2}(t)$.

Similarly to [?], in order to simplify the presentation, we make use of the following notations. We say that f satisfies the condition

(H1) at $t \in [a, b)_T$ if $f(t+h) \ominus f(\sigma(t))$ and $f(\sigma(t)) \ominus f(t-h)$ exist for h sufficiently small,

and we say that f satisfies the condition

(H2) at $t \in [a, b)_T$ if $f(\sigma(t)) \ominus f(t+h)$ and $f(t-h) \ominus f(\sigma(t))$ exist for h sufficiently small.

The following result extends Theorem 5 in [?] to time scales.

Theorem 3.4. Let $f : T \rightarrow \mathbb{R}$ be Δ -differentiable and $g : T \rightarrow \mathbb{R}_F$ be Δ_{GH} -differentiable, then

1. If $f(\sigma(t))f^\Delta(t) > 0$ and g is Δ_1 -differentiable, then fg is Δ_1 -differentiable and we have $(fg)^{\Delta_1}(t) = f^\Delta(t)g(t) + f(\sigma(t))g^{\Delta_1}(t)$.
2. If $f(\sigma(t))f^\Delta(t) < 0$ and g is Δ_2 -differentiable, then fg is Δ_2 -differentiable and we have $(fg)^{\Delta_2}(t) = f^\Delta(t)g(t) + f(\sigma(t))g^{\Delta_2}(t)$.
3. If $f(\sigma(t))f^\Delta(t) > 0$ and g is Δ_2 -differentiable and fg satisfies (H1) at t , then fg is Δ_{GH} -differentiable and we have $(fg)^{\Delta_{GH}}(t) = f^\Delta(t)g(t) \ominus (-f(\sigma(t)))g^{\Delta_{GH}}(t)$.
4. If $f(\sigma(t))f^\Delta(t) > 0$ and g is Δ_2 -differentiable and fg satisfies (H2) at t , then fg is Δ_{GH} -differentiable and we have $(fg)^{\Delta_{GH}}(t) = f(\sigma(t))g^{\Delta_{GH}}(t) \ominus (-f^\Delta(t))g(t)$.
5. If $f(\sigma(t))f^\Delta(t) < 0$ and g is Δ_1 -differentiable and fg satisfies (H1) at t , then fg is Δ_{GH} -differentiable and we have $(fg)^{\Delta_{GH}}(t) = f(\sigma(t))g^{\Delta_{GH}}(t) \ominus (-f^\Delta(t))g(t)$.
6. If $f(\sigma(t))f^\Delta(t) < 0$ and g is Δ_1 -differentiable and fg satisfies (H2) at t , then fg is Δ_{GH} -differentiable and we have $(fg)^{\Delta_{GH}}(t) = f^\Delta(t)g(t) \ominus (-f(\sigma(t)))g^{\Delta_{GH}}(t)$.

Proof. 1) Let $\epsilon \in (0, 1)$ and define $\epsilon^* = \epsilon [1 + |f(\sigma(t))| + \|g(t)\| + |f^\Delta(t)|]^{-1}$, then $\epsilon^* \in (0, 1)$. Thus, there exist neighborhoods $U_T^1(t, \delta)$ and $U_T^2(t, \delta)$ of t such that for all $0 < h < \delta$ with $t+h \in U_T^1(t, \delta)$, we have

$$|[f(\sigma(t)) - f(t+h)] - f^\Delta(t)(\mu(t) - h)| \leq \epsilon^*(h - \mu(t)), \quad (5)$$

and for all $0 < h < \delta$ with $t - h, t + h \in U_T^2(t, \delta)$, we have

$$D [(g(t+h) \ominus g(\sigma(t))), g^{\Delta_{GH}}(t)(h - \mu(t))] \leq \epsilon^*(h - \mu(t)), \quad (6)$$

$$D [(g(\sigma(t)) \ominus g(t-h)), g^{\Delta_{GH}}(t)(h + \mu(t))] \leq \epsilon^*(h + \mu(t)). \quad (7)$$

First, we show that the corresponding H-differences in definition of $(fg)^{\Delta_1}(t)$ exist. Since the H -difference $g(t+h) \ominus g(\sigma(t))$ exists for $h > 0$ sufficiently small, then there exists $u_1(\sigma(t), h) \in \mathbb{R}_F$ such that $g(t+h) = g(\sigma(t)) + u_1(\sigma(t), h)$. Also, since f is continuous, for $h > 0$ sufficiently small $f(\sigma(t))$, $f(t-h)$ and $f(t+h)$ have the same sign and we have $f(t+h) = f(\sigma(t)) + v_1(\sigma(t), h)$. Since $f(\sigma(t))f^\Delta(t) > 0$, then v_1 has the same sign as $f(\sigma(t))$ and $f(t+h)$ for sufficiently small $h > 0$. Therefore, by Remark ?? (6)-(7), we get

$$(fg)(t+h) = f(\sigma(t))g(\sigma(t)) + f(\sigma(t))u_1(\sigma(t), h) + g(\sigma(t))v_1(\sigma(t), h) + (u_1v_1)(\sigma(t), h),$$

i.e., the H -difference $f(t+h)g(t+h) \ominus f(\sigma(t))g(\sigma(t))$ exists.

On the other hand, by Δ_1 -differentiability of g at t , the H -difference $g(\sigma(t)) \ominus g(t-h)$ exists for $h > 0$ sufficiently small i.e., there exists $u_2(\sigma(t), h) \in \mathbb{R}_F$, such that $g(\sigma(t)) = g(t-h) + u_2(\sigma(t), h)$. Also, we can write $f(\sigma(t)) = f(t-h) + v_2(\sigma(t), h)$, where v_2 has the same sign as $f(\sigma(t))$ and $f(t-h)$ for sufficiently small $h > 0$. So, by Remark ?? (6)-(7), we get

$$(fg)(\sigma(t)) = f(t-h)g(t-h) + f(\sigma(t))u_2(\sigma(t), h) + g(\sigma(t))v_2(\sigma(t), h) + (u_2v_2)(\sigma(t), h),$$

i.e., the H -difference $f(\sigma(t))g(\sigma(t)) \ominus f(t-h)g(t-h)$ exists.

