The role of states in triangle algebras

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Abstract

In this paper, we enlarge the language of triangle algebra by adding a unary operation that describes properties of a state. These structure algebras are called state triangle algebra. The vital properties of these algebras are given. The notion of state interval-valued residuated lattice (IVRL)-filters are introduced and give some examples and properties of them. Using this concept, we define two types IVRL-extended σ-filters of a state triangle algebra.

Keywords: Triangle algebra, IVRL-filter, State operator, State IVRL-filter.

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1 Introduction

It is well known that certain information processing is based on the classical two-valued logic. Naturally, it is necessary to establish some rational logic systems as the logical foundation for uncertain information processing. For this reason, various kinds of non-classical logic systems have been extensively proposed and researched. To formalize the many-valued logics induced by continuous t-norms on the real unit interval [0, 1], in 1998. Various logical algebras have been proposed as the semantical systems of non-classical logic systems. Among these logical algebras, residuated lattices are important algebraic structures. The concept of a residuated lattice was firstly introduced by M. Ward and R.P. Dilworth as generalization of ideal lattices of rings. The lattice of filters of a residuated lattice was investigated by [10]. Van Gasse et al. introduced the notion of triangle algebras as a variety of residuated lattices equipped with unary operators ν and μ together with a third angular point u which is different from 0 and 1. They showed that these algebras serve as an equational representation of interval-valued residuated lattices (IVRLs). The authors defined triangle logic (TL) and showed that this logic is sound and complete with respect to the variety of triangle algebras [13]. The theory of triangle algebras has been enriched with filter theory. The same authors introduced the notion of IVRL-filters in triangle algebras and defined Boolean and prime IVRL-filters and revealed interesting properties of them [12]. Triangle algebras are different from the other algebraic structures, so triangle algebras play an important role in studying fuzzy logics and the related algebraic structures.

Semi-divisible residuated lattices are related to probability theory in the following way. In 1986 Mundici extended probability theory on MV-algebras by defining states on these algebras and investigated the advantages of such approach in quantum logic framework [6]. In, 2006 Mundici showed that his approach fits well to De Finetti’s subjective probabilities, too. Next, the notion of state was extended to even more general residuated structures and references thereon. A new approach to states on MV-algebras was presented by Flaminio and Montagna, they added a unary operation σ to the language of MV-algebras, which preserves the usual properties of states. The notion of states has been extended to other logical algebras such as BL-algebras, MTL-algebras and etc [3, 9]. Ciungu et al. defined a state operator and a strong state operator for a BL-algebra. They introduced the concept of state filters on a state BL-algebra and gave some properties of them [2].
We are going to study more properties of triangle algebras. To this scope, firstly we define the concept of state triangle algebras and prove some theorems that determine some properties of them. Since the filter theory plays an important role in studying these algebras, we introduce the notion of state IVRL-filters. These state IVRL-filters have different functions. Because of the existence of two operators \( \vee \) and \( \land \) and using them in the definition of state IVRL-filters, we introduce the notion of state IVRL-ideal. These state IVRL-ideals have an important role in studying these algebras. In this section, we introduce the state IVRL-\( \sigma \)-filters and some type of them as IVRL-extended (maximal, prime) \( \sigma \)-filter and study them in details. Also, we discuss the relationship between these state IVRL-filters.

2 Preliminaries

In this section, we summarize some definitions and results about residuated lattices and triangle algebras, which will be used in the following sections. Firstly, we recall the definition of bounded commutative residuated latticed.

**Definition 2.1.** \[\text{A bounded commutative residuated latticed is an algebra } L = (L, \vee, \land, *, \rightarrow, 0, 1) \text{ with four binary operations and two constants } 0, 1 \text{ such that:}
\]

- \((L, \vee, \land, 0, 1)\) is a bounded lattice,
- operation * is commutative and associative, with 1 as neutral element, and
- \(x \ast y \leq z\) iff \(x \leq y \rightarrow z\), for all \(x, y, z \in L\).

The ordering \(\leq\) and negation \(\neg\) in a residuated lattice \(L = (L, \vee, \land, *, \rightarrow, 0, 1)\) are defined as follows, for all \(x, y \in L\):

1. \(x \land y \leq (x \rightarrow y) \rightarrow y\) (in particular \(x \leq \neg \neg x\)),
2. \(x \rightarrow y \leq x \ast z \rightarrow y \ast z\),
3. \((x \rightarrow y) \ast (y \rightarrow z) \leq (x \rightarrow z)\),
4. If \(x \leq y\), then \(x \ast z \leq y \ast z\), \(z \rightarrow x \leq z \rightarrow y \) and \(y \rightarrow z \leq x \rightarrow z\),
5. \(x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z) = (x \ast y) \rightarrow z\),
6. \(x \rightarrow y \leq (y \rightarrow z) \rightarrow (x \rightarrow z)\).

**Definition 2.2.** \[\text{Let } L = (L, \vee, \land, *, \rightarrow, 0, 1) \text{ be a residuated lattice. Then the following properties are valid, for all } x, y, z \in L:\n\]

1. \(x \vee y \leq (x \rightarrow y) \rightarrow y\) (in particular \(x \leq \neg \neg x\)),
2. \(x \rightarrow y \leq x \ast z \rightarrow y \ast z\),
3. \((x \rightarrow y) \ast (y \rightarrow z) \leq (x \rightarrow z)\),
4. If \(x \leq y\), then \(x \ast z \leq y \ast z\), \(z \rightarrow x \leq z \rightarrow y \) and \(y \rightarrow z \leq x \rightarrow z\),
5. \(x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z) = (x \ast y) \rightarrow z\),
6. \(x \rightarrow y \leq (y \rightarrow z) \rightarrow (x \rightarrow z)\).

**Definition 2.3.** \[\text{Given a lattice } A = (A, \vee, \land), \text{ its triangularization } T(A) \text{ is the structure } T(A) = (\text{Int}(A), \vee, \land) \text{ defined by}\n\]

- \(\text{Int}(A) = \{(x_1, x_2) \mid (x_1, x_2) \in A^2 \text{ and } x_1 \leq x_2\}\),
- \([x_1, x_2] \land [y_1, y_2] = [x_1 \land y_1, x_2 \land y_2]\),
- \([x_1, x_2] \lor [y_1, y_2] = [x_1 \lor y_1, x_2 \lor y_2]\).

The set \(D_A = \{[x, x] : x \in L\}\) is called the diagonal of \(T(A)\).

**Definition 2.4.** \[\text{An interval-valued residuated lattice (IVRL) is a residuated lattice } (\text{Int}(A), \vee, \land,\n\]

\(\otimes, \rightarrow, [0, 0], [1, 1]) \text{ on the triangularization } T(A) \text{ of a bounded lattice } A, \text{ in which the diagonal } D_A \text{ is closed under } \otimes \text{ and } \rightarrow, \text{ i.e. } [x, x] \otimes [y, y] \in D_A \text{ and } [x, x] \rightarrow [y, y] \in D_A, \text{ for all } x, y \in L.\)

**Definition 2.5.** \[A \text{ triangle algebra is a structure } A = (A, \vee, \land, *, \nu, 0, 1) \text{ in which } (A, \vee, \land, *, \rightarrow, 0, 1) \text{ is a} \]
residuated lattice, \(v\) and \(\mu\) are unary operations on \(A\), \(u\) a constant, and satisfying the following conditions:

\[(T.1) \nu x \leq x,\]
\[(T.2) \nu x \leq \nu x,\]
\[(T.3) \nu(x \land y) = \nu x \land \nu y,\]
\[(T.4) \nu(x \lor y) = \nu x \lor \nu y,\]
\[(T.5) \nu u = 0,\]
\[(T.6) \nu \mu x = \mu x,\]
\[(T.7) \nu(x \to y) \leq \nu x \to \nu y,\]
\[(T.8)(\nu x \leftrightarrow \nu y) \land (\mu x \leftrightarrow \mu y) \leq (x \leftrightarrow y),\]
\[(T.9) \nu x \to \nu y \leq \nu(\nu x \to \nu y).\]

From now on \(A = (A, \lor, \land, \to, *, \nu, \mu, 0, u, 1)\) or simply \(A\) is a triangle algebra unless otherwise specified.

In triangle algebra \(A\), operator \(\nu\) (necessity) and \(\mu\) (possibility) are modal operators, and \(u\) (uncertainty, \(u \neq 0, u \neq 1\)) is a new constant. It turns out that triangle algebras are the equational representations of interval-valued residuated lattices (IVRLs).

**Theorem 2.6.** There is a one-to-one correspondence between the class of IVRLs and the class of triangle algebras. Every extended IVRL is a triangle algebra and conversely, every triangle algebra is isomorphic to an extended IVRL.

**Proposition 2.7.** Suppose \((A, \lor, \land, \to, 0, 1)\) is a residuated lattice such that \(\neg\) is involutive. If there exists an element \(u\) in \(A\) such that \(\neg u = u\), if \(\nu\) is a unary operator on \(A\) that satisfies \(T.1\) to \(T.6\), \(T.8\), \(T.9\) and if \((\nu x \leftrightarrow \nu y) \land (\mu x \leftrightarrow \mu y) \leq (x \leftrightarrow y)\), then \((A, \lor, \land, \to, *, \nu, \mu, 0, u, 1)\) is a triangle algebra if we define \(\mu x = \neg \nu x\).

