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The Sugeno fuzzy integral of concave functions

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Abstract

The fuzzy integrals are a kind of fuzzy measures acting on fuzzy sets. They can be viewed as an average membership value of fuzzy sets. The value of the fuzzy integral in a decision making environment where uncertainty is present has been well established. Most of the integral inequalities studied in the fuzzy integration context normally consider conditions such as monotonicity or comonotonicity. In this paper, we are trying to extend the fuzzy integrals to the concept of concavity. It is shown that the Hermite-Hadamard integral inequality for concave functions is not satisfied in the case of fuzzy integrals. We propose upper and lower bounds on the fuzzy integral of concave functions. We present a geometric interpretation and some examples in the framework of the Lebesgue measure to illustrate the results.

Keywords: Sugeno fuzzy integral; Hermite-Hadamard inequality; Concave function; Supergradient.

1 Introduction

Aggregation is a process of combining several numerical values into a single one which exists in many disciplines, such as image processing [18, 30], pattern recognition [29] and decision making [19, 20]. To obtain a consensus quantifiable judgments, some synthesizing functions have been proposed. For example, arithmetic mean, geometric mean and median can be regarded as a basic class, because they are often used and very classic. However, these operators are not able to model an interaction between criteria. For having a representation of interaction phenomena between criteria, fuzzy measures have been proposed [31]. The properties and applications of the fuzzy measures and fuzzy integrals have been studied by many authors. Ralescu and Adams [22] studied the several equivalent definitions of fuzzy integrals. Román-Flores *et al.* [15], [24–27] studied the level-continuity of fuzzy integrals, H-continuity of fuzzy measures and geometric inequalities for fuzzy measures and integral, respectively. Wang and Klir [33] had a general overview on fuzzy measurement and fuzzy integration theory. Two main classes of the fuzzy integrals are Choquet and Sugeno integrals. Choquet and Sugeno integrals are idempotent, continuous and monotone operators. Liu *et al.* [18] proposed a color image encryption scheme based on Choquet fuzzy integral and hyper chaotic system. Chen *et al.* [13] proposed a fusion recognition scheme based on nonlinear decision fusion, using fuzzy integral to fuse the objective evidence supplied by each modality. Seyedzadeh *et al.* [28] presented a new RGB color image encryption using keystream generator based on fuzzy integral. Zhang and Zheng [35] generalized Ying's model of linguistic quantifiers based on Sugeno fuzzy integral to interval-valued intuitionistic Sugeno fuzzy integral and evaluated the truth value of a quantified proposition by using interval-valued intuitionistic Sugeno fuzzy integral. Nemmour and Chibani [21] proposed a new support vector mixture in which Sugeno fuzzy integral is used as a gater to remove the time complexity induced by conventional gaters such as artificial neural networks. Moreover, fuzzy measuring and fuzzy integral is newly used to evaluate the interaction of scale factors and to compare the energy performance of buildings in different scale factors [17]. Considering the interactions between the weights of attributes of building energy performance, a multiple attribute decision-making approach, fuzzy measure and fuzzy integral, is adopted to rank the evaluated buildings. Recently, many authors have studied the most well known integral inequalities for fuzzy integral. Abbaszadeh *et al.* [1–4, 32] studied Jensen and Hadamard integral

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inequalities for nonlinear integrals. Agahi *el al.* [5–7] proved general Minkowski type inequalities, general extensions of Chebyshev type inequalities and general Barnes-Godunova-Levin type inequalities for fuzzy integrals (see also [14]). Caballero and Sadarangani [8–12] proved Hermite-Hadamard type inequalities, Chebyshev type inequalities and Cauchy type inequalities for fuzzy integrals. Kaluszka *et al.* [16] gave the necessary and sufficient conditions guaranteeing the validity of Chebyshev type inequalities for the generalized Sugeno fuzzy integrals in the case of functions belonging to a much wider class than the comonotone functions. Wu *et al.* [34] proved two inequalities for the Sugeno fuzzy integral on abstract spaces generalizing all previous Chebyshev's inequalities. Most of the integral inequalities studied in the Sugeno integration context normally consider conditions such as monotonicity or comonotonicity. The main purpose of this paper is to estimate the Sugeno fuzzy integral of concave functions using the classical Hermite-Hadamard inequality. The paper is organized as follows. Some necessary preliminaries and summarization of previous known results are presented in Section 2. In Section 3, the upper and lower bounds of the Sugeno fuzzy integral for concave functions are established and some examples are given. In Section 4, a geometric interpretation is presented to illustrate the results. Finally, a conclusion is given in Section 5.

