

## SEMI $\theta$ -COMPACTNESS IN INTUITIONISTIC FUZZY TOPOLOGICAL SPACES

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ABSTRACT. The purpose of this paper is to construct the concept of semi  $\theta$ -compactness in intuitionistic fuzzy topological spaces. We give some characterizations of semi  $\theta$ -compactness and locally semi  $\theta$ -compactness. Also, we compare these concepts with some other types of compactness in intuitionistic fuzzy topological spaces.

### 1. Introduction

The concept of a fuzzy set was introduced by Zadeh [11], and later Atanassov [1, 2] generalized this idea to intuitionistic fuzzy sets. On the other hand, Coker [3] introduced the notions of intuitionistic fuzzy topological spaces, fuzzy continuity and other related concepts. In this paper, we introduce the concepts of semi  $\theta$ -compactness and locally semi  $\theta$ -compactness in intuitionistic fuzzy topological spaces. We also consider some characterizations and basic properties for these concepts. For preliminary notions, we refer to the papers [1, 3, 5, 6, 8]. The words “neighbourhood”, “continuous” and “irresolute” will be abbreviated as respectively “*nb*”, “*cont.*” and “*i*”.

### 2. Preliminaries

We first review the fundamental definitions.

**Definition 2.1.** [2] Let  $X$  be a nonempty fixed set. An intuitionistic fuzzy set (IFS)  $U$  is an object having the form  $U = \{ \langle x, \mu_U(x), \gamma_U(x) \rangle : x \in X \}$  where the functions  $\mu_U : X \rightarrow I$  and  $\gamma_U : X \rightarrow I$  denote respectively the degree of membership (namely  $\mu_U(x)$ ) and the degree of nonmembership (namely  $\gamma_U(x)$ ) of each element  $x \in X$  to the set  $U$ , and  $0 \leq \mu_U(x) + \gamma_U(x) \leq 1$  for each  $x \in X$ .

The reader may consult [3, 4, 6] for the operations and properties on intuitionistic fuzzy points (IFP) and images and preimages of IFS's.

**Definition 2.2.** [3] An intuitionistic fuzzy topology (IFT) on a nonempty set  $X$  is a family  $\Psi$  of IFS's in  $X$  containing  $\underline{0}$ ,  $\underline{1}$  and closed under finite infima and arbitrary suprema.

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In this case the pair  $(X, \Psi)$  is called an intuitionistic fuzzy topological space (IFTS) and each IFS in  $\Psi$  is known as an intuitionistic fuzzy open set (IFOS) in  $X$ . The complement  $\bar{U}$  of an IFOS  $U$  in an IFTS  $(X, \Psi)$  is called an intuitionistic fuzzy closed set (IFCS, for short), in  $X$ .

**Definition 2.3.** [3] Let  $X$  be a nonempty set and the IFS's,  $U$  and  $V$  be of the form  $U = \{\langle x, \mu_U(x), \gamma_U(x) \rangle : x \in X\}$ ,  $V = \{\langle x, \mu_V(x), \gamma_V(x) \rangle : x \in X\}$  and let  $\{U_j : j \in J\}$  be an arbitrary family of IFS's in  $X$ . Then

- (i)  $U \leq V$  iff  $\mu_U(x) \leq \mu_V(x)$  and  $\gamma_U(x) \geq \gamma_V(x), \forall x \in X$ ;
- (ii)  $\bar{U} = \{\langle x, \gamma_U(x), \mu_U(x) \rangle : x \in X\}$ ;
- (iii)  $\cap U_j = \{\langle x, \wedge \mu_{U_j}(x), \vee \gamma_{U_j}(x) \rangle : x \in X\}$ ;
- (iv)  $\cup U_j = \{\langle x, \vee \mu_{U_j}(x), \wedge \gamma_{U_j}(x) \rangle : x \in X\}$ ;
- (v)  $\underline{1} = \{\langle x, 1, 0 \rangle : x \in X\}$  and  $\underline{0} = \{\langle x, 0, 1 \rangle : x \in X\}$ ;
- (vi)  $\bar{\bar{U}} = U, \bar{\underline{0}} = \underline{1}$  and  $\bar{\underline{1}} = \underline{0}$ ;
- (vii)  $\square U = \{\langle x, \mu_U(x), 1 - \mu_U(x) \rangle : x \in X\}$ ;
- (viii)  $\langle U = \{\langle x, 1 - \gamma_U(x), \gamma_U(x) \rangle : x \in X\}$ .

**Proposition 2.4.** [3] Let  $(X, \Psi)$  be an IFTS on  $X$ . Then, we can construct the following two IFTS's:

- (i)  $\Psi_{0,1} = \{\square U : U \in \Psi\}$ ;
- (ii)  $\Psi_{0,2} = \{\langle U : U \in \Psi\}$ .