**Definition 2.8.** A triangle algebra \(A\) is called an MTL-triangle algebra if \((x \to y) \lor (y \to x) = 1\). An MTL-triangle algebra \(A\) is called a BL-triangle algebra if \(x \land y = x \ast (x \to y)\), for all \(x, y \in A\).

**Definition 2.9.** A triangle algebra \(A\) is called RL-triangle algebra if \(x \land y = x \ast (x \to y)\), for all \(x, y \in A\).

**Definition 2.10.** An IVRL-filter of \(A\) is a non-empty subset \(F\) of \(A\) satisfying:

\((F.1)\) if \(x \in F, y \in A\) and \(x \leq y\), then \(y \in F\),
\((F.2)\) if \(x, y \in F\), then \(x \ast y \in F\),
\((F.3)\) if \(x \in F\), then \(\nu x \in F\).

An alternative definition for an IVRL-filter \(F\) of a triangle algebra \(A = (A, \lor, \land, *, \to, \nu, \mu, 0, u, 1)\) is the following:

* \(1 \in F\),
* for all \(x\) and \(y\) in \(A\): if \(x \in F\) and \(x \to y \in F\), then \(y \in F\),
* if \(x \in F\), then \(\nu x \in F\).

For all \(x, y \in A\), we write \(x \equiv_F y\) iff \(x \to y \in F\) and \(y \to x \in F\) are both in \(F\).

\(\equiv_F\) is always a congruence relation \(\equiv_F\). Note that \((F.3)\) is a necessary condition for this statement. Indeed, if \(\equiv_F\) is a congruence relation on a triangle algebra \(A = (A, \lor, \land, *, \to, \nu, \mu, 0, u, 1)\) and \(x \in F\), then \(x \equiv_F 1\) and therefore \(\nu x \equiv_F \nu 1 = 1\), which is equivalent with \(\nu x \in F\).

**Proposition 2.11.** In a triangle algebra \((A, \lor, \land, *, \nu, \mu, 0, u, 1)\), the implication \(\to\) and the product \(*\) are completely determined by their action on \(E(A)\) and the value of \(u \ast u\), where \(E(A) = \{x \in A \mid \nu x = x\}\). More specifically:

* \(\nu(x \to y) = (\nu x \to \nu y) \land (\mu x \to \mu y)\),
* \(\mu(x \to y) = (\mu x \to (\mu(u \ast u) \to \mu y)) \land (\nu x \to \mu y)\),
* \(\nu(x \ast y) = \nu x \ast \nu y\),
* \(\mu(x \ast y) = (\nu x \ast \nu y) \lor (\mu x \ast \nu y) \lor (\mu x \ast \mu y) \lor (\mu(u \ast u))\).

**Definition 2.12.** The order of \(x \in A\), denoted by \(\text{ord}(x)\), is the smallest \(n \in \mathbb{N}\) such that \(x^n = 0\). If there is no such \(n\), then \(\text{ord}(x) = \infty\).

**Theorem 2.13.** Let \(A\) be a triangle algebra. \(M\) is an IVRL-extended maximal filter of \(A\) iff for all \(x \in A, x \notin M\), there exist \(m \in M\), \(n \geq 1\) such that \((m \ast \nu x^n)^k = 0\).
Theorem 2.14. Let $A$ be an MTL-triangle algebra. Then
\[
\text{Rad}(A) = \{a \in A : va \geq (va^n) \text{ for any } n \in \mathbb{N}\}.
\]

Theorem 2.15. Let $F$ be an IVRL-filter of MTL-triangle algebra $A$. Then
\[
\text{Rad}(F) = \{a \in A : \neg(va^n) \rightarrow va \in F, \text{ for all } n \in \mathbb{N}\}
\]

Definition 2.16. A triangle algebra $A$ is said to be local iff has exactly one IVRL-extended maximal filter.

Definition 2.17. Let $B(A)$ be the set of all complemented elements of the triangle algebra $A$ (recall that an element $a \in A$ is called complemented if there is an element $b \in A$ such that $a \lor b = 1$ and $a \land b = 0$; if such an element $b$ exists, it is called a complement of $a$).

3 State triangle algebras

In this section, we enlarge the language of triangle algebras by introducing a new operator, an internal state and study some related properties of such operators. The special set $\text{Ker}(\sigma)$ has been introduced and discuss relations between this set and $B(A)$. Finally, we investigate the effect of $\sigma$ on operators $\lor, \land, \ast$ and $\Rightarrow$.

Definition 3.1. Let $A$ be a triangle algebra. A mapping $\sigma : A \rightarrow A$ such that, for all $x, y \in A$, we have

\begin{align*}
(ST_1) \quad & \sigma(0) = 0, \\
(ST_2) \quad & \sigma(x \rightarrow y) = \sigma(x) \rightarrow \sigma(x \land y), \\
(ST_3) \quad & \sigma(x \ast y) = \sigma(x) \ast \sigma(x \rightarrow (x \ast y)), \\
(ST_4) \quad & \sigma(\sigma(x) \ast \sigma(y)) = \sigma(x) \ast \sigma(y), \\
(ST_5) \quad & \sigma(\sigma(x) \rightarrow \sigma(y)) = \sigma(x) \rightarrow \sigma(y), \\
(ST_6) \quad & \sigma(x \land \sigma(y)) = \sigma(x) \land \sigma(y), \\
(ST_7) \quad & \sigma(\sigma(x) \lor \sigma(y)) = \sigma(x) \lor \sigma(y), \\
(ST_8) \quad & \sigma(\nu x) = \nu(\sigma(x)), \\
(ST_9) \quad & \sigma(\mu x) = \mu(\sigma(x)).
\end{align*}

is said to be a state operator on $A$ and the pair $(A, \sigma)$ is said to be a state triangle algebra.

If $\sigma$ is a state operator, then $\text{Ker}(\sigma) := \{x \in A \mid \sigma(\nu x) = 1\}$ is said to be the kernel of $\sigma$. A state operator $\sigma$ is said to be faithful if $\text{Ker}(\sigma) = \{1\}$.

Example 3.2. (i) Let $A$ be a triangle algebra. Then $(A, id_A)$ is a state triangle algebra.

(ii) Let $A = \{[0, 0], [0, a], [0, b], [a, a], [a, b], [b, b], [0, 1], [a, 1], [b, 1], [1, 1]\}$. Define $\lor$ and $\Rightarrow$ as follows:

\[
\begin{array}{cccc|cccc}
\lor & 0 & a & b & 1 \\
\hline
0 & 0 & 0 & 0 & 0 \\
a & 0 & 0 & a & a \\
b & 0 & a & b & b \\
1 & 0 & a & b & 1
\end{array}
\Rightarrow
\begin{array}{cccc|cccc}
\Rightarrow & 0 & a & b & 1 \\
\hline
0 & 1 & 1 & 1 & 1 \\
a & a & 1 & 1 & 1 \\
b & 0 & a & 1 & 1 \\
1 & 0 & a & b & 1
\end{array}
\]

And we define $\nu, \mu, \ast$ and $\Rightarrow$ as follows:

\[
\begin{align*}
u(x_1, x_2) & = [x_1, x_1], \quad \mu(x_1, x_2) = [x_2, x_2], \quad [x_1, x_2] \ast [y_1, y_2] = [x_1 \lor y_1, x_2 \lor y_2], \quad [x_1, x_2] \Rightarrow [y_1, y_2] = [(x_1 \Rightarrow y_1) \land (x_2 \Rightarrow y_2), x_2 \Rightarrow y_2].
\end{align*}
\]

Then $(A, \lor, \land, \Rightarrow, \nu, \mu, [0, 0], [0, 1], [1, 1])$ is a triangle algebra with $[0, 0]$ as the smallest and $[1, 1]$ as the greatest element. We define the unary operation $\sigma$ as follows:

\[
\sigma(x) = \begin{cases}
[0, 0], & x = [0, 0], [0, a], [0, b], [0, 1] \\
[a, a], & x = [a, a], [a, b], [a, 1] \\
[1, 1], & x = [b, b], [b, 1], [1, 1].
\end{cases}
\]

Then $\sigma$ is a state operator on $A$. So $(A, \sigma)$ is a state triangle algebra. Also, we have $\sigma(x \ast y) = \sigma(x) \ast \sigma(y)$ and $\sigma(x \rightarrow y) = \sigma(x) \rightarrow \sigma(y)$, for all $x, y \in A$. Thus $\sigma$ is an endomorphism and $\sigma(A) = \{[0, 0], [a, a], [1, 1]\}$. Also, we have $\text{Ker}(\sigma) = \{[b, b], [b, 1], [1, 1]\}$ so $\sigma$ is not faithful.
Let $\mathcal{L}^I = (L^I, \lor, \land, T_{\alpha}, I_{\alpha}, \nu, \mu, [0, 0], [0, 1], [1, 1])$ be a triangle algebra if:

for $a = [a_1, a_2], x = [x_1, x_2]$ and $y = [y_1, y_2]$ in $L^I$,

$$T_{\alpha}(x, y) = [T(x_1, y_1), \max(T(\alpha, x_2, y_2)), T(x_1, y_2), T(x_2, y_1)],$$

induces a residuated lattice on $\mathcal{L}^I$, with residual implicator

$$I_{\alpha}(x, y) = [\min(I_T(x_1, y_1), I_T(x_2, y_2)), \min(I_T(T(x_2, \alpha), y_2), I_T(x_2, y_2))].$$

And $\nu x = [x_1, x_1]$ and $\mu x = [x_2, x_2]$, for all $\alpha \in I$ and $x = [x_1, x_2] \in L^I$.