2 Preliminaries

In this section, we are going to review some well known results from the theory of non-additive integrals. Let X be a non-empty set and Σ be a σ-algebra of subsets of X.

Definition 2.1. (Rockafellar [23, p. 214]) *A function $f:[a,b]\longrightarrow\mathbb{R}$ has the supergradient m $(f'_-(x_0)\le m\le f'_+(x_0))$ at $x_0\in[a,b]$ if $f(x)\le f(x_0)+m(x-x_0)$ for all $x\in[a,b]$. This condition, which we refer to as the supergradient inequality, has a simple geometric meaning when f is finite at x_0: it says that the graph of the affine function $g(x)=f(x_0)+m(x-x_0)$ is a non-vertical supporting line to the concave set epif at the point $(x_0,f(x_0))$.*

The set of all supergradient of f at x_0 is called the superdifferential of f at x_0 and is denoted by $\partial f(x_0)$. Indeed,

$$\partial f(x_0)=\{m\in\mathbb{R}: f'_-(x_0)\le m\le f'_+(x_0)\}.$$

The following theorem shows that f is differentiable at x_0 if and only if $\partial f(x_0)$ is a singleton.

Theorem 2.2. (Rockafellar [23, p. 242]) *Let $f:(a,b)\longrightarrow\mathbb{R}$ be a concave function. Then f is differentiable at x_0 if and only if $f'(x_0)$ is the unique supergradient of f at x_0.*

Definition 2.3. (Ralescu [22]) *Suppose that $\mu:\Sigma\longrightarrow[0,\infty]$ is a set function. We say that μ is a fuzzy measure if it satisfies*

1. $\mu(\emptyset)=0$;
2. $E,F\in\Sigma$ *and* $E\subset F$ *imply* $\mu(E)\le\mu(F)$;
3. $\{E_n\}\subset\Sigma$, $E_1\subset E_2\subset\ldots$, *imply* $\lim_{n\to\infty}\mu(E_n)=\mu(\bigcup_{n=1}^{\infty}E_n)$ *(continuity from below);*
4. $\{E_n\}\subset\Sigma$, $E_1\supset E_2\supset\ldots$, $\mu(E_1)<\infty$, *imply* $\lim_{n\to\infty}\mu(E_n)=\mu(\bigcap_{n=1}^{\infty}E_n)$ *(continuity from above).*

The triple (X,Σ,μ) is called a fuzzy measure space.

If f is a non-negative real-valued function defined on X, we will denote by $L_\alpha f=\{x\in X: f(x)\ge\alpha\}=\{f\ge\alpha\}$ the α-level of f, for $\alpha>0$. Note that if $\alpha\le\beta$ then $L_\beta f\subset L_\alpha f$. Let (X,Σ,μ) be a fuzzy measure space. We denote by $\mathcal{F}^\mu(X)$ the set of all non-negative μ-measurable functions with respect to Σ.

Definition 2.4. (Sugeno [31]) *Let (X,Σ,μ) be a fuzzy measure space, $f\in\mathcal{F}^\mu(X)$ and $A\in\Sigma$, then the Sugeno fuzzy integral of f on A with respect to the fuzzy measure μ is defined by $⨍_A f\mathrm{d}\mu=\vee_{\alpha\ge0}\big(\alpha\wedge\mu(A\cap L_\alpha f)\big)$, where $\vee$ and $\wedge$ denote the operations sup and inf on $[0,\infty)$, respectively.*

It is well known that the Sugeno fuzzy integral is a type of nonlinear integral, i.e., for general cases

$$⨍_A(af+bg)\mathrm{d}\mu=a⨍_A f\mathrm{d}\mu+b⨍_A g\mathrm{d}\mu$$

does not hold.