Since g is Δ_1 -differentiable at t , then it is continuous at t . Thus there exists a neighborhood $U_T^3(t, \delta)$ of t such that for all $0 < h < \delta$ with $t - h, t + h \in U_T^3(t, \delta)$, we have $D[g(t+h), g(t)] \leq \epsilon^*$, (8)

$$\|g(t+h)\| = D[g(t+h), \hat{0}] \leq D[g(t), \hat{0}] + 1 = \|g(t)\| + 1, \quad (9)$$

$$\text{and } D[g(t-h), g(t)] \leq \epsilon^*, \quad (10)$$

$$\|g(t-h)\| = D[g(t-h), \hat{0}] \leq D[g(t), \hat{0}] + 1 = \|g(t)\| + 1. \quad (11)$$

Let $U_T(t, \delta) = U_T^1(t, \delta) \cap U_T^2(t, \delta) \cap U_T^3(t, \delta)$. Using Lemma ??, if we consider $c = f(t+h)g(t+h)$, $d = f(\sigma(t))g(\sigma(t))$, $a = f(\sigma(t))g^{\Delta_{GH}}(t)(h - \mu(t))$, $b = g(t)f^\Delta(t)(h - \mu(t))$, $w = g(t+h)f(\sigma(t))$ and $r = g(t+h)f^\Delta(t)(h - \mu(t))$, then for $t - h, t + h \in U_T(t, \delta)$, we obtain

$$D [(fg)(t+h) \ominus (fg)(\sigma(t)), (f(\sigma(t))g^{\Delta_{GH}}(t) + g(t)f^\Delta(t))(h - \mu(t))] \leq D [(g(t+h) \ominus g(\sigma(t)))f(\sigma(t)), f(\sigma(t))g^{\Delta_{GH}}(t)(h - \mu(t))] \\ + D [f(t+h)g(t+h), f(\sigma(t))g(t+h) + g(t+h)f^\Delta(t)(h - \mu(t))] + D [g(t)f^\Delta(t)(h - \mu(t)), g(t+h)f^\Delta(t)(h - \mu(t))].$$

Since $h - \mu(t) > 0$ and $f(\sigma(t))f^\Delta(t) > 0$, then $f^\Delta(t)(h - \mu(t))$ has the same sign as $f(\sigma(t))$. So, we have $f(\sigma(t))g(t+h) + f^\Delta(t)g(t+h)(h - \mu(t)) = g(t+h)[f(\sigma(t)) + f^\Delta(t)(h - \mu(t))]$. Therefore, we get

$$D [f(t+h)g(t+h), f(\sigma(t))g(t+h) + g(t+h)f^\Delta(t)(h - \mu(t))] = |f(t+h) - f(\sigma(t)) - f^\Delta(t)(h - \mu(t))| \|g(t+h)\|. \quad (12)$$

Then, by (5)-(6), (8)-(9) and (12), we have

$$D [(fg)(t+h) \ominus (fg)(\sigma(t)), (f(\sigma(t))g^{\Delta_{GH}}(t) + g(t)f^\Delta(t))(h - \mu(t))] \leq \epsilon^* |f(\sigma(t))(h - \mu(t)) + \epsilon^* \|g(t+h)\|(h - \mu(t))| \\ + D[g(t+h), g(t)]f^\Delta(t)(h - \mu(t)) \leq \epsilon^*(h - \mu(t))[|f(\sigma(t))| + 1 + \|g(t)\| + |f^\Delta(t)|] \leq \epsilon(h - \mu(t)). \quad (13)$$

Similarly, for the left-sided derivative, there exists neighborhood $U_T^1(t, \delta)$ of t such that for all $0 < h < \delta$ with $t - h \in U_T^1(t, \delta)$, we have

$$|[f(\sigma(t)) - f(t-h)] - f^\Delta(t)(h + \mu(t))| \leq \epsilon^*(h + \mu(t)). \quad (14)$$

Then, using Lemma ??, if we consider $c = f(\sigma(t))g(\sigma(t))$, $d = f(t-h)g(t-h)$, $a = f(\sigma(t))g^{\Delta GH}(t)(h + \mu(t))$, $b = g(t)f^{\Delta}(t)(h + \mu(t))$, $w = g(t-h)f(\sigma(t))$ and $r = g(t-h)f^{\Delta}(t)(h + \mu(t))$, we get

$$D \left[(fg)(\sigma(t)) \ominus (fg)(t-h), \left(f(\sigma(t))g^{\Delta GH}(t) + g(t)f^{\Delta}(t) \right) (h + \mu(t)) \right] \leq D \left[f(\sigma(t))g(\sigma(t)) \ominus g(t-h)f(\sigma(t)), f(\sigma(t))g^{\Delta GH}(t)(h + \mu(t)) \right] + D \left[g(t-h)f(\sigma(t)), f(t-h)g(t-h) + g(t-h)f^{\Delta}(t)(h + \mu(t)) \right] + D \left[g(t)f^{\Delta}(t)(h + \mu(t)), g(t-h)f^{\Delta}(t)(h + \mu(t)) \right].$$

By continuity of f , we know that $f(\sigma(t))$ and $f(t-h)$ have the same sign. Since $h + \mu(t) > 0$ and $f(\sigma(t))f^{\Delta}(t) > 0$, then $f^{\Delta}(t)(h + \mu(t))$ has the same sign as $f(\sigma(t))$. Then, by Remark ?? (6)-(7), we have $f(t-h)g(t-h) + f^{\Delta}(t)g(t-h)(h + \mu(t)) = g(t-h)[f(t-h) + f^{\Delta}(t)(h + \mu(t))]$. So, we get

$$D \left[g(t-h)f(\sigma(t)), f(t-h)g(t-h) + f^{\Delta}(t)g(t-h)(h + \mu(t)) \right] = |f(\sigma(t)) - f(t-h) - f^{\Delta}(t)(h + \mu(t))| \|g(t-h)\|. \tag{15}$$

Then, by (7), (14), (15), (10) and (11), we have

$$D \left[(fg)(\sigma(t)) \ominus (fg)(t-h), \left(f(\sigma(t))g^{\Delta GH}(t) + g(t)f^{\Delta}(t) \right) (h + \mu(t)) \right] \leq \epsilon^*(h + \mu(t))|f(\sigma(t))| + \epsilon^* \|g(t-h)\|(h + \mu(t)) + (h + \mu(t)) |f^{\Delta}(t)| |D[g(t-h), g(t)]| \leq \epsilon^*(h + \mu(t))[|f(\sigma(t))| + 1 + \|g(t)\| + |f^{\Delta}(t)|] \leq \epsilon(h + \mu(t)). \tag{16}$$

Therefore, by (13) and (16), we conclude $(fg)^{\Delta_1}(t) = f(\sigma(t))g^{\Delta_1}(t) + f^{\Delta}(t)g(t)$.