**Definition 2.5.** [8] Let  $X, Y$  be nonempty sets. Let  $U = \langle x, \mu_U(x), \gamma_U(x) \rangle$ ,

$V = \langle y, \mu_V(y), \gamma_V(y) \rangle$  be IFS's of  $X$  and  $Y$ , respectively. Then  $U \times V$  is an IFS of  $X \times Y$  defined by:

$$(U \times V)(x, y) = \langle (x, y), \min(\mu_U(x), \mu_V(y)), \max(\gamma_U(x), \gamma_V(y)) \rangle.$$

**Definition 2.6.** [8] Let  $(X, \Psi), (Y, \Phi)$  be IFTS's and  $A \in \Psi, B \in \Phi$ . We say that  $(X, \Psi)$  is a product related to  $(Y, \Phi)$  if for any IFS's  $U$  of  $X$  and  $V$  of  $Y$ , whenever  $(\bar{A} \not\geq U$  and  $\bar{B} \not\geq V) \Rightarrow (\bar{A} \times \underline{1} \cup \underline{1} \times \bar{B} \geq U \times V)$ , there exist  $A_1 \in \Psi, B_1 \in \Phi$  such that  $\bar{A}_1 \geq U$  or  $\bar{B}_1 \geq V$  and  $\bar{A}_1 \times \underline{1} \cup \underline{1} \times \bar{B}_1 = \bar{A} \times \underline{1} \cup \underline{1} \times \bar{B}$ .

**Definition 2.7.** [9] An IFP  $c(a, b)$  is said to be an intuitionistic fuzzy  $\theta$ -cluster point (IF $\theta$ -cluster point) of an IFOS  $U$  iff for each  $A \in N_\varepsilon^q(c(a, b)), cl(A) \not\geq U$ .

The set of all IF $\theta$ -cluster points of  $U$  is called the intuitionistic fuzzy  $\theta$ -closure of  $U$  and denoted by  $cl_\theta(U)$ . An IFS  $U$  will be called IF $\theta$ -closed (IF $\theta$ CS) iff  $U = cl_\theta(U)$ . The complement of an IF $\theta$ -closed set is IF $\theta$ -open (IF $\theta$ OS).

**Lemma 2.8.** [10] Let  $X, Y$  be IFTS's such that  $X$  is a product related to  $Y$ . Then the product  $U \times V$  of IF $\theta$ OS  $U$  of  $X$  and IF $\theta$ OS  $V$  of  $Y$  is an IF $\theta$ OS of  $X \times Y$ .

**Definition 2.9.** An IFS  $U$  of an IFTS  $X$  is called an  $\varepsilon$ -nbd [ $\varepsilon\theta$ -nbd] [10] of an IFP  $c(a, b)$ , if there exists an IFOS (IF $\theta$ OS)  $U$  in  $X$  such that  $c(a, b) \in U \leq U$ .

The family of all  $\varepsilon$ -nbd ( $\varepsilon\theta$ -nbd) of an IFP  $c(a, b)$  will be denoted by  $N_\varepsilon(N_\varepsilon^\theta)(c(a, b))$ .

**Definition 2.10.** An IFS  $U$  of an IFTS  $X$  is said to be an IFsemiopen(IFSOS)(IFpreopen(IFPOS)) iff  $U \leq cl(int(U))(U \leq int(cl(U)))$ .

**Definition 2.11.** Let  $(X, \Psi)$  and  $(Y, \Phi)$  be two IFTS's. A function  $f : X \rightarrow Y$  is said to be:

(i) IF-cont.[3](IFsemi-cont.(IFS-cont.,)[7]) if the preimage of each IFOS in  $Y$  is IFOS(IFSOS) in  $X$ .

(ii) IFi (IFsuper i) function if the preimage of each IFSOS in  $Y$  is IFSOS (IFOS) in  $X$ [10].

(iii) IFstrongly  $\theta$ -(resp. IF $\theta$ -, IFfaintly, IFS $\theta$ -)cont. if the preimage of each IFOS(resp. IF $\theta$ OS, IF $\theta$ OS, IFSOS) of  $Y$  is IF $\theta$ OS(resp. IF $\theta$ OS, IFOS, IF $\theta$ OS) in  $X$ [9][10].

(iv) IFweakly cont.[7] if for each IFOS  $V$  of  $Y$ ,  $f^{-1}(V) \leq int(f^{-1}(cl(V)))$ .

(v) IF-[10](resp. IFsemi-[10], IFpre-, IFsuper semi-, IF $\theta$ -, IFfaintly-[10])open if the image of each IFOS(resp. IFOS, IFOS, IFSOS, IF $\theta$ OS, IF $\theta$ OS) of  $X$  is IFOS(resp. IFSOS, IFPOS, IFOS, IF $\theta$ OS, IFOS) in  $Y$ .

**Definition 2.12.** An IFS  $U$  of an IFTS  $(X, \Psi)$  is said to be IF[3](IF $\theta$ -)compact relative to  $X$  iff every an IF( $\theta$ -)open cover of  $U$  has a finite subcover.

**Definition 2.13.** An IFTS  $(X, \Psi)$  is called :

(i) IFcompact [3](resp. IFS-compact, IF $\lambda$ -compact, IF $\theta$ -compact) iff every an IFopen (resp. semiopen,  $\lambda$ -open,  $\theta$ -open) cover of  $X$  has a finite subcover which covers  $X$ .

(ii) Locally IF( $\theta$ -)compact if for each IFP  $c(a, b)$  in  $X$ , there is  $U \in N_\varepsilon(c(a, b))$  such that  $\mu_U(c) = 1$ ,  $\gamma_U(c) = 0$  and  $U$  is an IF( $\theta$ -)compact relative to  $X$ .

(iii) IF-submaximal if each dense subset of  $X$  is IFOS.

(iv) IFS-closed iff every an IFsemiopen cover of  $X$  has a finite subfamily whose closures cover  $X$ .

(v) IF-regular iff for each  $U \in \Psi$ ,  $U = \vee\{U_j : U_j \in \Psi, cl(U_j) \leq U\}$ .