The following properties of the Sugeno fuzzy integral are well known and can be found in [33].

Theorem 2.5. Let (X, Σ, μ) be a fuzzy measure space, $A, B \in \Sigma$ and $f, g \in \mathcal{F}^\mu(X)$ then

1. $⨍_A f \mathrm{d}\mu \leq \mu(A)$;
2. $⨍_A k \mathrm{d}\mu = k \wedge \mu(A)$, k non-negative constant;
3. If $f \leq g$ on A then $⨍_A f \mathrm{d}\mu \leq ⨍_A g \mathrm{d}\mu$;
4. If $A \subset B$ then $⨍_A f \mathrm{d}\mu \leq ⨍_B f \mathrm{d}\mu$;
5. $\mu(A \cap L_\alpha f) \geq \alpha \Rightarrow ⨍_A f \mathrm{d}\mu \geq \alpha$;
6. $\mu(A \cap L_\alpha f) \leq \alpha \Rightarrow ⨍_A f \mathrm{d}\mu \leq \alpha$.

Remark 2.6. *Let F be the distribution function associated to f on A, that is, $F(\alpha) = \mu(A \cap L_\alpha f)$. Then by conditions (5) and (6) of Theorem* 2.5*, $F(\alpha) = \alpha \Rightarrow ⨍_A f \mathrm{d}\mu = \alpha$. Thus, from a numerical point of view, the Sugeno fuzzy integral can be calculated by solving the equation $F(\alpha) = \alpha$. On the other hand, the existence of the Sugeno fuzzy integral of a function $f \in \mathcal{F}^\mu(X)$ does not imply the existence of a solution of $F(\alpha) = \alpha$ (see* [9, Example 1]*).*

The classical Hermite-Hadamard inequality provides estimates of the mean value of a concave function $f : [a, b] \longrightarrow \mathbb{R}$, $\frac{f(a)+f(b)}{2} \leq \frac{1}{b-a} \int_a^b f(x)dx \leq f(\frac{a+b}{2})$. The above inequalities should be reversed if f is convex. The following theorem may be regarded as a refinement of the right-hand side of Hermite-Hadamard inequality for convex functions in the fuzzy context.

Theorem 2.7. *(Caballero and Sadarangani [8]). Let $f : [a, b] \longrightarrow [0, \infty)$ be a convex function, Σ be the Borel field and μ be the Lebesgue measure on $\mathbb{R}$. Then*
(a) If $f(a) < f(b)$ then $⨍_a^b f \mathrm{d}\mu \leq \left(\frac{f(b)(b-a)}{f(b)-f(a)+b-a}\right) \wedge (b-a)$.
(b) If $f(a) = f(b)$ then $⨍_a^b f \mathrm{d}\mu \leq f(a) \wedge (b-a)$.
(c) If $f(a) > f(b)$ then $⨍_a^b f \mathrm{d}\mu \leq \left(\frac{f(a)(b-a)}{f(a)-f(b)+b-a}\right) \wedge (b-a)$.

3 The main results

Hereafter, we will assume that (X, Σ, μ) is a general fuzzy measure space. To simplify the calculation of the fuzzy integral, for a given $f \in \mathcal{F}^\mu(X)$ and $A \in \Sigma$, we write $\Gamma = \{\alpha \mid \alpha \geq 0, \mu(A \cap L_\alpha f) > \mu(A \cap L_\beta f) \text{ for any } \beta > \alpha\}$. It is easy to see that $⨍_A f \mathrm{d}\mu = \bigvee_{\alpha \in \Gamma} \left(\alpha \wedge \mu(A \cap L_\alpha f)\right)$.