2) Let $\epsilon \in (0, 1)$ and define $\epsilon^* = \epsilon[1 + |f(\sigma(t))| + \|g(t)\| + |f^{\Delta}(t)|]^{-1}$, then $\epsilon^* \in (0, 1)$. Thus, there exist neighborhoods $U_T^1(t, \delta)$ and $U_T^2(t, \delta)$ of t such that for all $0 < h < \delta$ with $t + h \in U_T^1(t, \delta)$ we have

$$|[f(\sigma(t)) - f(t+h)] - f^{\Delta}(t)(\mu(t) - h)| \leq \epsilon^*(h - \mu(t)), \tag{17}$$

and for all $0 < h < \delta$ with $t-h, t+h \in U_T^2(t, \delta)$, we have

$$D[(g(\sigma(t)) \ominus g(t+h)), g^{\Delta GH}(t)(\mu(t) - h)] \leq \epsilon^*(h - \mu(t)), \tag{18}$$

$$D[(g(t-h) \ominus g(\sigma(t))), g^{\Delta GH}(t)(-(h + \mu(t)))] \leq \epsilon^*(h + \mu(t)). \tag{19}$$

By Δ_2 -differentiability of g at t , the H -difference $g(\sigma(t)) \ominus g(t+h)$ exists for $h > 0$ sufficiently small i.e., there exists $u_1(\sigma(t), h) \in \mathbb{R}_F$, such that $g(\sigma(t)) = g(t+h) + u_1(\sigma(t), h)$. Also, since f is continuous at t , then for $h > 0$ sufficiently small, $f(\sigma(t))$, $f(t-h)$ and $f(t+h)$ have the same sign and we have $f(\sigma(t)) = f(t+h) + v_1(\sigma(t), h)$, where v_1 has the same sign as $f(\sigma(t))$ and $f(t+h)$. By Remark ?? (6)-(7), we have

$$(fg)(\sigma(t)) = f(t+h)g(t+h) + f(\sigma(t))u_1(\sigma(t), h) + g(\sigma(t))v_1(\sigma(t), h) + (u_1v_1)(\sigma(t), h),$$

i.e., the H -difference $f(\sigma(t))g(\sigma(t)) \ominus f(t+h)g(t+h)$ exists. Similarly, the H -difference $g(t-h) \ominus g(\sigma(t))$ exists for $h > 0$ sufficiently small i.e., there exists $u_2(\sigma(t), h) \in \mathbb{R}_F$, such that $g(t-h) = g(\sigma(t)) + u_2(\sigma(t), h)$. Also, we can write $f(t-h) = f(\sigma(t)) + v_2(\sigma(t), h)$, where v_2 has the same sign as $f(\sigma(t))$ and $f(t-h)$ for sufficiently small $h > 0$. By Remark ?? (6,7), we get

$$(fg)(t-h) = f(\sigma(t))g(\sigma(t)) + f(\sigma(t))u_2(\sigma(t), h) + g(\sigma(t))v_2(\sigma(t), h) + (u_2v_2)(\sigma(t), h),$$

i.e., the H -difference $f(t-h)g(t-h) \ominus f(\sigma(t))g(\sigma(t))$ exists. Now, since g is Δ_2 -differentiable at t , then it is continuous at t and there exists a neighborhood $U_T^3(t, \delta)$ of t such that for all $0 < h < \delta$ with $t-h, t+h \in U_T^3(t, \delta)$, we have

$$D[g(t-h), g(t)] \leq \epsilon^*, \quad \|g(t-h)\| = D[g(t-h), \hat{0}] \leq D[g(t), \hat{0}] + 1 = \|g(t)\| + 1,$$

and $D[g(t+h), g(t)] \leq \epsilon^*$, $D[g(t+h), \hat{0}] \leq D[g(t), \hat{0}] + 1 = \|g(t)\| + 1$. Let $U_T(t, \delta) = U_T^1(t, \delta) \cap U_T^2(t, \delta) \cap U_T^3(t, \delta)$. Now, in Lemma ??, if we consider $c = f(t-h)g(t-h)$, $d = f(\sigma(t))g(\sigma(t))$, $a = f(\sigma(t))g^{\Delta GH}(t)(-(h + \mu(t)))$, $b = g(t)f^{\Delta}(t)(-(h + \mu(t)))$, $w = g(t-h)f(\sigma(t))$ and $r = g(t-h)f^{\Delta}(t)(-(h + \mu(t)))$, then for $t-h, t+h \in U_T(t, \delta)$, we obtain

$$D \left[(fg)(t-h) \ominus (fg)(\sigma(t)), \left(f(\sigma(t))g^{\Delta GH}(t) + g(t)f^{\Delta}(t) \right) (-(h + \mu(t))) \right] \leq D \left[g(t-h)f(\sigma(t)) \ominus g(\sigma(t))f(\sigma(t)), f(\sigma(t))g^{\Delta GH}(t)(-(h + \mu(t))) \right] + D \left[f(t-h)g(t-h), f(\sigma(t))g(t-h) + g(t-h)f^{\Delta}(t)(-(h + \mu(t))) \right] + D[g(t)f^{\Delta}(t)(-(h + \mu(t))), g(t-h)f^{\Delta}(t)(-(h + \mu(t)))].$$

By continuity of f , we know that $f(\sigma(t))$ and $f(t-h)$ have the same sign. Since $-(h + \mu(t)) \leq 0$ and $f(\sigma(t))f^{\Delta}(t) < 0$, then $f^{\Delta}(t)(-(h + \mu(t)))$ has the same sign as $f(\sigma(t))$. Then, we have

$$D \left[f(t-h)g(t-h), f(\sigma(t))g(t-h) + f^{\Delta}(t)g(t-h)(-(h + \mu(t))) \right] = |f(t-h) - f(\sigma(t)) - f^{\Delta}(t)(-(h + \mu(t)))| \|g(t-h)\|. \tag{20}$$

Thus, by (14), (19) and (20), we have

$$\begin{aligned} D \left[(fg)(t-h) \ominus (fg)(\sigma(t)), \left(f(\sigma(t))g^{\Delta_{GH}}(t) + g(t)f^{\Delta}(t) \right) (-h + \mu(t)) \right] &\leq D \left[g(t-h) \ominus g(\sigma(t)), g^{\Delta_{GH}}(t)(-h + \mu(t)) \right] |f(\sigma(t))| \\ &+ |f(t-h) - f(\sigma(t)) - f^{\Delta}(t)(-h + \mu(t))| \|g(t-h)\| + |f^{\Delta}(t)(-h + \mu(t))| D[g(t-h), g(t)] \leq \epsilon^* |f(\sigma(t))(h + \mu(t)) \\ &+ \epsilon^* \|g(t-h)\|(h + \mu(t)) + |f^{\Delta}(t)|(h + \mu(t)) D[g(t-h), g(t)] \leq \epsilon^*(h + \mu(t)) \left[|f(\sigma(t))| + 1 + \|g(t)\| + |f^{\Delta}(t)| \right] \leq \epsilon(h + \mu(t)). \end{aligned}$$

In a similar way, we can show that

$$D[(fg)(\sigma(t)) \ominus (fg)(t+h), (f(\sigma(t))g^{\Delta_{GH}}(t) + g(t)f^{\Delta}(t))(\mu(t) - h)] \leq \epsilon(h - \mu(t)).$$