**Lemma 2.14.** Let  $f : X \rightarrow Y$  be an IFS-cont. and IFpreopen function, then  $f^{-1}(V)$  is an IFSOS in  $X$  for each an IFSOS  $V$  in  $Y$ .

*Proof.* Let  $V$  be an IFSOS in  $Y$ , then there exists an IFOS  $U$  of  $X$  such that  $U \leq V \leq cl(U)$ . Now,  $f^{-1}(U) \leq f^{-1}(V) \leq f^{-1}(cl(U))$  and since  $f$  is an IFpreopen function we have,  $f^{-1}(U) \leq f^{-1}(V) \leq f^{-1}(cl(U)) \leq cl(f^{-1}(U))$ . Since  $f$  is an IFS cont.,  $f^{-1}(U)$  is an IFSOS in  $X$ , implies there is an IFOS  $G$  of  $X$  such that  $G \leq f^{-1}(U) \leq f^{-1}(V) \leq cl(f^{-1}(U)) \leq cl(G)$ . Hence  $f^{-1}(V)$  is an IFSOS in  $X$ .  $\square$

### 3. Semi $\theta$ -compactness in IFTS's

**Definition 3.1.** (i) A family  $\{\langle x, \mu_{U_j}, \gamma_{U_j} \rangle : j \in J\}$  of IFSOS's(IF $\theta$ OS's) in  $X$  such that  $\vee\{\langle x, \mu_U(x), \gamma_U(x) \rangle : x \in X\} = \underline{1}$ , is called an IFsemi( $\theta$ -)open cover of  $X$ .

(ii) A finite subfamily  $\{\langle x, \mu_{U_j}, \gamma_{U_j} \rangle : j = 1, 2, \dots, n\}$  of an IFsemi( $\theta$ -)open cover, which is also a semi( $\theta$ -)open cover, i.e.  $\bigvee_{j=1}^n \{\langle x, \mu_{U_j}, \gamma_{U_j} \rangle\} = \underline{1}$ , is called a finite subcover of  $\{\langle x, \mu_{U_j}, \gamma_{U_j} \rangle : j \in J\}$ .

**Definition 3.2.** A family  $\{\langle x, \mu_{U_j}, \gamma_{U_j} \rangle : j \in J\}$  of IFS's satisfies the  $\theta$ -finite intersection property ( $\theta$ -FIP) iff for every finite subfamily  $\{\langle x, \mu_{U_j}, \gamma_{U_j} \rangle : j = 1, 2, \dots, n\}$  of the family, we have  $\bigwedge_{j=1}^n \{\langle x, \mu_{U_j}, \gamma_{U_j} \rangle : j \in J\} \neq \underline{0}$ .

**Definition 3.3.** An IFTS( $X, \Psi$ ) is called fuzzy semi  $\theta$ -compact (IFS $\theta$ -compact) iff every an IFsemiopen cover of  $X$  has a finite subcollection(subcover) of IF $\theta$ OS's, which covers  $X$ .

**Definition 3.4.** An IFS  $U$  of an IFTS( $X, \Psi$ ) is said to be an IFS $\theta$ -compact relative to  $X$  if for every family  $\{U_j : j \in J\}$  of IFSOS's in  $X$  such that  $U \subseteq \bigvee_{j \in J} U_j$ , there is a finite subfamily  $\{U_j : j = 1, 2, \dots, n\}$  of IF $\theta$ OS's such that  $U \subseteq \bigvee_{j=1}^n U_j$ .

**Remark 3.5.** From the above definition and some other types of IF compactness, one can illustrate the following implications:

$$\text{IFS}\theta\text{-compact} \Rightarrow \text{IFS-compact} \Rightarrow \text{IF}\lambda\text{-compact} \Rightarrow \text{IF-compact} \Rightarrow \text{IF}\theta\text{-compact}$$

**Theorem 3.6.**  $(X, \Psi)$  is IFS $\theta$ -compact iff every family  $U = \{U_j : j \in J\}$  of IFSCS's in  $X$  having the  $\theta$ -FIP,  $\bigwedge_{j \in J} U_j \neq \underline{0}$ .

*Proof.* ( $\implies$ ): Let  $U = \{U_j : j \in J\}$  be a family of IFSCS's in  $X$  having the  $\theta$ -FIP. Suppose that  $\bigwedge_{j \in J} U_j = \underline{0}$ , then  $\bigvee_{j \in J} \overline{U_j} = \underline{1}$ . From the IFS $\theta$ -compactness and the fact that  $\{\overline{U_j} : j \in J\}$  is IFSOSs, there is a finite subfamily  $\{\overline{U_j} : j = 1, 2, \dots, n\}$  of IF $\theta$ OS's such that  $\bigvee_{j=1}^n \overline{U_j} = \underline{1}$ . Then  $\bigwedge_{j=1}^n U_j = \overline{\bigvee_{j=1}^n \overline{U_j}} = \underline{0}$ , which is a contradiction to the  $\theta$ -FIP. Hence  $\bigwedge_{j \in J} U_j \neq \underline{0}$ .