Let $T(x, y) = \min(x, y)$ and $I_T(x, y) = \begin{cases} 1 & x \leq y \\ y & y < x \end{cases}$, $\alpha = 1$. We define

$$\sigma_a(x) = \begin{cases} x & x \leq a \\ 1 & \text{otherwise}. \end{cases}$$

It is clear that $\sigma_a(x)$ is a state operator on $L^I$. Thus, $(L^I, \sigma_a(x))$ is a satat triangle algebra.

(iv) Let $A = \{[0, 0], [0, v], [0, a], [0, b], [0, 1], [v, v], [v, a], [v, b], [v, 1], [a, a], [a, 1], [b, 1], [1, 1]\}$. Define $\circ$ and $\Rightarrow$ as follows:

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Define $\nu, \mu, *$ and $\Rightarrow$ one as follows:

$$\nu[x_1, x_2] = [x_1, x_1], \mu[x_1, x_2] = [x_2, x_2],$$

$$[x_1, x_2] \ast [y_1, y_2] = [x_1 \cup y_1, x_2 \circ y_2],$$

$$[x_1, x_2] \Rightarrow [y_1, y_2] = [(x_1 \Rightarrow y_1) \land (x_2 \Rightarrow y_2), x_2 \Rightarrow y_2].$$

Then $(A, \lor, \land, \ast, \Rightarrow, \nu, \mu, [0, 0], [0, 1], [1, 1])$ is a triangle algebra with $[0, 0]$ as the smallest and $[1, 1]$ as the greatest element. We define the unary operation $\sigma$ as follows:

$$\sigma(x) = \begin{cases} [0, 0], & x = [0, 0], [0, v], [v, v], [v, 1] \\ [1, 1], & \text{otherwise}. \end{cases}$$

Then $\sigma$ is a state operator on $A$. So $(A, \sigma)$ is a state triangle algebra.
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Proposition 3.3. Let \((A, \sigma)\) be a state triangle algebra. Then the following statements hold:

(i) If \(\text{Ker}(\sigma) \subseteq B(A)\), then \(B(A/\text{Ker}(\sigma)) = B(A)/\text{Ker}(\sigma)\).

(ii) If \(a \in A\) is nilpotent element, then \(a/\text{Ker}(\sigma) \in A/\text{Ker}(\sigma)\) is a nilpotent element.

Proof. (i) Let \(\text{Ker}(\sigma) \subseteq B(A)\). Then

\[
B(A/\text{Ker}(\sigma)) = \{[e] \in A/\text{Ker}(\sigma) \mid [e] \lor [-e] = [1]\}
\]

\[
= \{[e] \in A/\text{Ker}(\sigma) \mid e/\text{Ker}(\sigma) \lor -e/\text{Ker}(\sigma) = 1/\text{Ker}(\sigma)\}
\]

\[
= \{[e] \in A/\text{Ker}(\sigma) \mid e \lor -e = 1\}
\]

\[
= B(A)/\text{Ker}(\sigma).
\]

(ii) It is clear.

\[\square\]

Lemma 3.4. Let \((A, \sigma)\) be a state triangle algebra. Then for all \(x, y \in A\), we have:

1. \(\sigma(1) = 1\).
2. \(\sigma(\neg x) = \neg(\sigma(x))\).
3. If \(x \leq y\), then \(\sigma(x) \leq \sigma(y)\).
4. \(\sigma(x \ast y) = \sigma(x) \ast \sigma(y)\) and if \(x \ast y = 0\), then \(\sigma(x \ast y) = \sigma(x) \ast \sigma(y)\).
5. \(\sigma(x \rightarrow y) \leq \sigma(y) \rightarrow \sigma(y)\) and if \(x, y\) are comparable, then \(\sigma(x \rightarrow y) = \sigma(x) \rightarrow \sigma(y)\).
6. \(\sigma(\sigma(x)) = \sigma(x)\).
7. \(\sigma(A)\) is a triangle subalgebra of \(A\).
8. \(\sigma(A) = \{x \in A \mid x = \sigma(x)\}\).
9. If \(\text{ord}(x) < \infty\), then \(\text{ord}(\sigma(x)) \leq \text{ord}(x)\) and in MTL-triangle algebras we have \(\sigma(x) \notin \text{Rad}(A)\).
10. \(\sigma(x \rightarrow y) = \sigma(x) \rightarrow \sigma(y)\) if \(\sigma(y \rightarrow x) = \sigma(y) \rightarrow \sigma(x)\).
11. If \(\sigma(A) = A\), then \(\sigma\) is the identity on \(A\).
12. If \(\sigma\) is faithful, then \(x < y\) implies \(\sigma(x) < \sigma(y)\).
13. If \(\sigma\) is faithful, then either \(\sigma(x) = x\) or \(\sigma(x)\) and \(x\) are not comparable.
14. If \(A\) is linear and \(\sigma\) faithful, then \(\sigma(x) = x\), for all \(x \in A\).
15. \(\sigma(v(x \ast y)) = \sigma(vx) \ast \sigma(vx \rightarrow v(x \ast y))\).
16. \(u \notin \text{Ker}(\sigma)\).

Proof. The proof of parts (1 – 6) and (8) is Similar to Proposition 3.5 of [6].

7. From \((ST_1),(ST_2),(ST_3),(ST_6),(ST_7),(ST_8),(ST_9)\) and part (1), we have \(\sigma(A)\) is closed under all operators \(*, \rightarrow, \lor, \land, \lor\) and \(\mu\). Thus \(\sigma(A)\) is a triangle subalgebra of \(A\).

9. Let \(\text{ord}(x) = n\). By part (4), \(0 = \sigma(x^n) \geq \sigma(x)^n\). So \(\text{ord}(\sigma(x)) \leq \text{ord}(x)\). From Theorem 2.15, we conclude any element of finite order cannot belong to \(\text{Rad}(A)\).

10. Let \(\sigma(x \rightarrow y) = \sigma(x) \rightarrow \sigma(y)\). Then by part (5), \(\sigma(y \rightarrow x) = \sigma(y) \rightarrow \sigma(y \land x) = \sigma(y) \rightarrow (\sigma(x) \ast (\sigma(x \rightarrow y)) = \sigma(y) \rightarrow (\sigma(x) \ast (\sigma(x) \rightarrow \sigma(y)))) = \sigma(y) \rightarrow \sigma(x) \land \sigma(y) = \sigma(y) \rightarrow \sigma(x)\). The converse implication is proved by exchanging \(x\) and \(y\) in the previous formulas.
(11) For all \( x \in A \), we have \( x = \sigma(x_0) \), for some \( x_0 \in A \). By part (7), we have \( \sigma(x) = \sigma(\sigma(x_0)) = \sigma(x_0) = x \).

(12) Let \( \sigma(x) = \sigma(y) \). Then \( \sigma(y \to x) = \sigma(y) \to \sigma(x) = 1 \). So \( y \leq x \) which is a contradiction.

(13) Let \( x \neq \sigma(x) \) and \( x \) and \( \sigma(x) \) be comparable. Then \( x < \sigma(x) \) or \( \sigma(x) < x \) giving \( \sigma(x) < \sigma(x) \), which is a contradiction.

(14) It is clear by (14).

(15) By Theorem 3.24 and \((ST_3)\), we have \( \sigma(\nu(x \ast y)) = \sigma(\nu x \ast \nu y) = \sigma(\nu x) \ast \sigma(\nu y) = \sigma(\nu x) \ast \sigma(\nu x \to \nu(x \ast y)) \).

(16) By (T.5), \( \sigma(\nu u) = \sigma(0) = 0 \) so \( u \notin \text{Ker}(\sigma) \).

\[ \Box \]

**Proposition 3.5.** The following hold:

1. If \( \sigma(x \land y) = \sigma(x) \land \sigma(y) \), then \( \sigma(x \to y) = \sigma(x) \to \sigma(y) \), for all \( x, y \in A \).
2. If \( \sigma(x \lor y) = \sigma(x) \lor \sigma(y) \), then \( \sigma(x \to y) = \sigma(x) \to \sigma(y) \), for all \( x, y \in A \).
3. If \( \sigma(x \to y) = \sigma(x) \to \sigma(y) \), then \( \sigma(x \ast y) = \sigma(x) \ast \sigma(y) \), for all \( x, y \in A \).

**Proof.**

1. We have \( \sigma(x \to y) = \sigma(x) \to \sigma(x \land y) = \sigma(x) \to (\sigma(x) \land \sigma(y)) = \sigma(x) \to \sigma(y) \).
2. Since \( (x \lor y) \to y = x \to y \), \( \sigma(x \to y) = \sigma((x \lor y) \to y) = \sigma(x \lor y) \to \sigma((x \lor y) \land y) = \sigma(x) \lor \sigma(y) \to \sigma(y) = \sigma(x) \to \sigma(y) \).
3. \( \sigma(x \ast y) \to \sigma(z) = \sigma((x \ast y) \to z) = \sigma(x \to (y \to z)) = \sigma(x) \to (\sigma(y) \to \sigma(z)) = (\sigma(x) \ast \sigma(y)) \to \sigma(z) \).