3) We consider $\epsilon \in (0, 1)$ and define $\epsilon^* = [2 + \|g(t)\| + |f^{\Delta}(t)| + |f(t)| + \|g^{\Delta_{GH}}(t)\|]^{-1}$, then $\epsilon^* \in (0, 1)$. Analogously to previous cases, there exist neighborhoods $U_T^1(t, \delta)$ and $U_T^2(t, \delta)$ of t such that for all $0 < h < \delta$ with $t+h \in U_T^1(t, \delta)$, we have (5) and for all $0 < h < \delta$ with $t-h, t+h \in U_T^2(t, \delta)$, we have (18)-(19). Since g is Δ_{GH} -differentiable at t , then its continuous at t . So, there exists a neighborhood $U_T^3(t, \delta)$ of t such that

$$D[g(\sigma(t)), g(t)] \leq \epsilon^*, \text{ and } \|g(\sigma(t))\| = D[g(\sigma(t)), \hat{0}] \leq D[g(t), \hat{0}] + 1 = \|g(t)\| + 1. \quad (21)$$

Since f is Δ -differentiable at t , then it is continuous at t , so there exists a neighborhood of t , such that for every $t+h \in U_T^4(t, \delta)$, we have

$$|f(\sigma(t)) - f(t+h)| < \epsilon^*, \quad 0 < h < \delta. \quad (22)$$

By condition (H1), we know that the H-difference $f(t+h)g(t+h) \ominus f(\sigma(t))g(\sigma(t))$ exists. Now, let $U_T(t, \delta) = U_T^1(t, \delta) \cap U_T^2(t, \delta) \cap U_T^3(t, \delta) \cap U_T^4(t, \delta)$. In Lemma ??, if we consider $c = f(t+h)g(t+h)$, $d = f(\sigma(t))g(\sigma(t))$, $a = g(t)f^{\Delta}(t)(h - \mu(t))$, $b = -f(\sigma(t))g^{\Delta_{GH}}(t)(h - \mu(t))$, $g = g(\sigma(t))f(t+h)$, $f = f^{\Delta}(t)g(\sigma(t))(h - \mu(t))$, $e = -f(t+h)g^{\Delta_{GH}}(t)(h - \mu(t))$, we get

$$\begin{aligned} D \left[(fg)(t+h) \ominus (fg)(\sigma(t)), \left(f^{\Delta}(t)g(t) \ominus (-f(\sigma(t))g^{\Delta_{GH}}(t))(h - \mu(t)) \right) \right] &\leq D \left[f(t+h)g(\sigma(t)), f(\sigma(t))g(\sigma(t)) + f^{\Delta}(t)g(\sigma(t))(h - \mu(t)) \right] \\ &+ D \left[f^{\Delta}(t)g(t)(h - \mu(t)), f^{\Delta}(t)g(\sigma(t))(h - \mu(t)) \right] + D \left[-f(t+h)g^{\Delta_{GH}}(t)(h - \mu(t)), f(t+h)g(\sigma(t)) \ominus f(t+h)g(t+h) \right] \\ &+ D \left[(-f(t+h))g^{\Delta_{GH}}(t)(h - \mu(t)), (-f(\sigma(t)))g^{\Delta_{GH}}(t)(h - \mu(t)) \right]. \end{aligned}$$

Since $f^{\Delta}(t)f(\sigma(t)) > 0$ and $h - \mu(t) > 0$, then $f^{\Delta}(t)(h - \mu(t))$ has the same sign as $f(\sigma(t))$, so we have

$$D \left[f(t+h)g(\sigma(t)), f(\sigma(t))g(\sigma(t)) + f^{\Delta}(t)g(\sigma(t))(h - \mu(t)) \right] = |f(t+h) - f(\sigma(t)) - (h - \mu(t))f^{\Delta}(t)| \|g(\sigma(t))\|.$$

Now, using this inequality and (21)-(22), we have

$$\begin{aligned} D \left[(fg)(t+h) \ominus (fg)(\sigma(t)), \left(f^{\Delta}(t)g(t) \ominus (-f(\sigma(t))g^{\Delta_{GH}}(t)(h - \mu(t))) \right) \right] &\leq |f(t+h) - f(\sigma(t)) - (h - \mu(t))f^{\Delta}(t)| \|g(\sigma(t))\| \\ &+ |f^{\Delta}(t)(h - \mu(t))| D[g(\sigma(t)), g(t)] + D \left[-g^{\Delta_{GH}}(t)(h - \mu(t)), g(\sigma(t)) \ominus g(t+h) \right] |f(t+h)| \\ &+ \|g^{\Delta_{GH}}(t)\|(h - \mu(t)) |f(t+h) - f(\sigma(t))| \leq \epsilon^*(h - \mu(t)) \|g(\sigma(t))\| + |f^{\Delta}(t)|(h - \mu(t))\epsilon^* + (h - \mu(t))\epsilon^* |f(t+h)| \\ &+ \|g^{\Delta_{GH}}(t)\|(h - \mu(t))\epsilon^* \leq \epsilon^*(h - \mu(t)) \left[1 + \|g(t)\| + |f^{\Delta}(t)| + 1 + |f(t)| + \|g^{\Delta_{GH}}(t)\| \right] \leq \epsilon(h - \mu(t)). \end{aligned}$$

In similar way, we can easily prove the left differentiability of fg at t . Therefore, we obtain

$$(fg)^{\Delta_{GH}}(t) = f^{\Delta}(t)g(t) \ominus (-f(\sigma(t))g^{\Delta_{GH}}(t)).$$

4) The proof is similar to case (3). Let $\epsilon \in (0, 1)$ and define $\epsilon^* = [1 + \|g(t)\| + |f^{\Delta}(t)| + |f(t)| + \|g^{\Delta_{GH}}(t)\|]^{-1}$, then $\epsilon^* \in (0, 1)$. By (H2), the H-difference $f(\sigma(t))g(\sigma(t)) \ominus f(t+h)g(t+h)$ exists. Now, in Lemma ??, if we consider $c = f(\sigma(t))g(\sigma(t))$, $d = f(t+h)g(t+h)$, $a = f(\sigma(t))g^{\Delta_{GH}}(t)(\mu(t) - h)$, $b = -f^{\Delta}(t)g(t)(\mu(t) - h)$, $w = f(\sigma(t))g(t+h)$, $r = -g(t+h)f^{\Delta}(t)(\mu(t) - h)$, we have

$$\begin{aligned} D \left[(fg)(\sigma(t)) \ominus (fg)(t+h), \left(f(\sigma(t))g^{\Delta_{GH}}(t) \ominus (-f^{\Delta}(t))g(t) \right) (\mu(t) - h) \right] &\leq D \left[f(\sigma(t))g(\sigma(t)) \ominus g(t+h)f(\sigma(t)), \left(f(\sigma(t))g^{\Delta_{GH}}(t) \right) (\mu(t) - h) \right] \\ &+ D \left[g(t+h)f(\sigma(t)) + (-f^{\Delta}(t))g(t+h)(\mu(t) - h), f(t+h)g(t+h) \right] + D \left[(-f^{\Delta}(t))g(t+h)(\mu(t) - h), (-f^{\Delta}(t))g(t)(\mu(t) - h) \right]. \end{aligned}$$