( $\impliedby$ ): Let  $U = \{U_j : j \in J\}$  be an IFsemiopen cover of  $X$ . Hence  $\{\overline{U_j} : j \in J\}$  is a family of IFSCSs having the  $\theta$ -FIP. Then from the hypothesis, we have  $\bigwedge_{j \in J} \overline{U_j} \neq \underline{0}$  which implies  $\bigvee_{j \in J} U_j \neq \underline{1}$  and in contradiction with  $\{U_j : j \in J\}$  being an IFsemiopen cover of  $X$ .  $\square$

**Theorem 3.7.** An IFTS( $X, \Psi$ ) is IFS $\theta$ -compact iff  $(X, \Psi_{0,1})$  is IFS $\theta$ -compact.

*Proof.* ( $\implies$ ) Let  $\{\square U_j : j \in J\}$  be an IFsemiopen cover of  $X$  in  $(X, \Psi_{0,1})$ . Hence  $\bigvee(\square U_j) = \underline{1} \implies \bigvee \mu_{U_j} = \underline{1}, \bigwedge \gamma_{U_j} = \underline{1} - \bigvee \mu_{U_j} = \underline{0}$ . Since  $(X, \Psi)$  is an IFS $\theta$ -compact, there is  $\{U_j : j = 1, 2, \dots, n\}$  of IF $\theta$ OS's such that  $\bigvee_{j=1}^n U_j = \underline{1}$ . Now we have,  $\bigvee_{j=1}^n \mu_{U_j} = \underline{1}$  and  $\bigwedge_{j=1}^n (1 - \mu_{U_j}) = 1 - \bigvee_{j=1}^n \mu_{U_j} = \underline{0}$ . Hence  $\{\square U_j : j \in J\}$  has a subcover of IF $\theta$ OS's and so  $(X, \Psi_{0,1})$  is an IFS $\theta$ -compact.

( $\Leftarrow$ ) Let  $\{U_j : j \in J\}$  be an IFsemiopen cover of  $X$  in  $(X, \Psi)$ . Since  $\vee U_j = \underline{1}$ , we have  $\vee \mu_{U_j} = \underline{1}$ ,  $\wedge \gamma_{U_j} = \underline{1} - \vee \mu_{U_j} = \underline{0}$ . Since  $(X, \Psi_{0,1})$  is an IFS $\theta$ -compact, there is a subfamily  $\{U_j : j = 1, 2, \dots, n\}$  of IF $\theta$ OS's such that  $\bigvee_{j=1}^n (\bigcap U_j) = \underline{1}$  i.e.,  $\bigvee_{j=1}^n \mu_{U_j} = \underline{1}$  and  $\bigwedge_{j=1}^n (1 - \mu_{U_j}) = \underline{0}$ . Hence  $\mu_{U_j} = \underline{1} - \gamma_{U_j} \implies \bigvee_{j=1}^n \mu_{U_j} = \bigvee_{j=1}^n (1 - \gamma_{U_j}) \implies \underline{1} = \underline{1} - \bigwedge_{j=1}^n \gamma_{U_j} \implies \bigwedge_{j=1}^n \gamma_{U_j} = \underline{0} \implies \bigvee_{j=1}^n U_j = \underline{1}$  i.e.,  $\{U_j : j \in J\}$  has a finite subcover of IF $\theta$ OS's. Hence  $(X, \Psi)$  is IFS $\theta$ -compact.  $\square$

**Theorem 3.8.** *An IFTS  $(X, \Psi)$  is IFS $\theta$ -compact iff  $(X, \Psi_{0,2})$  is IFS $\theta$ -compact.*

*Proof.* Similar to the proof of Theorem 3.7.  $\square$

**Theorem 3.9.** *Every IFS $\theta$ -compact space  $X$  which is submaximal regular is IFS-closed.*

*Proof.* Let  $\{U_j : j \in J\}$  be an IFsemiopen cover of  $X$ . Then  $cl(U_j) = cl(H_j)$  where  $H_j$  is an IFOS in  $X$ . Since  $X$  is submaximal regular space, then  $\{cl(U_j) : j \in J\}$  is an IFopen cover of  $X$  and consequently an IFsemiopen cover of  $X$ . Since  $X$  is an IFS $\theta$ -compact, then there is a subfamily  $\{cl(U_j) : j = 1, 2, \dots, n\}$  of IF $\theta$ OS's such that  $\bigvee_{j=1}^n cl(U_j) = \underline{1}$ . Hence  $X$  is IFS-closed.  $\square$

**Theorem 3.10.** *Every IF $\theta$ -compact space  $X$  which is submaximal regular is IFS $\theta$ -compact.*

*Proof.* Let  $\{U_j : j \in J\}$  be an IFsemiopen cover of  $X$ . Since every IFSOS in an IFsubmaximal regular  $X$  is an IF $\theta$ OS, then  $\{U_j : j \in J\}$  is an IF $\theta$ -open cover of  $X$ . Since  $X$  is IF $\theta$ -compact, then there is a subfamily  $\{U_j : j = 1, 2, \dots, n\}$  of IF $\theta$ OS's such that  $\bigvee_{j=1}^n U_j = \underline{1}$ . Hence  $X$  is IFS $\theta$ -compact.  $\square$

**Corollary 3.11.** *Every IF-(IFS-, IF $\lambda$ -)compact space  $X$  which is submaximal regular is IFS $\theta$ -compact.*

**Lemma 3.12.** *If  $U$  is an IF $\theta$ CS of an IFTS  $(X, \Psi)$  and  $c(a, b) \notin U$ , then there is an IFOS  $V$  of  $X$  such that  $c(a, b) \in cl(V)$ , for each  $a, b \in (0, 1)$ .*