The following examples show that the classical Hermite-Hadamard inequality is not valid in the fuzzy context.

Example 3.1. Let $X = [0, 1]$ and let μ be the usual Lebesgue measure on X. We take the non-negative and concave function $f(x) = \sqrt{x/3}$. To calculate the Sugeno fuzzy integral $⨍_0^1 \sqrt{x/3}\mathrm{d}\mu$, by Remark 2.6, consider the distribution function F associated to f on $[0, 1]$, this is

$$F(\alpha) = \mu\big([0, 1] \cap L_\alpha f\big) = \mu\left([0, 1] \cap \left\{\sqrt{x/3} \geq \alpha\right\}\right) = \mu\left([0, 1] \cap \left\{x \geq 3\alpha^2\right\}\right) = 1 - 3\alpha^2.$$

We solve the equation $1-3\alpha^2 = \alpha$ and get $\alpha = \frac{-1+\sqrt{13}}{6} \approx 0,434$. By Remark 2.6, we have $⨍_0^1 f\mathrm{d}\mu = ⨍_0^1 \sqrt{x/3}\mathrm{d}\mu = 0,434$. On the other hand, $f(1/2) = 1/\sqrt{6} \approx 0,408$. This proves that the right part of Hermite-Hadamard inequality is not satisfied in the fuzzy context.

Example 3.2. Let $X = [0, 1]$ and let μ be the usual Lebesgue measure on X. Then for the concave function $f(x) = 2\sqrt{x}$ and using a similar argument that in Example 3.1, we can get $⨍_0^1 f\mathrm{d}\mu = ⨍_0^1 2\sqrt{x}\mathrm{d}\mu = -2 + 2\sqrt{2} \approx 0,828$. On the other hand, $\frac{f(0)+f(1)}{2} = 1$ and this proves that the left-hand side of Hermite-Hadamard inequality is not satisfied for Sugeno fuzzy integrals.

Now, we prove the following result concerning the Sugeno fuzzy integral of concave functions. Indeed, we are dealing with the left-hand side of Hermite-Hadamard inequality for the Sugeno fuzzy integral of concave functions.

Theorem 3.3. *Let $f:[a,b]\longrightarrow[0,\infty)$ be a concave function with $f(a)\neq f(b)$. If α is a solution of the equation*

$$\mu\left([a,b]\cap\left\{\left(1-\frac{x-a}{b-a}\right)f(a)+\frac{x-a}{b-a}f(b)\geq\alpha\right\}\right)=\alpha$$

with respect to the fuzzy measure μ, where $\alpha\in\big[f(a),f(b)\big)$ for $f(b)>f(a)$ and $\alpha\in\big(f(b),f(a)\big]$ for $f(a)>f(b)$, then

$$\alpha\leq\fint_a^b f\mathrm{d}\mu.$$

Proof. Using the concavity of f, for $x\in[a,b]$ we have

$$f(x)=f\left(\left(1-\frac{x-a}{b-a}\right)a+\frac{x-a}{b-a}b\right)\geq\left(1-\frac{x-a}{b-a}\right)f(a)+\frac{x-a}{b-a}f(b)=g(x)$$

and by (3) of Theorem 2.5, we get $\fint_a^b f\mathrm{d}\mu\geq\fint_a^b\left(\left(1-\frac{x-a}{b-a}\right)f(a)+\frac{x-a}{b-a}f(b)\right)\mathrm{d}\mu=\fint_a^b g\mathrm{d}\mu$. For calculating the integral in the right-hand part of the last inequality, we consider the distribution function G given by $G(\alpha)=\mu\big([a,b]\cap L_\alpha g\big)$. If $f(b)>f(a)$, then

$$G(\alpha)=\mu\left([a,b]\cap\left\{\left(1-\frac{x-a}{b-a}\right)f(a)+\frac{x-a}{b-a}f(b)\geq\alpha\right\}\right)=\mu\left([a,b]\cap\left\{x\geq\frac{\alpha(b-a)+af(b)-bf(a)}{f(b)-f(a)}\right\}\right).$$