Since f is Δ -differentiable, then it is continuous, so $f(\sigma(t))$ and $f(t+h)$ have the same sign. Since $\mu(t)-h \leq 0$ and $f(\sigma(t))f^\Delta(t) > 0$, then $-f^\Delta(t)(\mu(t)-h)$ has the same sign as $f(\sigma(t))$. Then, by Remark ?? (6)-(7), we have

$$D[g(t+h)f(t+h), g(t+h)f(\sigma(t)) + (-f^\Delta(t))g(t+h)(\mu(t)-h)] = |f(t+h) - f(\sigma(t)) - (f^\Delta(t))(h - \mu(t))| \|g(t+h)\|.$$

$$\begin{aligned} & D \left[(fg)(\sigma(t)) \ominus (fg)(t+h), \left(f(\sigma(t))g^{\Delta_{GH}}(t) \ominus (-f^\Delta(t))g(t) \right) (\mu(t)-h) \right] \leq D \left[g(\sigma(t)) \ominus g(t+h), g^{\Delta_{GH}}(t)(\mu(t)-h) \right] |f(\sigma(t))| \\ & + |f(t+h) - f(\sigma(t)) - (f^\Delta(t))(h - \mu(t))| \|g(t+h)\| + |(-f^\Delta(t))(\mu(t)-h)| D[g(t+h), g(t)] \\ & \leq \epsilon^* |f(\sigma(t))| \|\mu(t)-h\| + \epsilon^* |\mu(t)-h| \|g(t+h)\| + \epsilon^* |(-f^\Delta(t))| \|\mu(t)-h\| \leq \epsilon^* |\mu(t)-h| \left(|f(\sigma(t))| + 1 + \|g(t)\| + | -f^\Delta(t) | \right) \\ & \leq \epsilon(h - \mu(t)). \end{aligned}$$

Similarly, we can show the left-sided differentiability for case (4). The proofs of cases (5)-(6) are similar to the previous cases and we omit them here. \square

Theorem 3.5. [?] *Let $f : T \rightarrow \mathbb{R}_F$ be a fuzzy-valued function. Let $t \in T^k$ and denote $[f(t)]_\alpha = [f_\alpha(t), f^\alpha(t)]$ for each $\alpha \in [0, 1]$.*

1) *If f is Δ_1 -differentiable, then $[f^{\Delta_{GH}}(t)]_\alpha = [f_\alpha^\Delta(t), \overline{f_\alpha}^\Delta(t)]$.*

2) *If f is Δ_2 -differentiable, then $[f^{\Delta_{GH}}(t)]_\alpha = [\overline{f_\alpha}^\Delta(t), f_\alpha^\Delta(t)]$.*

Let $f : T \rightarrow \mathbb{R}_F$ be a fuzzy-valued function and let $P : a = t_0 < t_1 < \dots < t_n = b$ be a partition of $[a, b]_T$. In each interval $[t_{i-1}, t_i]_T$, where $1 \leq i \leq n$, we choose an arbitrary point ξ_i and form the sum $S = \sum_{i=1}^n (t_i - t_{i-1})f(\xi_i)$. We call S a Riemann Δ -sum of f corresponding to the partition P .

Definition 3.6. [?] *We say that a fuzzy-valued function $f : T \rightarrow \mathbb{R}_F$ is Riemann Δ -integrable from a to b if there is a fuzzy number u such that for each $\epsilon > 0$ there is $\delta > 0$ such that $D(S, u) < \epsilon$, for every Riemann Δ -sum S of f corresponding to a partition P , independent of the way in which we choose $\xi_i \in [t_{i-1}, t_i]_T$, $i = 1, 2, \dots, n$. It is easily seen that $u \in \mathbb{R}_F$ is unique. The number $u \in \mathbb{R}_F$ is called the Riemann Δ -integral of f from a to b , and we denote it by $\int_a^b f(t)\Delta t$.*

Theorem 3.7. [?] *Every rd-continuous fuzzy function on $[a, b]_T$ is Riemann Δ -integrable.*

Similarly to [?, ?], we have the following result.

Theorem 3.8. *Let $f : T \rightarrow \mathbb{R}_F$ be an rd-continuous fuzzy-valued function and $\alpha \in \mathbb{R}_F$. Then*

1) $F(t) = \alpha + \int_a^t f(\tau)\Delta\tau$ is Δ_1 -differentiable and we have $F^{\Delta_{GH}}(t) = f(t)$.

2) $F(t) = \alpha \ominus \int_a^t -f(\tau)\Delta\tau$ is Δ_2 -differentiable and we have $F^{\Delta_{GH}}(t) = f(t)$.

4 Fuzzy differential equations on time scales

Let us consider the fuzzy initial-value problem

$$\begin{cases} y^{\Delta_{GH}}(t) = a(t)y(t) + b(t), \\ y(t_0) = y_0, \end{cases} \quad (23)$$

where $a : [t_0, t_1]_T \rightarrow \mathbb{R}$, $y_0 \in \mathbb{R}_F$ and $b : [t_0, t_1]_T \in \mathbb{R}_F$. It is easy to check that generally this problem is not equivalent to the following other two problems

$$\begin{cases} y^{\Delta_{GH}}(t) + (-a(t))y(t) = b(t), \\ y(t_0) = y_0, \end{cases} \quad (24)$$

and

$$\begin{cases} y^{\Delta_{GH}}(t) + (-b(t)) = a(t)y(t), \\ y(t_0) = y_0. \end{cases} \quad (25)$$

Also, in the general case, problems (24) and (25) are not equivalent. Let us remark that the problem

$$\begin{cases} y^{\Delta_{GH}}(t) + (-b(t)) + (-a(t))y(t) = \hat{0}, \\ y(t_0) = y_0. \end{cases} \tag{26}$$

has no fuzzy solution. In the following, we study the generalized differentiable solutions of (23)-(25).

Theorem 4.1. *Let*

$$y_1(t) = e_a(t, t_0) \left(y_0 + \int_{t_0}^t b(\tau)e_{\ominus a}(\sigma(\tau), t_0)\Delta\tau \right), \tag{27}$$

and

$$y_2(t) = e_a(t, t_0) \left(y_0 \ominus \int_{t_0}^t (-b(\tau))e_{\ominus a}(\sigma(\tau), t_0)\Delta\tau \right), \tag{28}$$

provided that the H -difference $y_0 \ominus \int_{t_0}^t (-b(\tau))e_{\ominus a}(\sigma(\tau), t_0)\Delta\tau$ exists.