*Proof.* Let  $U$  be an IF $\theta$ CS and  $c(a, b) \notin U$ . Hence  $c(1 - a, 1 - b)q\bar{U}$ , for each  $a, b \in (0, 1)$ . From the definition of IF $\theta$ OS  $\bar{U}$ , there is an IFOS  $V = \langle x, \mu_V, \gamma_V \rangle$  of  $x$  such that  $c(1 - a, 1 - b)qcl(V) \leq \bar{U}$ , where  $cl(V) = \langle x, \wedge \mu_{G_j}, \vee \gamma_{G_j} \rangle$  and  $\{\langle x, \mu_{G_j}, \gamma_{G_j} \rangle : j \in J\}$  is the family of IFCS's containing  $V$ . Hence  $1 - a > \vee \gamma_{G_j}$  or  $1 - b < \wedge \mu_{G_j}$  which implies  $a < \wedge \mu_{G_j}$  and  $b > \vee \gamma_{G_j}$ . Hence  $c(a, b) \in cl(V)$ .  $\square$

**Theorem 3.13.** *If  $U$  is IF $\theta$ -closed in an IFS $\theta$ -compact space  $X$ , then  $U$  is IFS $\theta$ -compact relative to  $X$ .*

*Proof.* Let  $V = \{V_j : j \in J\}$  where  $V_j = \langle y, \mu_{V_j}, \gamma_{V_j} \rangle : j \in J$ , is an IFsemiopen cover of  $U$ . For  $x(a, b) \notin U$  and by Lemma 3.12, there is an IFOS  $G$  of  $X$  such that  $x(a, b) \in G$ . Hence  $\{V_j : j \in J\} \vee \{cl(G(x)) : x \in \bar{U}\}$  is an IFsemiopen cover

of  $X$ . Since  $X$  is IFS $\theta$ -compact, there is a finite subfamily  $\{V_j : j = 1, 2, \dots, n\} \vee \{cl(G(x_i)) : i = 1, 2, \dots, m\}$  of IF $\theta$ OS's which covers  $X$  and consequently  $\{V_j : j = 1, 2, \dots, n\}$  covers  $U$ . Hence  $U$  is IFS $\theta$ -compact relative to  $X$ .  $\square$

**Theorem 3.14.** *Let  $(X, \Psi_1)$  and  $(Y, \Psi_2)$  be IFTS's such that  $(X, \Psi_1)$  is a product related to  $(Y, \Psi_2)$ , then  $X$  and  $Y$  are IFS $\theta$ -compact iff the product  $X \times Y$  is IFS $\theta$ -compact.*

*Proof.* ( $\Rightarrow$ ) Let  $\{U_j \times V_j : j \in J\}$  be an IFsemiopen cover of  $X \times Y$ , where  $U_j$ 's and  $V_j$ 's are IFSOS's in  $X$  and  $Y$ , respectively. Then  $\{U_j : j \in J\}$  and  $\{V_j : j \in J\}$  are IFsemiopen covers of  $X$  and  $Y$ , respectively. Thus there exist subfamilies  $\{U_j : j = 1, 2, \dots, n\}$  and  $\{V_j : j = 1, 2, \dots, n\}$  of IF $\theta$ OS's such that  $\bigvee_{j=1}^n U_j = \underline{1}_X$  and  $\bigvee_{j=1}^n V_j = \underline{1}_Y$ . From the related product of  $X$ ,  $Y$  and Lemma 2.8, we have  $\bigvee_{j \in J_1 \vee J_2} U_j \times V_j = \bigvee_{j \in J_1 \vee J_2} U_j \times \bigvee_{j \in J_1 \vee J_2} V_j = \underline{1}_X \times \underline{1}_Y$ . Thus  $X \times Y$  is IFS $\theta$ -compact.

( $\Leftarrow$ ) Let  $\{U_j : j \in J\}$  and  $\{V_j : j \in J\}$  be IFsemiopen covers of  $X$  and  $Y$ , respectively, where  $U_j$ 's and  $V_j$ 's are IFSOS's in  $X$  and  $Y$ , respectively. Then  $\{U_j \times V_j : j \in J\}$  is an IFsemiopen cover of  $X \times Y$ . Since  $X \times Y$  is IFS $\theta$ -compact, there exists a subfamily  $\{U_j \times V_j : j = 1, 2, \dots, n\}$  of IF $\theta$ OS's such that  $\bigvee_{j=1}^n U_j \times V_j = \underline{1} = \underline{1}_X \times \underline{1}_Y$ . Now, from the related product of  $X, Y$  and Lemma 2.8, we have:  $\bigvee_{j=1}^n U_j \times V_j = \bigvee_{j=1}^n U_j \times \bigvee_{j=1}^n V_j = \underline{1}_X \times \underline{1}_Y$ . Thus there are subfamilies  $\{U_j : j = 1, 2, \dots, n\}$  and  $\{V_j : j = 1, 2, \dots, n\}$  of IF $\theta$ OS's such that  $\bigvee_{j=1}^n U_j = \underline{1}_X$  and  $\bigvee_{j=1}^n V_j = \underline{1}_Y$ . Hence  $X$  and  $Y$  are IFS $\theta$ -compact spaces.  $\square$

#### 4. Functions and IFS $\theta$ -compact Space

**Theorem 4.1.** *If  $f : X \rightarrow Y$  is an IFS-cont. surjection function and  $U$  is IFS $\theta$ -compact relative to  $X$ , then  $f(U)$  is IF-compact relative to  $Y$ .*