Let \( z = \sigma(x) \ast \sigma(y) \). Then \( \sigma(x \ast y) \to \sigma(x) \ast \sigma(y) = \sigma(x) \ast \sigma(y) \to \sigma(x) \ast \sigma(y) = \sigma(x) \ast \sigma(y) \to \sigma(x) \ast \sigma(y) = 1 \). Thus \( \sigma(x \ast y) \to \sigma(x) \ast \sigma(y) = \sigma(x \ast y) \to \sigma(x) \ast \sigma(y) = 1 \) and \( \sigma(x \ast y) \leq \sigma(x) \ast \sigma(y) \). By Lemma 3.24 (4), we have \( \sigma(x \ast y) \geq \sigma(x) \ast \sigma(y) \) so \( \sigma(x \ast y) = \sigma(x) \ast \sigma(y) \).

In the following, we consider that under which conditions the converse of above proposition hold.

**Proposition 3.6.** Let \( A \) be a RL-triangle algebra. Then we have

1. \( \sigma(x \land y) = \sigma(x) \ast \sigma(x \to y) \).
2. If \( \sigma(x \to y) = \sigma(x) \to \sigma(y) \), then \( \sigma(x \land y) = \sigma(x) \land \sigma(y) \), for all \( x, y \in A \).
3. If \( \sigma(x \to y) = \sigma(x) \to \sigma(y) \), then \( \sigma(x \lor y) = \sigma(x) \lor \sigma(y) \).

**Proof.** Similar to Proposition 3.6 of [8].

In the following example we show that the converse of part (3) of Proposition 3.6 is not generally true.

**Example 3.7.** Let \( A = \{0, u, 1\} \) be a chain. We define operations \( \nu, \mu, \ast, \to \) as follows:

<table>
<thead>
<tr>
<th>( x )</th>
<th>( \nu x )</th>
<th>( \mu x )</th>
<th>( \ast )</th>
<th>( 0 )</th>
<th>( u )</th>
<th>( 1 )</th>
<th>( \to )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>u</td>
<td>0</td>
<td>u</td>
<td>1</td>
<td>u</td>
<td>u</td>
<td>u</td>
<td>u</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

We define the unary operation \( \sigma \) as follows:

\[ \sigma(x) = \begin{cases} 0 & \text{if } x = 0, u \\ 1 & \text{if } x = 1. \end{cases} \]

Then \( (A, \lor, \land, \ast, \to, \nu, \mu, 0, u, 1) \) is a state triangle algebra. We have \( \sigma(u \ast 0) = \sigma(u) \ast \sigma(0) \) but \( \sigma(u \to 0) \neq \sigma(u) \to \sigma(0) \).

\[ \Box \]

4 State IVRL-filters in state triangle algebras

From now on \( (A, \sigma) \) is a state triangle algebra unless otherwise specified.

**Definition 4.1.** A non-empty subset \( F \subseteq A \) is called a state IVRL-filter of \( (A, \sigma) \) if \( F \) is an IVRL-filter of \( A \) and \( \sigma(\nu x) \in F \), for all \( x \in F \).

We will denote the set of \( \sigma \)-IVRL-filters of \( (A, \sigma) \) by \( F_{\sigma}(A) \).

**Example 4.2.** In Example 3.4 (iii), clearly \( F = \{[1, 1]\} \) is a state IVRL-filter of \( (A, \sigma) \).
Corollary 4.3. Let \((A, \sigma)\) be a state triangle algebras and \(F,G\) be two IVRL-filters of \(A\) such that \(F \subseteq G\). If \(F\) is a state IVRL-filter of \((A, \sigma)\), then for all \(x \in A\), we have \(\sigma(ux) \in F \subseteq G\), hence \(\sigma(ux) \in G\). So \(G\) is a state IVRL-filter of \((A, \sigma)\).

Proposition 4.4. Let \((A, \sigma)\) be a state triangle algebra and \(F\) be a state IVRL-filter of \((A, \sigma)\). Then \(\sigma(A/F) = \sigma(A)/F\).

Proof. By Lemma 4.3, we have

\[
\sigma(A/F) = \{ [x] \in A/F \mid [x] = \sigma([x]) \} = \{ [x] \in A/F \mid x = \sigma(x) \} = \sigma(A)/F.
\]

Proposition 4.5. Let \((A, \sigma)\) be a state triangle algebra. Then \(Ker(\sigma)\) is a state IVRL-filter of \(A\).

Proof. We have that \(\sigma(1) = 1\), and so \(1 \in Ker(\sigma)\). Let \(x, y \in Ker(\sigma)\). Then \(\sigma(\nu x) = \sigma((\nu x) \ast (\nu y)) \geq \sigma(\nu x) \ast \sigma(\nu y) = 1\), \(\sigma(\nu x \ast y) = 1\). Thus \(x, y \in Ker(\sigma)\). If \(x \in Ker(\sigma)\) and \(y \in A\), \(x \leq y\), then \(1 = \sigma(ux) \leq \sigma(\nu y)\). So \(y \in Ker(\sigma)\). Also, let \(x \in Ker(\sigma)\). Then we have \(\sigma(ux) = 1\). Since \(\nu u x = \nu x, \sigma(\nu u x) = 1\). Thus \(\nu x \in Ker(\sigma)\). Therefore, \(Ker(\sigma)\) is an IVRL-filter of \(A\).

Definition 4.6. A proper \(\sigma\)-IVRL-filter of \((A, \sigma)\) is called an IVRL-extended maximal filter of \((A, \sigma)\) if it is not strictly contain in any proper \(\sigma\)-IVRL-filter of \((A, \sigma)\).

We denote the set of IVRL-extended maximal \(\sigma\)-filters of \((A, \sigma)\) by \(Max_{\sigma}(A)\) and \(Rad_{\sigma}(A) = \cap_{F \in Max_{\sigma}(A)} F\).

Remark 4.7. If \((F_i)_{i \in I}\) is a family of IVRL-\(\sigma\)-factors of a triangle \(\sigma\)-algebra \((A, \sigma)\), then \(\cap_{i \in I} F_i\) is a \(\sigma\)-IVRL-filter of \(A\).

Proposition 4.8. If \(F\) is an \(\sigma\)-IVRL-filter of \((A, \sigma)\) and \(x \in A\), then the \(\sigma\)-IVRL-filter generated by \(F\) and \(x\) is the set \([F, x]_{\sigma} = \{ y \in A \mid \nu y \geq (f_1 \ast (\nu x \ast \sigma(\nu x))^{n_1}) \ast \ldots \ast (f_k \ast (\nu x \ast \sigma(\nu x))^{n_k}), \text{ for some } f_1 \in F_i, n_i \in \mathbb{N}, i \in \{1, \ldots, k\}, k \geq 1 \} \). If \(y \in Y\), \(\nu y \geq (f_1 \ast (\nu x \ast \sigma(\nu x))^{n_1}) \ast \ldots \ast (f_k \ast (\nu x \ast \sigma(\nu x))^{n_k})\). Then by Lemma 4.3, \(\sigma(\nu y) \geq \sigma((f_1 \ast (\nu x \ast \sigma(\nu x))^{n_1}) \ast \ldots \ast (f_k \ast (\nu x \ast \sigma(\nu x))^{n_k}))\) and \(\sigma(f_i) \ast \sigma(\nu x \ast \sigma(\nu x))^{(n_i)} \ast \ldots \ast \sigma(f_k) \ast \sigma(\nu x \ast \sigma(\nu x))^{(n_k)}\). Since \(\sigma(f_i) \in F\), \(i \in \{1, \ldots, k\}\), it follows that \(\sigma(\nu y) \in Y\). Thus \(y \in [A]_{\sigma}\). If \(F' \subseteq [A]_{\sigma}\) such that \(F' \subseteq F'\), \(x \in F'\) and \(y \in Y\). Then \(\nu y \geq (f_1 \ast (\nu x \ast \sigma(\nu x))^{n_1}) \ast \ldots \ast (f_k \ast (\nu x \ast \sigma(\nu x))^{n_k})\). But \((f_1 \ast (\nu x \ast \sigma(\nu x))^{n_1}) \ast \ldots \ast (f_k \ast (\nu x \ast \sigma(\nu x))^{n_k})\) \(\in F'\).

Thus \(y \in F'\) and \(Y = [F, x]_{\sigma}\).

Proposition 4.9. If \(a, b \in A\), then the following hold:

1. The \(\sigma\)-IVRL-filter generated by \(a\) is the set \([a]_{\sigma} = \{ x \in A \mid \nu x \geq (\nu a \ast \sigma(\nu a))^n, n \geq 1 \}\).
2. If \(a \leq b\), then \([b]_{\sigma} \subseteq [a]_{\sigma}\).
3. \(\sigma(a) / \sigma \subseteq [a]_{\sigma}\).
4. \(\sigma(a) / \sigma = \nu(a) / \sigma\).
5. \(a_{\sigma} \vee b_{\sigma} = (\nu(a) \ast \sigma(\nu a))_{\sigma} \ast (\nu(b) \ast \sigma(\nu b))_{\sigma}\).
6. \(a_{\sigma} \cap b_{\sigma} = (\nu(a) \ast \sigma(\nu a)) \vee (\nu(b) \ast \sigma(\nu b))_{\sigma}\).

Proof. (1) It follows from Proposition 4.3.

Properties (2) – (4) are obvious.