- 1) If $a(t) > 0$ for $t \in [t_0, t_1)_T$, then y_1 is Δ_1 -differentiable and it is a solution of the problem (23) in the interval $[t_0, t_1)_T$.
- 2) If $a(t) < 0$ for $t \in [t_0, t_1)_T$ and the H -difference $y_0 \ominus \int_{t_0}^t (-b(\tau))e_{\ominus a}(\sigma(\tau), t_0)\Delta\tau$ exists for $t \in [t_0, t_1)_T$, then y_2 is Δ_2 -differentiable and it is a solution of the problem (23) in the interval $[t_0, t_1)_T$.
- 3) If $a(t) < 0$ for $t \in [t_0, t_1)_T$ and y_1 satisfies condition (H1) or (H2) for $t \in [t_0, t_1)_T$, then it is a solution of (24) or (25), respectively.
- 4) If $a(t) > 0$ for $t \in [t_0, t_1)_T$, the H -difference $y_0 \ominus \int_{t_0}^t (-b(\tau))e_{\ominus a}(\sigma(\tau), t_0)\Delta\tau$ exists for $t \in [t_0, t_1)_T$ and y_2 satisfies condition (H1) or (H2) for $t \in [t_0, t_1)_T$, then y_2 is a solution of (25) or (24), respectively.

Proof. A constant function is differentiable in any case of differentiability. Then, by Theorem ??, we have $y_0 + \int_{t_0}^t b(\tau)e_{\ominus a}(\sigma(\tau), t_0)\Delta\tau$ is Δ_1 -differentiable and

$$\left(y_0 + \int_{t_0}^t b(\tau)e_{\ominus a}(\sigma(\tau), t_0)\Delta\tau \right)^{\Delta_1} = b(t)e_{\ominus a}(\sigma(t), t_0). \tag{29}$$

If the H -difference $y_0 \ominus \int_{t_0}^t (-b(\tau))e_{\ominus a}(\sigma(\tau), t_0)\Delta\tau$ exists, then by Theorem ??, it is Δ_2 -differentiable and we have

$$\left(y_0 \ominus \int_{t_0}^t (-b(\tau))e_{\ominus a}(\sigma(\tau), t_0)\Delta\tau \right)^{\Delta_2} = b(t)e_{\ominus a}(\sigma(t), t_0). \tag{30}$$

- 1) Consider $f(t) = e_a(t, t_0)$, $g_1(t) = y_0 + \int_{t_0}^t b(\tau)e_{\ominus a}(\sigma(\tau), t_0)\Delta\tau$. Since for every $t \in [t_0, t_1)_T$, $a(t) > 0$ and $e_a^{\Delta}(t, t_0) = a(t)e_a(t, t_0)$, then we have $f(\sigma(t))f^{\Delta}(t) = a(t)e_a(\sigma(t), t_0)e_a(t, t_0) > 0$, and the conditions in Theorem ?? (1) are satisfied by (29). Then, since $e_a(\sigma(t), t_0)e_{\ominus a}(\sigma(t), t_0) = 1$, we have

$$\begin{aligned} y_1^{\Delta_1}(t) &= (fg_1)^{\Delta_1}(t) = f(\sigma(t))g_1^{\Delta_1}(t) + g_1(t)f^{\Delta}(t) = e_a(\sigma(t), t_0) (b(t)e_{\ominus a}(\sigma(t), t_0)) \\ &+ \left(y_0 + \int_{t_0}^t b(\tau)e_{\ominus a}(\sigma(\tau), t_0)\Delta\tau \right) e_a(t, t_0)a(t) = b(t) + a(t)y_1(t), \end{aligned}$$

which means that y_1 is a solution of (23).

2) We denote $f(t) = e_a(t, t_0)$, $g_2(t) = y_0 \ominus \int_{t_0}^t (-b(\tau))e_{\ominus a}(\sigma(\tau), t_0)\Delta\tau$. Since for all $t \in [t_0, t_1)_T$, $a(t) < 0$, then we have $f(\sigma(t))f^\Delta(t) = a(t)e_a(\sigma(t), t_0)e_a(t, t_0) < 0$, and the conditions in Theorem ?? (2) are satisfied by (30). Then, we obtain

$$y_2^{\Delta^2}(t) = (fg_2)^{\Delta^2}(t) = f(\sigma(t))g_2^{\Delta^2}(t) + g_2(t)f^\Delta(t) = e_a(\sigma(t), t_0) (b(t)e_{\ominus a}(\sigma(t), t_0)) + \left(y_0 \ominus \int_{t_0}^t (-b(\tau))e_{\ominus a}(\sigma(\tau), t_0)\Delta\tau \right) e_a(t, t_0)a(t) = b(t) + a(t)y_2(t),$$

i.e., y_2 is a solution of (23).

3) First, we suppose that y_1 satisfies condition (H1). We denote $f(t) = e_a(t, t_0)$ and $g_1(t) = y_0 + \int_{t_0}^t b(\tau)e_{\ominus a}(\sigma(\tau), t_0)\Delta\tau$.

Since for all $t \in [t_0, t_1)_T$, $a(t) < 0$, then $f(\sigma(t))f^\Delta(t) = a(t)e_a(\sigma(t), t_0)e_a(t, t_0) < 0$ and the hypothesis in Theorem ?? (5) are satisfied and we have

$$y_1^\Delta(t) = (fg_1)^{\Delta_{GH}}(t) = f(\sigma(t))(g_1)^{\Delta_{GH}}(t) \ominus (-f^\Delta(t))g_1(t) = e_a(\sigma(t), t_0) (b(t)e_{\ominus a}(\sigma(t), t_0)) \ominus (-a(t)e_a(t, t_0)) \left(y_0 + \int_{t_0}^t b(\tau)e_{\ominus a}(\sigma(\tau), t_0)\Delta\tau \right).$$

Therefore, $(y_1)^{\Delta_{GH}}(t) + (-a(t))y_1(t) = b(t)$, i.e., y_1 is a solution of (24). Now, suppose that y_1 satisfies condition (H2). If f and g_1 are as above, then the hypothesis in Theorem ?? (6) are satisfied and we have

$$y_1^\Delta(t) = (fg_1)^{\Delta_{GH}}(t) = f^\Delta(t)g_1(t) \ominus (-f(\sigma(t)))(g_1)^{\Delta_{GH}}(t) = a(t)e_a(t, t_0) \left(y_0 + \int_{t_0}^t b(\tau)e_{\ominus a}(\sigma(\tau), t_0)\Delta\tau \right) \ominus (-e_a(\sigma(t), t_0)) (b(t)e_{\ominus a}(\sigma(t), t_0)) = a(t)y_1(t) \ominus (-b(t)).$$

Thus, we have $(y_1)^{\Delta_{GH}}(t) + (-b(t)) = a(t)y_1(t)$, which means that y_1 is a solution of (25).