*Proof.* Let  $V = \{V_j : j \in J\}$  where  $V_j = \{\langle y, \mu_{V_j}, \gamma_{V_j} \rangle : j \in J\}$ , is an IFopen cover of  $f(U)$ . Since  $f$  is IFS cont., then  $\{f^{-1}(V_j) : j \in J\}$  is an IFsemiopen cover of  $U$ . Since  $U$  is IFS $\theta$ -compact, there is a finite subfamily  $\{V_j : j = 1, 2, \dots, n\}$  of IF $\theta$ OS's such that  $U \subseteq \bigvee_{j=1}^n V_j$ , which implies that  $f(U) \subseteq \bigvee_{j=1}^n f f^{-1}(V_j) = \bigvee_{j=1}^n V_j$ . Hence  $f(U)$  is IF-compact relative to  $Y$ .  $\square$

**Corollary 4.2.** *If  $f : X \rightarrow Y$  is an IFS-cont. surjection function and  $X$  is IFS $\theta$ -compact, then  $Y$  is IF-compact.*

**Corollary 4.3.** *If  $f : X \rightarrow Y$  is an IF-cont. surjection function and  $U$  is IFS $\theta$ -compact relative to  $X$ , then  $f(U)$  is IF-compact relative to  $Y$ .*

**Corollary 4.4.** *If  $f : X \rightarrow Y$  is an IF-cont. surjection function and  $X$  is IFS $\theta$ -compact, then  $Y$  is IF-compact.*

**Theorem 4.5.** *Let  $f : X \rightarrow Y$  be an IFS $\theta$ -cont. and IF $\theta$ -open function. If  $U$  is IF $\theta$ -compact relative to  $X$ , then  $f(U)$  is IFS $\theta$ -compact relative to  $Y$ .*

*Proof.* Let  $V = \{V_j : j \in J\}$  where  $V_j = \{\langle y, \mu_{V_j}, \gamma_{V_j} \rangle : j \in J\}$ , be an IFsemiopen cover of  $f(U)$ . Since  $f$  is IFS $\theta$ -cont., then the family  $\{f^{-1}(V_j) : j \in J\}$  of IF $\theta$ OS's covers  $U$  [ Note  $\theta$ -open  $\Rightarrow$  open  $\Rightarrow$  semiopen ]. Since  $U$  is IF $\theta$ -compact, there is a finite subfamily  $\{f^{-1}(V_j) : j = 1, 2, \dots, n\}$  of IF $\theta$ OS's such that  $U \subseteq \bigvee_{j=1}^n f^{-1}(V_j)$ .

Since  $f$  is IF $\theta$ -open, we have  $f(U) \subseteq f(\bigvee_{j=1}^n f^{-1}(V_j)) = \bigvee_{j=1}^n ff^{-1}(V_j) = \bigvee_{j=1}^n V_j$ . Hence  $f(U)$  is IFS $\theta$ -compact relative to  $Y$ .  $\square$

**Corollary 4.6.** *Let  $f : X \rightarrow Y$  be an IFS $\theta$ -cont. and IF $\theta$ -open function. If  $U$  is IFS $\theta$ -compact relative to  $X$ , then  $f(U)$  is IFS $\theta$ -compact relative to  $Y$ .*

**Corollary 4.7.** *Let  $f : X \rightarrow Y$  be an IFS $\theta$ -cont. and IF $\theta$ -open function. If  $X$  is IF $\theta$ -compact, then  $Y$  is IFS $\theta$ -compact.*

**Corollary 4.8.** *Let  $f : X \rightarrow Y$  be an IFS $\theta$ -cont. and IF $\theta$ -open function. If  $X$  is IFS $\theta$ -compact, then so is  $Y$ .*

**Theorem 4.9.** *If  $f : X \rightarrow Y$  is an IFfaintly cont. function and  $U$  is IFS $\theta$ -compact relative to  $X$ , then  $f(U)$  is IF $\theta$ -compact relative to  $Y$ .*

*Proof.* Similar to the proof of Theorem 4.1.  $\square$

**Corollary 4.10.** *If  $f : X \rightarrow Y$  is an IFfaintly cont. function and  $X$  is IFS $\theta$ -compact, then  $Y$  is IF $\theta$ -compact.*

**Theorem 4.11.** *If  $f : X \rightarrow Y$  is an IFsuper  $i$  function and  $U$  is IFcompact relative to  $X$ , then  $f(U)$  is IFS $\theta$ -compact relative to  $Y$ .*

*Proof.* Similar to the proof of Theorem 4.1.  $\square$

**Corollary 4.12.** *If  $f : X \rightarrow Y$  is an IFsuper  $i$  function and  $X$  is IFcompact, then  $Y$  is IFS $\theta$ -compact.*

**Theorem 4.13.** *Let  $f : X \rightarrow Y$  be an IFsuper semiopen and IFstrongly  $\theta$ -cont. bijective function. If  $Y$  is IFcompact, then  $X$  is IFS $\theta$ -compact.*