(5) Let \(u = \nu a \ast \sigma(\nu a), v = \nu b \ast \sigma(\nu b)\). Since \(u \ast v \leq a, u \ast v \leq b\). By (2), we have \([a]_{\sigma} \subseteq [u \ast v]_{\sigma}, [b]_{\sigma} \subseteq [u \ast v]_{\sigma}\). Let \(f \in [A]_{\sigma}\) so that \([a]_{\sigma} \subseteq F, [b]_{\sigma} \subseteq F\). Then \(a, b \in F\) so \(u \ast v \in F\). Thus \([u \ast v]_{\sigma} \subseteq F\) and \([a]_{\sigma} \vee [b]_{\sigma} = [u \ast v]_{\sigma}\).

(6) Since \(\nu a \ast \sigma(\nu a) \leq (\nu(a) \ast \sigma(\nu a)) \vee (\nu(b) \ast \sigma(\nu b))_{\sigma}\), by (2) we have \([(\nu(a) \ast \sigma(\nu a)) \vee (\nu(b) \ast \sigma(\nu b))_{\sigma}] \subseteq [\nu(a) \ast \sigma(\nu a)]_{\sigma} = [a]_{\sigma}\).

Similarly, \([(\nu(a) \ast \sigma(\nu a)) \vee (\nu(b) \ast \sigma(\nu b))_{\sigma}] \subseteq [b]_{\sigma}\), so \([(\nu(a) \ast \sigma(\nu a)) \vee (\nu(b) \ast \sigma(\nu b))_{\sigma}] \subseteq [a]_{\sigma} \cap [b]_{\sigma}\). Let \(x \in [a]_{\sigma} \cap [b]_{\sigma}\). Then there are \(m, n \geq 1\) such that \(x \geq (\nu(a) \ast \sigma(\nu a))^m\) and \(x \geq (\nu(b) \ast \sigma(\nu b))^n\). So \(x \geq (\nu(a) \ast \sigma(\nu a))^m \vee (\nu(b) \ast \sigma(\nu b))^n\). Since \(x \geq (\nu(a) \ast \sigma(\nu a))^m \vee (\nu(b) \ast \sigma(\nu b))^n\), \(x \in [\nu(a) \ast \sigma(\nu a)] \vee (\nu(b) \ast \sigma(\nu b))_{\sigma}\).
The role of states in triangle algebras

Example 4.10. In Example (ii),

- \([b, b] \subseteq \{a, a\} = A\). But \([a, a] \not\subseteq [b, b]\).
- \([b, 1] = \{b, b, 1\} \subseteq \{a, a\} = A\). But \([a, a] \not\subseteq [b, 1]\).

Proposition 4.11. Let \((A, \sigma)\) be a state triangle algebra and \(F_1, F_2 \subseteq F_\sigma(A)\). Then \([F_1 \cup F_2] \subseteq F_1 \cup F_2 = \{x \in A | \nu x \geq (f_1 \ast f_2)^k, f_1 \in F_1, f_2 \in F_2, k \geq 1\} = \{x \in A | \nu x \geq (f_1 \ast f_2)^k, f_1 \in F_1, f_2 \in F_2, k \geq 1\}.

Proof. Let \(Y = \{x \in A | \nu x \geq (f_1 \ast f_2)^k, f_1 \in F_1, f_2 \in F_2, k \geq 1\} = \{x \in A | \nu x \geq (f_1 \ast f_2)^k, f_1 \in F_1, f_2 \in F_2, k \geq 1\}.\) If \(x, y \in Y\), then \(\nu x \geq (f_1 \ast f_2)^k, \nu y \geq (g_1 \ast g_2)^k, f_1, g_1 \in F_1, f_2, g_2 \in F_2, k \geq 1\). By Lemma 2, we have \(\nu (x \ast y) = \nu x \ast \nu y \geq (f_1 \ast f_2)^k, (g_1 \ast g_2)^k, (f_1 \ast g_1) \ast (f_2 \ast g_2)^k\). Since \(f_1 \ast g_1, f_2 \ast g_2 \in F_2\), \((x \ast y) \in Y\). If \(x \in Y\) and \(x \not\in y, y \in A\), then \(y \in Y\). So \(\nu (x \ast y) \geq (f_1 \ast f_2)^k, f_1 \in F_1, f_2 \in F_2, k \geq 1\). But \((f_1 \ast f_2)^k, f_1 \in F_1, f_2 \in F_2, k \geq 1\). But \((f_1 \ast f_2)^k, f_1 \in F_1, f_2 \in F_2, k \geq 1\).

Theorem 4.12. Let \(F\) be a proper \(\sigma\)-IVRL-filter of \((A, \sigma)\). Then \(F\) is an IVRL-extended maximal \(\sigma\)-filter iff for all \(x \in A \setminus F\), there are \(f \in F\) and \(n, m \geq 1\) such that \((f \ast (\nu x)^n)^m = 0\).

Proof. Let \(F\) be an IVRL-extended maximal \(\sigma\)-filter and \(x \in A \setminus F\). Then \([f, x] \subseteq A\), where \([f, x] = \{y \in A | \nu y \geq f_1 \ast (\nu x)^n \ast \cdots \ast f_k \ast (\nu x)^n, f_1 \in F, n, i \in \{1, \ldots, k\}, k \geq 1\}\). Thus there are \(f_1 \in F, i \in \{1, \ldots, k\}\) such that \(f_1 \ast (\nu x)^n \ast \cdots \ast f_k \ast (\nu x)^n \leq f_1 \ast (\nu x)^n \ast \cdots \ast f_k \ast (\nu x)^n = 0\). Let \(g = f_1 \ast \cdots \ast f_k \in F, b = \max \{n_1, \ldots, n_k\}\) so we have \((g \ast (\nu x)^n)^k = 0\). Then \((g \ast (\nu x)^n)^k = (\sigma(0) = 0) = 0\).

Proposition 4.13. Let \((A, \sigma)\) be a state triangle algebra. Then we have

(i) If \(F\) is an IVRL-extended maximal \(\sigma\)-filter of \((A, \sigma)\), then \(\sigma^{-1}(F)\) is an IVRL-extended maximal \(\sigma\)-filter of \((A, \sigma)\).

(ii) If \(F\) is an IVRL-extended maximal \(\sigma\)-filter of \((A, \sigma)\), then \(\sigma(F)\) is an IVRL-extended maximal \(\sigma\)-filter of \((A, \sigma)\).

Proposition 4.14. A proper IVRL-\(\sigma\)-filter \(P\) of \((A, \sigma)\) is called an IVRL-extended prime \(\sigma\)-filter if \(a, b \in A\) such that \((\nu a \ast (\nu a) \vee (\nu b \ast (\nu b)) \in P, then \nu a \in P \vee \nu b \in P\).

We will denote the set of IVRL-extended prime \(\sigma\)-filters of \((A, \sigma)\) by \(\text{Spec}_\sigma(A)\).

Proposition 4.15. Let \(P\) be a proper IVRL-\(\sigma\)-filter of \((A, \sigma)\). Then the following are equivalent:

(i) If \(P_1, P_2 \subseteq F_\sigma(A)\) and \(P = P_1 \cap P_2\), then \(P = P_1 \cup P_2\).

(ii) \(P\) is an IVRL-extended prime \(\sigma\)-filter of \((A, \sigma)\).
Proof. (i $\Rightarrow$ ii) Let $(va * \sigma(va)) \lor (vb * \sigma(vb)) \in P$, for all $a, b \in A$. Let $P_1 = [P, a]_{\sigma}$ and $P_2 = [P, b]_{\sigma}$. Clearly, $P \subseteq P_1 \cap P_2$. Let $x \in P_1 \cap P_2$ and $u = va * \sigma(va)$ and $v = vb * \sigma(vb)$. By Proposition 4.15, there are $p_i, q_j \in P$ and $m_i, n_j \geq 1, i \in \{1, ..., k\}, j \in \{1, ..., l\}, k, l \geq 1$, such that $vx \geq p_1 * u^{m_1} * ... * p_k * u^{m_k}$ and $vx \geq q_1 * v^{n_1} * ... * q_l * v^{n_l}$. Then

$$vx \geq (p_1 * u^{m_1} * ... * p_k * u^{m_k}) \lor (q_1 * v^{n_1} * ... * q_l * v^{n_l})$$

$$\geq \sum_{1 \leq i \leq k} \sum_{1 \leq j \leq l} ((p_i \lor q_j) * (p_i \lor v^{n_i}) * (u^{m_i} \lor q_j) * (u^{m_i} \lor v^{n_i}))$$

But $p_i \lor q_j, p_i \lor v^{n_i}, u^{m_i} \lor q_j, (u \lor v)^{m_i n_j} \in P, i \in \{1, ..., k\}, j \in \{1, ..., l\}$. So $x \in P$ and $P = P_1 \cap P_2$. Thus $P = P_1$ or $P = P_2$, that is $va \in P$ or $vb \in P$.

(ii $\Rightarrow$ i) Let $P_1, P_2 \in \mathcal{F}_\sigma(A)$ such that $P = P_1 \cap P_2$. If $P \neq P_1$ and $P \neq P_2$ and $a \in P_1 \setminus P$ and $b \in P_2 \setminus P$, then $va * \sigma(va) \in P_1, vb * \sigma(vb) \in P_2$. So $(va * \sigma(va)) \lor (vb * \sigma(vb)) \in P_1 \cap P_2 = P$. Thus $va \in P$ or $vb \in P$, a contradiction. Thus $P = P_1$ or $P = P_2$. \qed

Proposition 4.16. Let $F$ be a proper $\sigma$-IVRL-filter of $(A, \sigma)$. Then there exists an IVRL-extended maximal $\sigma$-filter $F_0$ of $(A, \sigma)$ such that $F \subseteq F_0$.