We have $L_\alpha g=\left\{x\geq\frac{\alpha(b-a)+af(b)-bf(a)}{f(b)-f(a)}\right\}=\left[\frac{\alpha(b-a)+af(b)-bf(a)}{f(b)-f(a)},b\right]$. Thus, $\Gamma=\big[f(a),f(b)\big)$ and we only need to consider $\alpha\in\big[f(a),f(b)\big)$. If $f(a)>f(b)$, then

$$G(\alpha)=\mu\left([a,b]\cap\left\{\left(1-\frac{x-a}{b-a}\right)f(a)+\frac{x-a}{b-a}f(b)\geq\alpha\right\}\right)=\mu\left([a,b]\cap\left\{x\leq\frac{\alpha(b-a)+af(b)-bf(a)}{f(b)-f(a)}\right\}\right).$$

We have $L_\alpha g=\left\{x\leq\frac{\alpha(b-a)+af(b)-bf(a)}{f(b)-f(a)}\right\}=\left[a,\frac{\alpha(b-a)+af(b)-bf(a)}{f(b)-f(a)}\right]$. Thus, $\Gamma=\big(f(b),f(a)\big]$ and we only need to consider $\alpha\in\big(f(b),f(a)\big]$. Taking into account (1) of Theorem 2.5 and Remark 2.6, we get $\fint_a^b g\mathrm{d}\mu=\alpha\leq\fint_a^b f\mathrm{d}\mu$,where α is the solution of the equation $G(\alpha)=\alpha$ with respect to the fuzzy measure μ. □

Remark 3.4. In the case $f(a)=f(b)$, we have $g(x)=f(a)$ and using (2) and (3) of Theorem 2.5, we get

$$\fint_a^b f\mathrm{d}\mu\geq\fint_a^b g\mathrm{d}\mu=\fint_a^b f(a)\mathrm{d}\mu=f(a)\wedge\mu([a,b]).$$

Corollary 3.5. Let $f:[a,b]\longrightarrow[0,\infty)$ be a concave function, Σ be the Borel field and μ be the Lebesgue measure on $\mathbb{R}$. Then

(*a*) If $f(a)<f(b)$, then $\frac{f(b)(b-a)}{f(b)-f(a)+b-a}\leq\fint_a^b f\mathrm{d}\mu$;
(*b*) If $f(a)=f(b)$, then $f(a)\wedge(b-a)\leq\fint_a^b f\mathrm{d}\mu$;
(*c*) If $f(a)>f(b)$, then $\frac{f(a)(b-a)}{f(a)-f(b)+b-a}\leq\fint_a^b f\mathrm{d}\mu$.

In the next theorem, we establish an upper bound on the Sugeno fuzzy integral of the concave function f defined on the interval $[a,b]$. Indeed, we are dealing with the right-hand side of Hermite-Hadamard inequality for the Sugeno fuzzy integral of concave functions.

Theorem 3.6. *Suppose that $f:[a,b]\longrightarrow[0,\infty)$ is a concave function and $0\neq m\in\partial f\left(\frac{a+b}{2}\right)$. If α is a solution of the equation*

$$\mu\left([a,b]\cap\left\{f\left(\frac{a+b}{2}\right)+m\left(x-\frac{a+b}{2}\right)\geq\alpha\right\}\right)=\alpha$$

with respect to the fuzzy measure μ, where

$$\begin{cases}\alpha\in\left[-\frac{b-a}{2}m+f\left(\frac{a+b}{2}\right),\frac{b-a}{2}m+f\left(\frac{a+b}{2}\right)\right), & f(b)>f(a);\\[2mm] \alpha\in\left(\frac{b-a}{2}m+f\left(\frac{a+b}{2}\right),-\frac{b-a}{2}m+f\left(\frac{a+b}{2}\right)\right], & f(a)>f(b),\end{cases}$$

then $\fint_a^b f\mathrm{d}\mu\leq\alpha$.