*Proof.* Let  $\{U_j : j \in J\}$  be an IFsemiopen cover of  $X$ . Since  $f$  is IFsuper semiopen, then the family  $\{f(U_j) : j \in J\}$  is an IFopen cover of  $Y$ . Since  $Y$  is IFcompact, there is a subfamily  $\{f(U_j) : j = 1, 2, \dots, n\}$  of IFOS's which cover  $Y$ . Now,  $\{U_j : j = 1, 2, \dots, n\} = \{f^{-1}(f(U_j)) : j = 1, 2, \dots, n\}$  is an IF $\theta$ -open cover in  $X$  (since  $f$  is IFstrongly  $\theta$ -cont. bijective function). Hence  $X$  is IFS $\theta$ -compact.  $\square$

**Corollary 4.14.** *Let  $f : X \rightarrow Y$  be an IFsuper semiopen and IFstrongly  $\theta$ -cont. bijective function. If  $V$  is IFcompact relative to  $Y$ , then  $f^{-1}(V)$  is IFS $\theta$ -compact relative to  $X$ .*

**Theorem 4.15.** *Let  $f : X \rightarrow Y$  be an IFsemiopen and IFfaintly cont. surjection function. If  $Y$  is IFS $\theta$ -compact, then  $X$  is IFcompact.*

*Proof.* Similar to the proof of Theorem 4.13.  $\square$

**Corollary 4.16.** *Let  $f : X \rightarrow Y$  be an IFsemiopen and IFfaintly cont. surjection function. If  $V$  is IFS $\theta$ -compact relative to  $Y$ , then  $f^{-1}(V)$  is IF-compact relative to  $X$ .*

**Corollary 4.17.** *Let  $f : X \rightarrow Y$  be an IFsemiopen and IFfaintly cont. surjection function. If  $V$  is IFS $\theta$ -compact relative to  $Y$ , then  $f^{-1}(V)$  is IF $\theta$ -compact relative to  $X$ .*

**Theorem 4.18.** *Let  $f : X \rightarrow Y$  be an IFfaintly open and IF $\theta$ - cont. surjection function. If  $Y$  is IFS $\theta$ -compact, then  $X$  is IF $\theta$ -compact.*

*Proof.* Similar to the proof of Theorem 4.13.  $\square$

**Corollary 4.19.** *Let  $f : X \rightarrow Y$  be an IFfaintly open and IF $\theta$ - cont. surjection function. If  $V$  is IFS $\theta$ -compact relative to  $Y$ , then  $f^{-1}(V)$  is IF $\theta$ -compact relative to  $X$ .*

**Theorem 4.20.** *Let  $Y$  be an IF-submaximal regular space and  $f : X \rightarrow Y$  be an IFpreopen surjection function. If  $f$  is IFS-cont. and  $X$  is an IFS $\theta$ -compact, then  $Y$  is so.*

*Proof.* Let  $\{V_j : j \in J\}$  be an IFsemiopen cover of  $Y$ . Since  $f$  is IFS-cont. and IFpreopen, then by Lemma 2.14 the family  $\{f^{-1}(V_j) : j \in J\}$  is an IFsemiopen cover of  $X$ . Since  $X$  is an IFS $\theta$ -compact, there is a subfamily  $\{f^{-1}(V_j) : j = 1, 2, \dots, n\}$  of IF $\theta$ OS's which covers  $X$ . Now,  $\{V_j : j = 1, 2, \dots, n\} = \{ff^{-1}(V_j) : j = 1, 2, \dots, n\}$  is an IF $\theta$ -open cover in  $Y$ , since  $Y$  is an IF-submaximal regular space. Hence  $Y$  is an IFS $\theta$ -compact.  $\square$

**Corollary 4.21.** *Let  $Y$  be an IF-submaximal regular space and  $f : X \rightarrow Y$  be an IF $i$  function. If  $X$  is IFS $\theta$ -compact, then so is  $Y$ .*

**Corollary 4.22.** *Let  $Y$  be an IF-submaximal regular space and  $f : X \rightarrow Y$  be an IF $i$  function. If  $U$  is IFS $\theta$ -compact relative to  $X$ , then  $f(U)$  is IFS $\theta$ -compact relative to  $Y$ .*

## 5. Locally IFS $\theta$ -compact

**Definition 5.1.** An IFTS  $(X, \Psi)$  is said to be locally IFS $\theta$ -compact if for each IFP  $c(a, b)$  in  $X$ , there is  $U \in N_\varepsilon(c(a, b))$  such that  $\mu_U(c) = 1$ ,  $\gamma_U(c) = 0$  and  $U$  is IFS $\theta$ -compact relative to  $X$ .

**Remark 5.2.** Every IFS $\theta$ -compact space is locally IFS $\theta$ -compact but the converse may not be true.

**Example 5.3.** An infinite discrete IFTS is locally IFS $\theta$ -compact but not IFS $\theta$ -compact.

**Remark 5.4.** Every locally IFS $\theta$ -compact space is locally IF-compact but the converse may not be true.



**Example 5.5.** A cofinite IFTS is locally IF-compact but not locally IFS $\theta$ -compact.

**Theorem 5.6.** *Let  $Y$  be an IF-submaximal regular space and  $f : X \rightarrow Y$  be an IF-open surjection function. If  $f$  is an IF $i$  function and  $X$  is locally IFS $\theta$ -compact, then so is  $Y$ .*