Proof. Clearly, the set $A_F = \{ F' | F'$ is a proper $\sigma$-IVRL-filter and $F \subseteq F' \}$ is nonvoid and inductively ordered by inclusion. So by Zorn lemma, $A_F$ has a maximal element $F_0$. We are going to prove that $F_0$ is an IVRL-extended maximal $\sigma$-filter of $(A, \sigma)$. If $F_1$ is a proper $\sigma$-IVRL-filter of $(A, \sigma)$ such that $F_0 \subseteq F_1$, then $F_1 \in A_F$ and the maximality of $F_0$ implies that $F_1 = F_0$. \qed

Proposition 4.17. Let $a \in A, a < 1$. Then there is an IVRL-extended prime $\sigma$-filter $P$ of $(A, \sigma)$, that $a \notin P$.

Proof. Let $F_a = \{ F | F$ is an IVRL-$\sigma$-filter and $a \notin F \}$. Then $\{1\} \in F_a$, so $F_a \neq \emptyset$. We show that $F_a$ is inductively ordered and by Zorn lemma, there is a maximal element $P$ of $F_a$. We will prove that $P$ is an IVRL-extended prime $\sigma$-filter. Let $x, y \in A$ such that $(\nu x * \sigma(\nu x)) \lor (\nu y * \sigma(\nu y)) \in P$ and $\nu x, \nu y \notin P$. Considering the sets $[P, x]_\sigma$ and $[P, y]_\sigma$. Then $P$ is strictly contained in $[P, x]_\sigma$ and $[P, y]_\sigma$. The maximality of $P$ implies $a \in [P, x]_\sigma \cap [P, y]_\sigma$. Then there are $a_i, b_j \in P$ and $m_i, n_j \geq 1, i \in \{1, ..., k\}, j \in \{1, ..., l\}, k, l \geq 1$ such that $va \geq a_1 * (\nu x * \sigma(\nu x))^{m_1} * ... * a_k * (\nu x * \sigma(\nu x))^{m_k}$ and $va \geq b_1 * (\nu y * \sigma(\nu y))^{n_1} * ... * b_l * (\nu y * \sigma(\nu y))^{n_l}$. Similar to proposition 4.15, we have $va \notin P$, a contradiction. So $P$ is an IVRL-extended prime $\sigma$-filter. \qed

In the following example we show that the concepts of IVRL-extended maximal $\sigma$-filters and IVRL-extended prime $\sigma$-filters on state triangle algebras are different from the concepts of IVRL-extended maximal filters and IVRL-extended prime filters on triangle algebras.

Example 4.18. Let $A = \{ [0, 0], [0, a], [a, a], [0, b], [a, b], [b, b], [a, c], [b, c], [c, c], [0, d], [a, d], [b, d], [c, d], [d, d], [0, 1], [1, 1] \}$. Define $\odot$ and $\Rightarrow$ as follows:

$$\begin{array}{cccccc}
\odot & 0 & a & b & c & d & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
a & 0 & a & 0 & 0 & a & a \\
b & 0 & a & b & 0 & a & b \\
c & 0 & 0 & c & c & c & c \\
d & 0 & 0 & a & c & d & d \\
1 & 0 & a & b & c & d & 1 \\
\Rightarrow & 0 & a & b & c & d & 1 \\
0 & 1 & 1 & 1 & 1 & 1 & 1 \\
a & d & 1 & 1 & d & 1 & 1 \\
b & c & d & 1 & c & d & 1 \\
c & b & b & 1 & 1 & 1 & 1 \\
d & a & b & b & d & 1 & 1 \\
1 & 0 & a & b & c & d & 1
\end{array}$$

And we define $\nu, \mu, *$ and $\rightarrow$ as follows:

$$\nu[x_1, x_2] = [x_1 \Rightarrow x_1, \mu[x_1, x_2] = [x_2 \Rightarrow x_2, [x_1, x_2] * [y_1, y_2] = [x_1 \odot y_1, x_2 \odot y_2], [x_1, x_2] \rightarrow [y_1, y_2] = [(x_1 \Rightarrow y_1) \land (x_2 \Rightarrow y_2), x_2 \Rightarrow y_2].$$
Then \((A, V, A, S, \to, \nu, \mu, [0, 0], [0, 1], [1, 1])\) is a triangle algebra with \([0, 0]\) as smallest and \([1, 1]\) as greatest element. We define the unary operation \(\sigma\) as follows:

\[
\sigma(x) = \begin{cases} 
[1, 1], & x = [c, c], [d, d], [c, d], [c, 1], [d, 1], [1, 1] \\
[0, 0], & \text{otherwise}.
\end{cases}
\]

Then \(\sigma\) is a state operator on \(A\). So \((A, \sigma)\) is a state triangle algebra. The IVRL-filters of \(A\) are \([1, 1]\), \([b, b]\), \([b, 1]\), \([1, 1]\), \([c, c]\), \([c, d]\), \([c, 1]\), \([d, 1]\), \([1, 1]\). The IVRL-\(\sigma\)-filters of \((A, \sigma)\) are \([1, 1]\), \([c, c]\), \([c, d]\), \([c, 1]\), \([d, 1]\), \([1, 1]\). Since \([1, 1]\) = \([b, b]\), \([b, 1]\), \([1, 1]\) \(\cap\) \([c, c]\), \([c, d]\), \([c, 1]\), \([d, 1]\), \([1, 1]\), the IVRL-filter \([1, 1]\) is not an IVRL-extended prime filter of \(A\). But \([1, 1]\) is an IVRL-extended prime \(\sigma\)-filter of \((A, \sigma)\). Thus the notion of IVRL-extended prime filter and IVRL-extended prime \(\sigma\)-filter of \((A, \sigma)\) are not the same.

**Proposition 4.19.** Let \((A, \sigma)\) be a state triangle algebra, \(F \in F_\sigma(A)\) and \(a \in A \setminus F\). Then the following hold:

(i) There is \(P \in \Spec_\sigma(A)\) such that \(F \subseteq P\) and \(a \notin P\).

(ii) \(F = \bigcap_{P \in \Spec_\sigma(A), F \subseteq P} P\).

(iii) \(F = \bigcap_{P \in \Spec_\sigma(A)} P = \{1\}\).

**Proof.** (i) It follows from Proposition 4.14.

(ii) Clearly, \(F \subseteq \bigcap_{P \in \Spec_\sigma(A), F \subseteq P} P\). Let \(a \in \bigcap_{P \in \Spec_\sigma(A), F \subseteq P} P\) and \(a \notin F\). Then from (i) it follows that there is \(P \in \Spec_\sigma(A)\) such that \(F \subseteq P\) and \(a \notin P\), a contradiction.

(iii) It follows from (ii), for \(F = \{1\}\). \(\square\)

**Proposition 4.20.** Let \((A, \sigma)\) be a state triangle algebra and \(P\) be a proper IVRL-\(\sigma\)-filter of \((A, \sigma)\). Then the following are equivalent:

(i) \(P \in \Spec_\sigma(A)\),

(ii) for every \(x, y \in A\) for which \(\nu x \ast \sigma(\nu x), \nu y \ast \sigma(\nu y) \in A \setminus P\), there is \(z \in A \setminus P\) such that \(\nu x \ast \sigma(\nu x) \leq \nu z\) and \(\nu y \ast \sigma(\nu y) \leq \nu z\),

(iii) if \(F_\sigma(x) \cap F_\sigma(y) \subseteq P\), then \(x \in P\) or \(y \in P\).

**Proof.** (i) \(\Leftrightarrow\) (ii) Let \(x, y \in A\) such that \(\nu x \ast \sigma(\nu x), \nu y \ast \sigma(\nu y) \in A \setminus P\) and \(z = (\nu x \ast \sigma(\nu x)) \lor (\nu y \ast \sigma(\nu y))\). If \(z \in P\), then \(\nu x \in P\) or \(\nu y \in P\). So \(\nu x \ast \sigma(\nu x) \in P\) or \(\nu y \ast \sigma(\nu y) \in P\), a contradiction. Thus \(x \in A \setminus P\) and \(\nu x \ast \sigma(\nu x) \leq \nu z\) and \(\nu y \ast \sigma(\nu y) \leq \nu z\).

Conversely, let \(x, y \in A\) such that \((\nu x \ast \sigma(\nu x)) \lor (\nu y \ast \sigma(\nu y)) \in A \setminus P\), \(x \notin P\) and \(y \notin P\). Then \(\nu x \ast \sigma(\nu x), \nu y \ast \sigma(\nu y) \in A \setminus P\). So there is \(z \in A \setminus P\) such that \(\nu x \ast \sigma(\nu x) \leq z\) and \(\nu y \ast \sigma(\nu y) \leq z\). But \((\nu x \ast \sigma(\nu x)) \lor (\nu y \ast \sigma(\nu y)) \leq z\), that is \(z \in P\), a contradiction.