Proof. For $0 \neq m \in \partial f\left(\frac{a+b}{2}\right)$, define the function $g : [a,b] \longrightarrow [0,\infty)$ as $g(x) = f(\frac{a+b}{2}) + m(x - \frac{a+b}{2})$ supporting f at $x_0 = \frac{a+b}{2}$. Utilizing (3) of Theorem 2.5, we get

$$\fint_a^b f\mathrm{d}\mu \leq \fint_a^b g\mathrm{d}\mu = \fint_a^b \left(f\left(\frac{a+b}{2}\right) + m\left(x - \frac{a+b}{2}\right)\right)\mathrm{d}\mu.$$

Let $m > 0$. We consider the distribution function G defined by

$$G(\alpha) = \mu\big([a,b] \cap L_\alpha g\big) = \mu\left([a,b] \cap \left\{f\left(\frac{a+b}{2}\right) + m\left(x - \frac{a+b}{2}\right) \geq \alpha\right\}\right) = \mu\left([a,b] \cap \left\{x \geq \frac{a+b}{2} + \frac{\alpha}{m} - \frac{1}{m}f\left(\frac{a+b}{2}\right)\right\}\right).$$

We have

$$L_\alpha g = \left\{x \geq \frac{a+b}{2} + \frac{\alpha}{m} - \frac{1}{m}f\left(\frac{a+b}{2}\right)\right\} = \left[\frac{a+b}{2} + \frac{\alpha}{m} - \frac{1}{m}f\left(\frac{a+b}{2}\right), b\right].$$

Thus, $\Gamma = \left[-\frac{b-a}{2}m + f\left(\frac{a+b}{2}\right), \frac{b-a}{2}m + f\left(\frac{a+b}{2}\right)\right)$, and we only need to consider $\alpha \in \left[-\frac{b-a}{2}m + f\left(\frac{a+b}{2}\right), \frac{b-a}{2}m + f\left(\frac{a+b}{2}\right)\right)$.

For $m < 0$, the function G is defined by

$$G(\alpha) = \mu\big([a,b] \cap L_\alpha g\big) = \mu\left([a,b] \cap \left\{f\left(\frac{a+b}{2}\right) + m\left(x - \frac{a+b}{2}\right) \geq \alpha\right\}\right) = \mu\left([a,b] \cap \left\{x \leq \frac{a+b}{2} + \frac{\alpha}{m} - \frac{1}{m}f\left(\frac{a+b}{2}\right)\right\}\right).$$

We have

$$L_\alpha g = \left\{x \leq \frac{a+b}{2} + \frac{\alpha}{m} - \frac{1}{m}f\left(\frac{a+b}{2}\right)\right\} = \left[a, \frac{a+b}{2} + \frac{\alpha}{m} - \frac{1}{m}f\left(\frac{a+b}{2}\right)\right].$$

Thus, $\Gamma = \left(\frac{b-a}{2}m + f\left(\frac{a+b}{2}\right), -\frac{b-a}{2}m + f\left(\frac{a+b}{2}\right)\right]$, and we only need to consider $\alpha \in \left(\frac{b-a}{2}m + f\left(\frac{a+b}{2}\right), -\frac{b-a}{2}m + f\left(\frac{a+b}{2}\right)\right]$. We can apply (1) of Theorem 2.5 and Remark 2.6 to get $\fint_a^b f\mathrm{d}\mu \leq \fint_a^b g\mathrm{d}\mu = \alpha$, where α is a solution of the equation $G(\alpha) = \alpha$ with respect to the fuzzy measure μ. $\square$

Remark 3.7. *In the case $m = 0$ we have $g(x) = f\left(\frac{a+b}{2}\right)$. By (2) and (3) of Theorem 2.5, we conclude that*

$$\fint_a^b f\mathrm{d}\mu \leq \fint_a^b g\mathrm{d}\mu = \fint_a^b f\left(\frac{a+b}{2}\right)\mathrm{d}\mu = f\left(\frac{a+b}{2}\right) \wedge \mu\big([a,b]\big).$$