4) Let f and g_2 be as above. Since for all $t \in [t_0, t_1)_T$ $a(t) > 0$, then $f(\sigma(t))f^\Delta(t) = a(t)e_a(\sigma(t), t_0)e_a(t, t_0) > 0$ and g_2 is Δ_2 -differentiable by (30). If y_2 satisfies condition (H1), then the conditions in Theorem ?? (3) are satisfied and we have

$$y_2^\Delta(t) = (fg_2)^{\Delta_{GH}}(t) = f^\Delta(t)g_2(t) \ominus (-f(\sigma(t)))g_2^{\Delta_{GH}}(t) = a(t)e_a(t, t_0) \left(y_0 \ominus \int_{t_0}^t (-b(\tau))e_{\ominus a}(\sigma(\tau), t_0)\Delta\tau \right) \ominus (-e_a(\sigma(t), t_0))b(t)e_{\ominus a}(\sigma(t), t_0).$$

Therefore, we obtain $(y_2)^{\Delta_{GH}}(t) + (-b(t)) = a(t)y_2(t)$, i.e., y_1 is a solution of (25). Similarly, y_2 satisfies condition (H2), then the conditions in Theorem ?? (4) are satisfied and we have

$$y_2^\Delta(t) = (fg_2)^{\Delta_{GH}}(t) = f(\sigma(t))g_2^{\Delta_{GH}}(t) \ominus (-f^\Delta(t))g_2(t) = e_a(\sigma(t), t_0)b(t)e_{\ominus a}(\sigma(t), t_0) \ominus (-a(t)e_a(t, t_0)) \left(y_0 \ominus \int_{t_0}^t (-b(\tau))e_{\ominus a}(\sigma(\tau), t_0)\Delta\tau \right) = b(t) \ominus (-a(t))y_2(t).$$

So, $(y_2)^{\Delta_{GH}}(t) + (-a(t))y_2(t) = b(t)$, i.e., y_2 is a solution of (24). □

In the following, we present two examples to illustrate the applicability of our results.

Example 4.2. Consider the first order fuzzy dynamic equation $\begin{cases} y^\Delta(t) = (-1)y(t) + t, \\ y(0) = (1, 2, 3). \end{cases}$ First, we consider the case

$T = \mathbb{R}$. Then, the H -difference $y_0 \ominus \int_0^t (-\tau)e^\tau d\tau$ exists and using Theorem ?? (2), we get

$$\begin{aligned} y_2(t) &= e^{-t} \left[y_0 \ominus \int_0^t (-\tau)e^\tau d\tau \right] = e^{-t} \left[(1, 2, 3) \ominus \int_0^t (-\tau)e^\tau d\tau \right] \\ &= e^{-t}(1, 2, 3) \ominus e^{-t}(-te^t + e^t - 1) = (2e^{-t} + t - 1, 3e^{-t} + t - 1, 4e^{-t} + t - 1), \end{aligned}$$

which is consistent with the result of [?] and it is shown in Figure 1. Next, we consider the case $T = h\mathbb{Z}$ with $h \neq 1$. Since the H -difference $y_0 \ominus \int_0^t (-\tau)e_{\ominus a}(\sigma(\tau), t_0)\Delta\tau$ exists, by Theorem 4.1 (2) we have

$$\begin{aligned} y_2(t) &= (1-h)^{\frac{t}{h}} \left[(1, 2, 3) \ominus \int_0^t (-\tau) \left(\frac{1}{1-h} \right)^{\frac{\tau+h}{h}} \Delta\tau \right] = (1-h)^{\frac{t}{h}} \left[(1, 2, 3) \ominus \int_0^t (-\tau)(1-h)^{-\left(\frac{\tau}{h}+1\right)} \Delta\tau \right] \\ &= (1-h)^{\frac{t}{h}} \left[(1, 2, 3) \ominus \left(\frac{-h}{1-h} \right) \sum_{\tau=0}^{t-1} \tau(1-h)^{-\frac{\tau}{h}} \right] = (1-h)^{\frac{t}{h}}(1, 2, 3) \ominus (-1) \left[(1-h)^{\frac{t}{h}} + t - 1 \right]. \end{aligned}$$

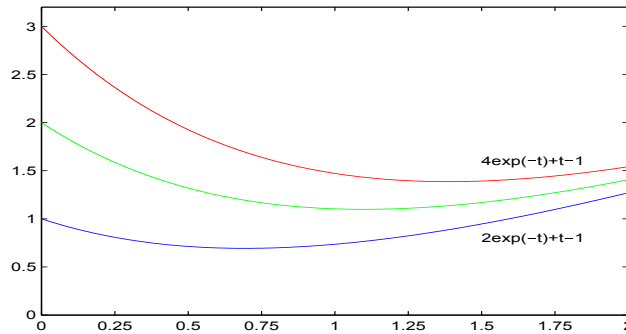


Figure 1: Δ_2 -differentiable solution of the Example 4.2 for $T = \mathbb{R}$.

Example 4.3. Let us consider the fuzzy initial value problem $\begin{cases} y^\Delta(t) = y(t) + t, \\ y(0) = (1, 2, 3). \end{cases}$ First, we consider the case $T = \mathbb{R}$.

By Theorem ?? (1), we get

$$y_1(t) = e^t \left[y_0 + \int_0^t (\tau)e^{-\tau} d\tau \right] = e^t \left[(1, 2, 3) + (-te^{-t} - e^{-t} + 1) \right] = (2e^t - t - 1, 3e^t - t - 1, 4e^t - t - 1).$$

This solution is shown in Figure 2.

Now, if we consider $T = h\mathbb{Z}$, then $[0, t]_T = \{0, h, 2h, \dots, nh\}$, where $n \in \mathbb{Z}$ and we have $e_a(t, 0) = (1 + ah)^{\frac{t}{h}}$ and $e_{\ominus a}(t, 0) = \frac{1}{(1+ah)^{\frac{t}{h}}}$. So, by Theorem ?? (1), we obtain

$$\begin{aligned} y_1(t) &= (1+h)^{\frac{t}{h}} \left((1, 2, 3) + \int_0^t \tau \frac{1}{(1+h)^{\frac{\tau+h}{h}}} \Delta\tau \right) = (1+h)^{\frac{t}{h}} \left((1, 2, 3) + \frac{h}{1+h} \sum_{\tau=0}^{t-1} \tau(1+h)^{-\frac{\tau}{h}} \right) \\ &= (1+h)^{\frac{t}{h}}(1, 2, 3) + (1+h)^{\frac{t}{h}} - t - 1. \end{aligned}$$

5 Conclusions

In this study, we provided solutions to fuzzy initial value problems for first order linear fuzzy dynamic equations on time scales under generalized Δ -Hukuhara differentiability. We studied fuzzy dynamic equations in three inequivalent

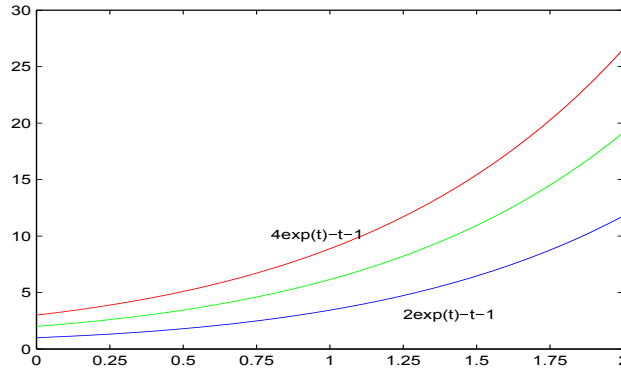


Figure 2: Δ_1 -differentiable solution of the Example 4.3 for $T = \mathbb{R}$.

forms and presented the general form of solutions. For further research, we propose the study of two-point boundary value problems for fuzzy dynamic equations on time scales. In addition, we propose to extend the results of the present paper to [?] to consider fuzzy dynamic equations in general cases.