(i) \(\Leftrightarrow\) (iii) By Proposition 4.19(6), we have \(\nu x \cap \nu y = \nu x = (\nu x \ast \nu x) \lor (\nu y \ast \nu y)) = (\nu x \ast \sigma(\nu x) \lor (\nu y \ast \sigma(\nu y)) \in P\) and \(\nu z \in P\) or \(\nu y \in P\).

Conversely, if \((\nu x \ast \sigma(\nu x)) \lor (\nu y \ast \sigma(\nu y)) \in P\), \(x, y \in A\), then \((\nu x \ast \sigma(\nu x) \lor (\nu y \ast \sigma(\nu y))) \subseteq P\). Hence \([x, y] \subseteq P\) and \(\nu x \in P\) or \(\nu y \in P\). \(\square\)

**Definition 4.21.** Let \((A, \sigma)\) be a state triangle algebra. A subalgebra \(B\) of \(A\) is called closed relative to \(\sigma\) if \(x \in B\) implies \(\sigma(x) \in B\).
Example 4.22. In Example 4.22 (ii), $B = \{[a, a], [1, 1]\}$ is closed relative to $\sigma$.

Proposition 4.23. Let $(A, \sigma)$ be a state triangle algebra and $B$ be a subalgebra of $A$ closed relative to $\sigma$ and $F \in F(A)$. Then $B \cap F \in [B]_\sigma$.

Proof. Let $x, y \in B \cap F$. Then $\nu x * \nu y \in B \cap F$. If $x \in B \cap F$ and $x \leq y, y \in B$, then $\nu y \in B$. Thus $\nu y \in B$. Thus $B \cap F$ is an IVRL-filter of $B$. Let $x \in B \cap F$. Then $\sigma(x) \in B \cap F$ and $B \cap F \in [B]_\sigma$.

It is clear that $|F(B)| \leq |F(A)|$, $|\text{Spec}(B)| \leq |\text{Spec}(A)|$ and $|\text{Max}(B)| \leq |\text{Max}(A)|$. So we have $||B|_\sigma| \leq ||A|_\sigma|$, $|\text{Spec}_\sigma(B)| \leq |\text{Spec}_\sigma(A)|$ and $|\text{Max}_{\sigma}(B)| \leq |\text{Max}_{\sigma}(A)|$.

Proposition 4.24. Let $(A, \sigma)$ be a state triangle algebra. Then $|\text{Max}_\sigma(A)| = |\text{Max}(\sigma(A))|$.

Proof. Clearly, if $B$ is a subalgebra of $A$ closed relative to $\sigma$, then $|\text{Max}_\sigma(B)| \leq |\text{Max}(\sigma(A))|$. If $B = \sigma(A)$, we have $|\text{Max}_\sigma(\sigma(A))| \leq |\text{Max}_\sigma(A)|$. Since on $\sigma(A)$ the concepts of IVRL-filter and $\sigma$-IVRL-filter are the same, $\text{Max}_\sigma(\sigma(A)) = \text{Max}(\sigma(A))$. Thus $|\text{Max}(\sigma(A))| \leq |\text{Max}_\sigma(A)|$.

Conversely, let $F \in \text{Max}_\sigma(A)$. We must prove that $F \cap \sigma(A) \in \text{Max}(\sigma(A))$. If $F \cap \sigma(A) = \sigma(A)$, then $0 \in F$, a contradiction. Thus $F \cap \sigma(A) = \text{Max}(\sigma(A)) (F \cap \sigma(A)) = \sigma(A) \setminus F$, since $F \in \text{Max}_\sigma(A)$, by Theorem 4.3, there are $f \in F$ and $m, n \geq 1$ such that $(f * \sigma(\nu x))^m = 0$. Thus $\sigma((f * \sigma(\nu x))^m) = 0$ and so $(\sigma(f) * \sigma(\nu x))^m = 0$. Since $\sigma(x) = x$, $(\sigma(f) * \nu x)^m = 0$. We know that $\sigma(f) \in F \cap \sigma(A)$ so by Theorem 4.3, $F \cap \sigma(A) \in \text{Max}(\sigma(A))$. We define function $Q : \text{Max}_\sigma(A) \to \text{Max}(\sigma(A))$ that $Q(F) = F \cap \sigma(A)$. If $F_1, F_2 \in \text{Max}_\sigma(A)$, then $Q(F_1) = Q(F_2)$. Let there is $x \in F_1 \setminus F_2$. Since $x \in F_1, \sigma(x) \in F_1$. So $\sigma(x) \in F_1 \cap \sigma(A) = F_2 \cap \sigma(A)$. Thus $\sigma(x) \in F_2$. Since $x \notin F_2$, from the maximality of $F_2$, there are $m, n \geq 1$ and $f \in F_2$ such that $(f * \sigma(\nu x))^m = 0$. But $(f * \sigma(\nu x))^m \in F_2$ thus $0 \in F_2$ and $F_2 = A$, a contradiction. Therefore $F_1 \subseteq F_2$, similarly $F_2 \subseteq F_1$ so $F_1 = F_2$. Hence $Q$ is injective and $|\text{Max}_\sigma(A)| \leq |\text{Max}(\sigma(A))|$.

Definition 4.25. A triangle algebra $A$ is called simple if it has only two IVRL-filters $\{1\}$ and $A$. A state triangle algebra $(A, \sigma)$ is called simple if $\sigma(A)$ is simple. A state triangle algebra $(A, \sigma)$ is called simple relative to $F_\alpha(A)$ if it has only two state IVRL-filter $\{1\}$ and $A$.

Example 4.26. In Example 4.26, clearly $(A, \sigma)$ is simple state triangle algebra.

Proposition 4.27. If $(A, \sigma)$ is a state triangle algebra that $A$ is simple, then $\sigma(A)$ is simple and $(A, \sigma)$ is simple.

Proof. Let $F$ be an IVRL-filter of $\sigma(A), F \neq \{1\}$. By Proposition 4.27 (i), the set $\sigma^{-1}(F)$ is an IVRL-filter of $A$. So $\sigma^{-1}(F) = \{1\}$ or $\sigma^{-1}(F) = A$. If $x \in F$, then $\sigma(x) = x$ and $x \in \sigma^{-1}(F)$. So $F \subseteq \sigma^{-1}(F)$ and $\sigma^{-1}(F) = A$. It follows that $0 \in \sigma^{-1}(F)$ and $\sigma(0) \in F$, thus $0 \in F, F = \sigma(A)$.

Proposition 4.28. Let $(A, \sigma)$ be a state triangle algebra and $(A, \sigma)$ is simple relative to $[A]_\sigma$. Then $(A, \sigma)$ is simple.

Proof. Let $F$ be an IVRL-filter of $\sigma(A), F \neq \{1\}$. By Proposition 4.28 (i) the set $-\sigma(F)$ is an IVRL-$\sigma$-filter of $(A, \sigma)$. Since $F \subseteq -\sigma(F)$, $-\sigma(F) \neq \{1\}$. So $-\sigma(F) = A$ and $0 \in -\sigma(F)$. Hence $0 = \sigma(0) \in F$ and $F = \sigma(A)$.

Corollary 4.29. Let $(A, \sigma)$ be a simple state triangle algebra relative to $F_\alpha(A)$. We know that $\text{Ker}(\sigma) \in [A]_\sigma$ and $\text{Ker}(\sigma) \neq A$. Thus $\text{Ker}(\sigma) = \{1\}$ and $\sigma$ is faithful.

Proposition 4.30. Let $(A, \sigma)$ be a state triangle algebra. Then $\text{Rad}(\text{Ker}(\sigma)) = \sigma^{-1}(\text{Rad}(\{1\}))$.

Proof. By Lemma 4.3 (6), for all $n \in \mathbb{N}$, we have

\[
a \in \text{Rad}(\text{Ker}(\sigma)) \iff \sigma(-\nu a^n) \rightarrow \nu a \in \text{Ker}(\sigma)
\]

\[
\iff \sigma(-\nu a^n) \rightarrow \nu a = 1
\]

\[
\iff \sigma(-\nu a^n) \rightarrow \nu a = 1
\]

\[
\iff \sigma(a) \in \text{Rad}(\{1\})
\]

\[
\iff a \in \sigma^{-1}(\text{Rad}(\{1\}))
\]

Definition 4.31. A triangle algebra $A$ is called semisimple if $\text{Rad}(A) = \{1\}$. A state triangle algebra $(A, \sigma)$ is called semisimple if $\sigma(A)$ is simple, that is $\text{Rad}(\sigma(A)) = \{1\}$. A state triangle algebra $(A, \sigma)$ is called semisimple relative to $[A]_\sigma$ if $\text{Rad}_\sigma(A) = \{1\}$.
Example 4.32. In Example 4.31, \( \sigma(A) = \{[0,0], [1,1]\} \) is simple, but \( \ker(\sigma) = \{[c,c], [c,d], [d,d], [d,1], [1,1]\} \) is not simple.

Theorem 4.33. Let \((A, \sigma)\) be a state triangle algebra. Then the following are equivalent:

(i) \((A, \sigma)\) is simple relative to \([A]_{\sigma}\).

(ii) \((A, \sigma)\) is simple and \(\sigma\) is faithful.

Proof. (i \( \Rightarrow \) ii) It is clear by Proposition 4.28 and corollary 4.29.