Corollary 3.8. *Suppose that $f : [a,b] \longrightarrow [0,\infty)$ is a concave function, $m \in \partial f\left(\frac{a+b}{2}\right)$, Σ is the Borel field and μ is the Lebesgue measure on $\mathbb{R}$. Then*

$$\fint_a^b f\mathrm{d}\mu \leq \begin{cases} \frac{m}{2(m+1)}(b-a) + \frac{1}{m+1}f(\frac{a+b}{2}), & m > 0; \\ \frac{a+b}{2} \wedge (b-a), & m = 0; \\ \frac{m}{2(m-1)}(b-a) - \frac{1}{m-1}f(\frac{a+b}{2}), & m < 0, \end{cases}$$

Example 3.9. Let μ be the Lebesgue measure on $\mathbb{R}$. Let $X = [0,2]$ and consider the function $f(x) = x + \arctan(x)$. This function is, obviously, concave and non-negative on $[0,2]$. Using Theorem 2.2, we have $m = f^{'}(1) = 3/2 > 0$. By Corollary 3.8, we obtain

$$\fint_0^2 \big(x + \arctan(x)\big)\mathrm{d}\mu \leq \frac{3/2}{2(3/2+1)}(2-0) + \frac{1}{3/2+1}\left(1 + \frac{\pi}{4}\right) \approx 1,315.$$

On the other hand, as $f(0) = 0$, $f(2) = 2 + \arctan(2)$ and $f(2) > f(0)$, by (a) of Corollary 3.5 we can get the following estimate

$$\fint_0^2 \big(x + \arctan(x)\big)\mathrm{d}\mu \geq \frac{f(2)(2-0)}{f(2) - f(0) + 2 - 0} \approx 1,216.$$

By the above inequalities, we conclude that the Sugeno fuzzy integral of f on $[0,2]$ is approximated by $1,216 \leq \fint_1^3 \big(x + \arctan(x)\big)\mathrm{d}\mu \leq 1,315$.

Example 3.10. Let μ be the Lebesgue measure on $\mathbb{R}$. Consider the function $f(x) = \ln(x+2) - x$ on $X = [-1, 1]$. This function is, obviously, non-negative and concave on $[-1, 1]$. Using Theorem 2.2, we have $m = f^{'}(0) = \frac{-1}{2} < 0$. By Corollary 3.8, we obtain

$$-\!\!\!\!\!\!\int_{-1}^{1} \big(\ln(x+2) - x\big) \mathrm{d}\mu \leq \frac{-1/2}{2(-1/2-1)}(1-(-1)) - \frac{1}{-1/2-1}\ln(2) \approx 0,796.$$

On the other hand, as $f(-1) = 1$, $f(1) = \ln(3) - 1$ and $f(-1) > f(1)$, by (c) of Corollary 3.5 we deduce

$$-\!\!\!\!\!\!\int_{-1}^{1} \big(\ln(x+2) - x\big) \mathrm{d}\mu \geq \frac{f(-1)(1-(-1))}{f(-1) - f(1) + 1 - (-1)} \approx 0,689.$$

Thus, we may approximate the Sugeno fuzzy integral of f on $[-1, 1]$ by $0,689 \leq -\!\!\!\!\!\int_{-1}^{1} \big(\ln(x+2) - x\big) \mathrm{d}\mu \leq 0,796$.