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References

- [1] O. Abu Arqub, M. AL-Smadi, S. Momani, T. Hayat, *Numerical solutions of fuzzy differential equations using reproducing kernel Hilbert space method*, *Soft Computing*, **20** (2016), 3283–3302.
- [2] O. Abu Arqub, M. AL-Smadi, S. Momani, T. Hayat, *Application of reproducing kernel algorithm for solving second-order two-point fuzzy boundary value problems*, *Soft Computing*, **21** (2017), 7191–7206.
- [3] O. Abu Arqub, *Adaptation of reproducing kernel algorithm for solving fuzzy Fredholm-Volterra integrodifferential equations*, *Neural Computing & Applications*, **28** (2017), 1591–1610.
- [4] R.P. Agarwal, *Difference Equations and Inequalities: Theory, Methods, and Applications*, Marcel Dekker, New York, 1992.
- [5] S.E. Amrahov, A. Khastan, N. Gasilov, A.G. Fatullayev, *Relationship between Bede–Gal differentiable set-valued functions and their associated support functions*, *Fuzzy Sets and Systems*, **295** (2016), 57–71.
- [6] B. Bede, S.G. Gal, *Generalizations of the differentiability of fuzzy number value functions with applications to fuzzy differential equations*, *Fuzzy Sets and Systems*, **151** (2005), 581–599.
- [7] B. Bede, I. J. Rudas, A. L. Bencsik, *First order linear fuzzy differential equations under generalized differentiability*, *Information Sciences*, **177** (2007), 1648–1662.
- [8] B. Bede, *Mathematics of Fuzzy Sets and Fuzzy Logic*, Springer, London, 2013.
- [9] M. Bohner, A. Peterson, *Dynamic equations on time scales: An introduction with applications*, Springer Science & Business Media, 2012.
- [10] K. A. Chrysafis, B. K. Papadopoulos, G. Pappaschinopoulos, *On the fuzzy difference equations of finance*, *Fuzzy Sets and Systems*, **159** (2008), 3259–3270.
- [11] E. Deeba, A. Korvin, E. L. Koh, *A fuzzy difference equation with and Application*, *Journal of Difference Equations and applications*, **2** (1996), 365–374.

- [12] P. Diamond, P. Kloeden, *Metric Spaces of Fuzzy Sets*, World Scientific, Singapore, 1994.
- [13] O. S. Fard, T. A. Bidgoli, *Calculus of fuzzy functions on time scales (I)*, *Soft Computing*, **19** (2014), 293–305.
- [14] O. S. Fard, D. F. M. Torres, M. R. Zadeh, *A Hukuhara approach to study of hybrid fuzzy systems on time scales*, *Applicable Analysis and Discrete Mathematics*, **10** (2016), 152–167.
- [15] G. S. Guseinov, *Integration on time scales*, *Journal of Mathematical Analysis and Applications*, **285** (2003), 107–127.
- [16] S. Hilger, *Analysis on measure chains - a unified approach to continuous and discrete calculus*, *Results in Mathematics*, **18** (1990), 18–56.
- [17] S. Hong, *Differentiability of multivalued functions on time scales and applications to multivalued dynamic equations*, *Nonlinear Analysis: Theory, Methods & Applications*, **71** (2009), 3622–3637.
- [18] S. Hong, Y. Peng, *Almost periodicity of set-valued functions and set dynamic equations on time scales*, *Information Sciences*, **330** (2016), 157–174.
- [19] S. Hong, J. Gao, Y. Peng, *Solvability and stability of impulsive set dynamic equations on time scales*, *Abstract Appl. Anal.*, vol. 2014, Article ID 610365, 19 pages, 2014. doi:10.1155/2014/610365.
- [20] A. Khastan, J.J. Nieto, R. Rodríguez-López, *Variation of constant formula for first order fuzzy differential equations*, *Fuzzy Sets and Systems*, **177** (2011), 20–33.
- [21] A. Khastan, R. Rodríguez-López, *On the solutions to first order linear fuzzy differential equations*, *Fuzzy Sets and Systems*, **295** (2016), 114–135.
- [22] A. Khastan, *New solutions for first order linear fuzzy difference equations*, *Journal of Computational and Applied Mathematics*, **312** (2017), 156–166.
- [23] V. Lakshmikantham, A. S. Vatsala, *Basic Theory of Fuzzy Difference Equations*, *Journal of Fuzzy Difference Equations*, **8** (2002), 957–968.
- [24] W. T. Li, H. R. Sun, *Dynamics of rational difference equation*, *Applied Mathematics and Computation*, **163** (2005), 577–591.
- [25] V. Lupulescu, *Hukuhara differentiability of interval-valued functions and interval differential equations on time scales*, *Information Sciences*, **248** (2013), 50–67.
- [26] M. Najariyan, M. Mazandarani, V.E. Balas, *Solving first order linear fuzzy differential equations system*, In: Balas V., Jain L., Balas M. (eds) *Soft Computing Applications. SOFA 2016. Advances in Intelligent Systems and Computing*, vol 634. Springer, Cham, 2018.
- [27] J. J. Nieto, R. Rodríguez-López, D. Franco, *Linear first order fuzzy differential equations*, *International Journal of Uncertainty Fuzziness Knowledge-Based Syst*, **14** (2006), 687–709.
- [28] C. Vasavi, G. S. Kumar, M. S. N. Murty, *Generalized differentiability and integrability for fuzzy set-valued functions on time scales*, *Soft Computing*, **20** (2016), 1093–1104.
- [29] C. Vasavi, G. S. Kumar, M. S. N. Murty, *Fuzzy dynamic equations on time scales under second type Hukuhara delta derivative*, *International Journal of Chemical Science*, **14** (2016), 49–66.
- [30] C. Vasavi, G.S. Kumar, M. S. N. Murty, *Fuzzy Hukuhara delta differential and applications to fuzzy dynamic equations on time scales*, *Journal of Uncertain Systems*, **10** (2016), 163–180.
- [31] Q. Zhang, L. Yang, D. Liao, *On the first fuzzy Riccati difference equation*, *Information Sciences*, **270** (2014), 226–236.