*Proof.* Let  $y(m, n)$  be an IFP in  $Y$ . Then  $y(m, n) = f(x(a, b))$  for some  $x(a, b)$  in  $X$ . Since  $X$  is locally IFS $\theta$ -compact, there is  $U \in N_\varepsilon(x(a, b))$  such that  $\mu_U(x) = 1$ ,  $\gamma_U(x) = 0$  and  $U$  is IFS $\theta$ -compact relative to  $X$ . Since  $f$  is an IF-open function,  $f(U) \in N_\varepsilon(y(m, n))$  with  $(f(U))(y) = \bigvee_{x \in f^{-1}(y)} U(x) = \underline{1}$  and by Corollary 4.22,  $f(U)$  is IFS $\theta$ -compact relative to  $Y$ . Hence  $Y$  is a locally IFS $\theta$ -compact space.  $\square$

**Corollary 5.7.** *Let  $Y$  be an IF-submaximal regular space and  $f : X \rightarrow Y$  be an IF-open surjection function. If  $f$  is an IFsuper  $i$  function and  $X$  is locally IFS $\theta$ -compact, then so is  $Y$ .*

*Proof.* Since every IFsuper  $i$  function is IF $i$ , the proof follows from Theorem 5.6.  $\square$

**Theorem 5.8.** *Let  $f : X \rightarrow Y$  be an IF-cont. and IF-open surjection function. If  $X$  is locally IFS $\theta$ -compact, then  $Y$  is locally IF-compact.*

*Proof.* Let  $y(m, n)$  be an IFP in  $Y$ . Then  $y(m, n) = f(x(a, b))$  for some  $x(a, b)$  in  $X$ . Since  $X$  is locally IFS $\theta$ -compact, there is  $U \in N_\varepsilon(x(a, b))$  such that  $\mu_U(x) = 1$ ,  $\gamma_U(x) = 0$  and  $U$  is IFS $\theta$ -compact relative to  $X$ . Since  $f$  is an IF-open function,  $f(U) \in N_\varepsilon(y(m, n))$  with  $(f(U))(y) = \bigvee_{x \in f^{-1}(y)} U(x) = \underline{1}$  and by Corollary 4.3,  $f(U)$  is an IF-compact relative to  $Y$ . Hence  $Y$  is locally IF-compact space.  $\square$

**Corollary 5.9.** *Let  $f : X \rightarrow Y$  be an IF-cont. and IF-open surjection function. If  $X$  is locally IFS $\theta$ -compact, then  $Y$  is locally IF $\theta$ -compact.*

*Proof.* Obvious, since every locally IF-compact is locally IF $\theta$ -compact.  $\square$

**Corollary 5.10.** *Let  $Y$  be an IF-regular space and  $f : X \rightarrow Y$  be an IF-open surjection function. If  $f$  is an IFweakly cont. function and  $X$  is locally IFS $\theta$ -compact, then  $Y$  is locally IF-compact.*

*Proof.* This follows from the above Theorem and the fact that every IFweakly cont. function is an IF-cont. in an IF-regular space.  $\square$

**Theorem 5.11.** *Let  $X$  be an IF-regular space and  $f : X \rightarrow Y$  be an IF $\theta$ -open bijective function. If  $f$  is IFS $\theta$ -cont. and  $X$  is locally IFS $\theta$ -compact, then so is  $Y$ .*

*Proof.* Using Corollary 4.6, the proof is similar to the proof of Theorem 5.6.  $\square$

**Theorem 5.12.** *Let  $f : X \rightarrow Y$  be an IFsuper semiopen and IFstrongly  $\theta$ -cont. surjection function. If  $Y$  is locally IFcompact, then  $X$  is locally IFS $\theta$ -compact.*

*Proof.* Let  $x(a, b)$  be an IFP in  $X$ . Since  $f$  is surjective, there is  $y(m, n)$  such that  $f(x(a, b)) = y(m, n)$ . Since  $Y$  is locally IFcompact, there is  $V \in N_\varepsilon(y(m, n))$  such that  $\mu_V(y) = 1$ ,  $\gamma_V(y) = 0$  and  $V$  is IF-compact relative to  $Y$ . Using Theorem 4.13,  $f^{-1}(V)$  is IFS $\theta$ -compact relative to  $X$ . Since  $f$  is IFstrongly  $\theta$ -cont, then

$f^{-1}(V) \in N_{\varepsilon}^{\theta}(x(a, b))$  and hence  $f^{-1}(V) \in N_{\varepsilon}(x(a, b))$ . Therefore  $f^{-1}(V)(x) = V(f(x)) = V(y) = \underline{1}$ . Hence for  $x(a, b)$  in  $X$ , there is  $f^{-1}(V) \in N_{\varepsilon}(x(a, b))$  such that  $f^{-1}(V)(x) = \underline{1}$  and  $f^{-1}(V)$  is an IFS $\theta$ -compact relative to  $X$ . Hence  $X$  is a locally IFS $\theta$ -compact.  $\square$

**Corollary 5.13.** *Let  $f : X \rightarrow Y$  be an IFsuper semiopen and IFstrongly  $\theta$ -cont. surjection function. If  $Y$  is locally IFcompact, then  $X$  is locally IFcompact.*

**Theorem 5.14.** *Let  $f : X \rightarrow Y$  be an IFsemiopen and IFfaintly cont. surjection function. If  $Y$  is locally IFS $\theta$ -compact, then  $X$  is locally IFcompact.*

*Proof.* Using Corollary 4.17, the proof is similar to proof of Theorem 5.6.  $\square$

**Proposition 5.15.** *Let  $f : X \rightarrow Y$  be an IFfaintly open and IF $\theta$ -cont. surjection function. If  $Y$  is locally IFS $\theta$ -compact, then  $X$  is locally IF $\theta$ -compact.*

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