(ii \( \Rightarrow \) i) Let \(F \in [A]_{\sigma}\), \(F \neq \{1\}\). Then \(\sigma(F) = F \cap (\sigma(A))\) is an IVRL-filter of \(A\). Since \(\sigma(A)\) is simple, \(\sigma(A) = \{1\}\) or \(\sigma(F) = \sigma(A)\). But \(\sigma\) is faithful and \(F \neq \{1\}\). So \(\sigma(F) \neq \{1\}\) and \(\sigma(F) = \sigma(A)\). Thus \(0 \in \sigma(F)\) and \(0 \in F\). Therefore \(F = A\).

Proposition 4.34. Let \((A, \sigma)\) be a state triangle algebra. Then \(\operatorname{Rad}(\sigma(A)) = \sigma(\operatorname{Rad}_\sigma(A))\).

Proof. Let \(x \in \operatorname{Rad}_\sigma(A)\). Then \(x = \sigma(y), y \in \operatorname{Rad}_\sigma(A)\). Let \(M\) be an IVRL-extended maximal filter of \(\sigma(A)\). By Proposition 4.30 (i), \(\neg \sigma(M)\) is an IVRL-extended maximal \(\sigma\)-filter of \((A, \sigma)\). Then \(y \in \neg \sigma(M), \sigma(y) \in M\). So \(x \in M\) and \(x \in \operatorname{Rad}(\sigma(A))\). Conversely, let \(x \in \operatorname{Rad}(\sigma(A))\) and \(M\) be an IVRL-extended maximal \(\sigma\)-filter of \((A, \sigma)\). By Proposition 4.30 (ii), \(\sigma(M)\) is an IVRL-extended maximal filter of \(\sigma(A)\). So \(x \in \sigma(M) = M \cap \sigma(A)\) and \(x \in M\). Thus \(x \in \operatorname{Rad}_\sigma(A)\). Since \(x = \sigma(x), x \in \sigma(\operatorname{Rad}_\sigma(A))\).

Proposition 4.35. Let \((A, \sigma)\) be a state triangle algebra. Then \((A, \sigma)\) is semisimple and \(\sigma\) is faithful iff \((A, \sigma)\) is semisimple relative to \([A]_{\sigma}\).

Proof. By Proposition 4.32, \(\sigma(\operatorname{Rad}_\sigma(A)) = \sigma(\operatorname{Rad}(\sigma(A))) = \{1\}\). So \(\operatorname{Rad}_\sigma(A) \subseteq \ker(\sigma) = \{1\}\) and \(\operatorname{Rad}_\sigma(A) = \{1\}\).

Conversely, \(\operatorname{Rad}(\sigma(A)) = \sigma(\operatorname{Rad}_\sigma(A)) = \{1\}\), so \((A, \sigma)\) is semisimple. Let \(x \in A\). Then \(\sigma(x) = 1\). If \(x \neq 1\), then \(x \notin \operatorname{Rad}_\sigma(A)\). There is an IVRL-extended maximal \(\sigma\)-filter \(M\) so that \(x \notin M\). By Theorem 4.12, there are \(f \in M\) and \(m, n \geq 1\) such that \((f + \sigma(\nu x))^m = 0\). Therefore \(f^m = 0\) and \(0 \in F\), a contradiction. So \(\nu x = 1, x = 1\) and \(\sigma\) is faithful.

Definition 4.36. A state triangle algebra \((A, \sigma)\) is called local if \(\sigma(A)\) is local. A state triangle algebra \((A, \sigma)\) is called local relative to \([A]_{\sigma}\) if it has only an IVRL-extended maximal \(\sigma\)-filter.

Example 4.37. In Example 4.31, \((A, \sigma)\) is local.

Theorem 4.38. Let \((A, \sigma)\) be a state triangle algebra. \((A, \sigma)\) is local relative to \([A]_{\sigma}\) iff \((A, \sigma)\) is local.

Proof. Let \(F\) be the only IVRL-extended maximal \(\sigma\)-filter of \((A, \sigma)\). We must show that \(\sigma(F)\) is the only IVRL-extended maximal filter of \(\sigma(A)\). If \(\sigma(F) = \sigma(A)\), then \(0 \in \sigma(F)\) and \(0 \in F\), a contradiction. So \(\sigma(F)\) is an proper IVRL-filter of \(\sigma(A)\). Let \(G\) be an IVRL-filter of \(\sigma(A)\), \(G \neq \sigma(A)\) and \(x \in G\). By Proposition 4.30 (ii), \(\neg \sigma(G)\) is an IVRL-extended filter of \((A, \sigma)\). Let \(- \sigma(G) = A\). Then \(0 \in \neg \sigma(G)\) and \(0 \in G\), a contradiction. So \(- \sigma(G)\) is an proper \(\sigma\)-filter of \((A, \sigma)\). Thus \(- \sigma(G) \subseteq F\). Since \(\nu x = \sigma(\nu x) \in G\), \(\nu x \in \neg \sigma(G)\) so \(\nu x \in F\). We know that \(x = \sigma(x), \nu x \in \sigma(F)\). Thus \(G \subseteq \sigma(F)\) and \((A, \sigma)\) is local.

Conversely, let \(G\) be the only IVRL-extended maximal \(\sigma\)-filter of \(\sigma(A)\). By Proposition 4.30 (ii), \(- \sigma(G)\) is an IVRL-extended maximal \(\sigma\)-filter of \((A, \sigma)\). We prove that \(- \sigma(G)\) is only IVRL-extended maximal \(\sigma\)-filter. Let \(F \in [A]_{\sigma}\), \(F \neq A\). Then \(F \cap \sigma(A) = \sigma(F)\) is a proper IVRL-filter of \(\sigma(A)\). Thus \(F \cap \sigma(A) \subseteq G\). If \(x \in F\), then \(\sigma(\nu x) \in F \cap \sigma(A) \subseteq G\). So \(\nu x \in \neg \sigma(G)\) and \(F \subseteq \neg \sigma(G)\).

Proposition 4.39. Let \((A, \sigma)\) be a state triangle algebra and \(F\) be a proper \(\sigma\)-IVRL-filter of \((A, \sigma)\). Then \(F \subseteq \sigma(M_\sigma(A)) := \{x \in A \mid \operatorname{ord}(\sigma(\nu x)) = \infty\}\).

Proof. Let \(x \in F\). Then \((x + \sigma(\nu x))^m \in F\) for every \(n \in \mathbb{N}\). If \(x \notin \sigma(M_\sigma(A))\), then there is \(m \geq 1\) such that \((\sigma(\nu x))^m = 0\). We have \((x + \sigma(\nu x))^m \leq (\sigma(\nu x))^m\), so \((x + \sigma(\nu x))^m = 0\). Thus \(0 \notin F\), which is a contradiction. Therefore \(F \subseteq \sigma(M_\sigma(A))\).

Theorem 4.40. Let \((A, \sigma)\) be a state triangle algebra. Then \((A, \sigma)\) is local iff \(M_\sigma(A)\) is a \(\sigma\)-IVRL-filter of \((A, \sigma)\). In this case \(M_\sigma(A)\) is the only IVRL-extended maximal \(\sigma\)-filter of \((A, \sigma)\).

Proof. Let \((A, \sigma)\) be local. Then \((A, \sigma)\) has only an IVRL-extended maximal \(\sigma\)-filter \(F\). By Proposition 4.30 (ii), \(F \subseteq M_\sigma(A)\). If \(x \in M_\sigma(A)\) and \(x \notin F\), then \(x \notin \sigma(M_\sigma(A))\). Since \(F\) is the only IVRL-extended maximal filter, \([x]_{\sigma} = A\) so \(0 \in [x]_{\sigma}\) thus there is \(n \geq 1\) such that \(\sigma((x + \sigma(\nu x))^n) \geq (\sigma(\nu x))^n \geq (\sigma(x + \sigma(\nu x))^n) \geq (\sigma(x + \sigma(\nu x))^n \geq 2^n(\nu x)^n)\). Thus \(2^n(\nu x)^n = 0\), that is \(x \notin \sigma(M_\sigma(A))\), which is a contradiction. So \(\sigma(M_\sigma(A)) \subseteq F\) and \(M_\sigma(A) = F\). Therefore \(M_\sigma(A)\) is an IVRL-\(\sigma\)-filter of \((A, \sigma)\).

Conversely, let \(M_\sigma(A)\) be an IVRL-\(\sigma\)-filter of \((A, \sigma)\) and \(F\) be a proper IVRL-\(\sigma\)-filter of \((A, \sigma)\). By Proposition 4.30 (ii), \(F \subseteq M_\sigma(A)\), so \(M_\sigma(A)\) is the only IVRL-extended maximal \(\sigma\)-filter of \((A, \sigma)\). Hence \((A, \sigma)\) is local relative to \([A]_{\sigma}\) and so \((A, \sigma)\) is local.
5 Conclusions

In this paper, motivated by the previous research of triangle algebras, we extended the concept of state triangle algebras. We introduce and study these algebras and their state IVRL-filters. Using the notion of state IVRL-filters, we characterize kinds of state IVRL-filters and focus on these and the relations between these IVRL-filters were determined. As an application of state triangle algebras, we show the relation between these algebras and some special sets of triangle algebras such as radical of an IVRL-filter and local triangle algebras and etc. We proved that \((A, \sigma)\) is local relative to \([A]_\sigma\) iff \((A, \sigma)\) is local.

The investigation of other such generalizations can be an interesting object for further work.

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References