Example 3.11. Let μ be the Lebesgue measure on $\mathbb{R}$. Consider the function $f(x) = x - x\ln(x)$ on the interval $X = [1/3, 5/3]$. Obviously this function is non-negative and concave on $[1/3, 5/3]$. Using Theorem 2.2, we have $m = f^{'}(1) = 0$. By Corollary 3.8, we obtain

$$-\!\!\!\!\!\!\int_{1/3}^{5/3} \big(x - x\ln(x)\big) \mathrm{d}\mu \leq f(1) \wedge (5/3 - 1/3) = 1.$$

On the other hand, as $f(1/3) = 1/3 - 1/3\ln(1/3)$, $f(5/3) = 5/3 - 5/3\ln(5/3)$ and $f(5/3) > f(1/3)$, by (a) of Corollary 3.5 we conclude that

$$-\!\!\!\!\!\!\int_{1/3}^{5/3} \big(x - x\ln(x)\big) \mathrm{d}\mu \geq \frac{f(5/3)(5/3 - 1/3)}{f(5/3) - f(1/3) + 5/3 - 1/3} \approx 0,750.$$

Therefore, we have the following estimate $0,750 \leq -\!\!\!\!\!\int_{1/3}^{5/3} \big(x - x\ln(x)\big) \mathrm{d}\mu \leq 1$.

4 Geometric interpretation

Let $X = \mathbb{R}$, Σ be the Borel field, μ the Lebesgue measure and $f : X \longrightarrow [0, \infty)$ be a continuous function. Then the geometric significance of $-\!\!\!\!\!\int_A f \mathrm{d}\mu$ is the edges length of the largest square between the curve of $f(x)$ and the x-axis. Now we consider the concave function $f(x) = \sqrt[3]{x} + 1$ on the interval $[0, 2]$. According to Theorems 3.3 and 3.6, respectively, we define the functions $g_1, g_2 : [0, 2] \longrightarrow [0, \infty)$ by $g_1(x) = \sqrt[3]{2}x/2 + 1$, $\quad g_2(x) = (x-1)/2 + 2$. We have $g_1(x) \leq f(x) \leq g_2(x)$ and by (3) of Theorem 2.5,

$$-\!\!\!\!\!\!\int_{0}^{2} \Big(\sqrt[3]{2}x/2 + 1\Big) \mathrm{d}\mu \leq -\!\!\!\!\!\!\int_{0}^{2} \big(\sqrt[3]{x} + 1\big) \mathrm{d}\mu \leq -\!\!\!\!\!\!\int_{0}^{2} \big((x-1)/2 + 2\big) \mathrm{d}\mu. \tag{1}$$

Geometric interpretation of (1) is shown in Figure 1. The lengths of the lines 1 , 2 and 3 are the solutions of the integrals in left, middle and right hand side of (1), respectively.

5 Conclusions

The importance of fuzzy integrals toward applications is hidden in their capability to express the possible interaction among single parts of a universe X for a global representation of a function (describing the real acting of an observed system) by means of a single value. This phenomenon cannot be captured by the standard Lebesgue integral, though it always plays a prominent role also in our discussed framework. In this paper, the fuzzy integrals are extended to the concept of concavity. The classical Hermite-Hadamard integral inequality is the first fundamental result for concave functions defined on an interval of real numbers with a natural geometrical interpretation and a loose number of applications for particular inequalities. We proved the Hermite-Hadamard integral inequality for concave functions in the case of Sugeno fuzzy integrals. There are numerous applications of fuzzy integral, and thus the study of Hermite-Hadamard and similar inequalities for the fuzzy integral is an important and interesting topic for the further research.

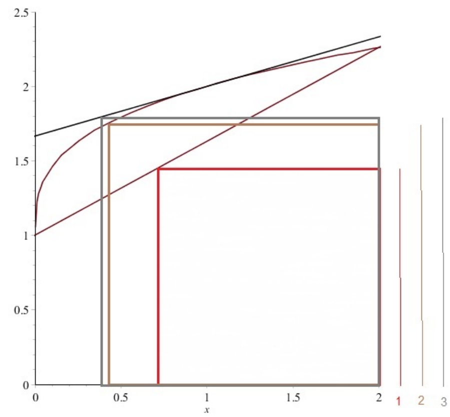

Figure 1: Geometric interpretation of the Sugeno fuzzy integral of a concave function.

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