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Scattered data approximation of fully fuzzy data by quasi-interpolation

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Abstract

Fuzzy quasi-interpolations help to reduce the complexity of solving a linear system of equations compared with fuzzy interpolations. Almost all fuzzy quasi-interpolations are focused on the form of $\widetilde{f}^* : \mathbb{R} \to F(\mathbb{R})$ or $\widetilde{f}^* : F(\mathbb{R}) \to \mathbb{R}$. In this paper, we intend to offer a novel fuzzy radial basis function by the concept of source distance. Then, we will construct a fuzzy linear combination of such basis functions in order to introduce a fully fuzzy quasi-interpolation in the form of $\widetilde{f}^* : F(\mathbb{R}) \to F(\mathbb{R})$. Also the error estimation of the proposed method is proved in terms of the fully fuzzy modulus of continuity which will be introduced in this paper. Finally some examples have been given to emphasize the acceptable accuracy of our method.

Keywords: Quasi-interpolation, Fuzzy interpolation, Fuzzy Logic, Radial basis function, Linear reproducing property.

1 Introduction

Starting in 1965 with the introduction of fuzzy sets by Zadeh [27], the theory of fuzzy sets has been being developed in variety of fields such as artificial intelligence, computer science, medicine, control engineering, decision theory, management science, robotics, etc. A fuzzy interpolation problem is proposed as below:

Suppose that we have $n+1$ distinct crisp numbers $x_0, ..., x_n$ and for each of them a fuzzy set in $F(\mathbb{R})$, rather than a crisp one, is given. There are many approaches to the problem of fuzzy interpolation, They differ one from the other one by restrictions on interpolation functions. We enumerate some approaches as below:

The Lagrange interpolation problem is introduced by Lowen [19]. Kaleva [16] investigated some properties of Lagrange and cubic spline interpolation. Abbasbandy and Babolian [4] have proposed the interpolation of fuzzy data by a natural spline. Abbasbandy [1] used the complete splines in the interpolation of fuzzy data. He measured the interpolation of fuzzy data using the fuzzy-valued piecewise quadric polynomial induced the cubic spline function. Abbasbandy and Amirfakhrian [2] presented a fuzzy polynomial approximation as a universal approximation of a fuzzy function on a discrete set of points. Below is a list of the most popular approaches of fuzzy interpolation:

Analogy-based interpolation [9], analytic fuzzy relation- based interpolation [11], fuzzy interpolation based on geometric transformations [13], and interpolation by convex completion [20]. In this paper, we present a new approach based on combination of fuzzy logic and quasi-interpolation. Quasi-interpolation is an appropriate tool in approximation theory and its applications. The most important advantage of quasi-interpolation is that one can evaluate the approximation directly without the need to solve any linear system of equations. Powell [21] constructed a quasi-interpolation function (Lf) by shifts of multiquadric basis function of first degree to scattered data in a one-dimensional space. The operator (Lf) satisfies the linear polynomial reproduction property. Hardy [12] proposed multiquadric (MQ) functions in 1968 as a Radial Basis Function (RBF). For the first time, Kansa [19] used successfully modified MQ functions for solving Partial Differential Equations (PDE). In order to approximate a function on an interval, Betson and Powell [10] proposed three univariate multiquadric quasi-interpolation operators, and named them L_A, L_B and L_C. Subsequently, Schaback and Wu [26] proposed a multiquadric quasi-interpolation L_D to improve L_A, L_B and L_C. The Multiquadric quasi-interpolation operator L_D possesses preserving monotonicity, convexity and linear reproducing on $[x_0, x_n]$, however

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L_A and L_B cannot preserve both linearity and convexity[15]. In this research, operator L_D was used as a multiquadric RBF due to its wide amount of the applications. Multiquadric is used in many fields such as geodesy(approximating the Earth's gravitational field), geophysics(approximating the Earth's crustal movement), photogrammetry(reconstructing images), hydrology(approximating the average rainfall in a region) and geology(approximating ground water levels from well-logs)[15]. The rest of this paper is arranged as below: In Section 2 we introduce the basic concepts of fuzzy logic. Section 3, describes the quasi-interpolation method. Mathematical formulation of our method is explained in Section 4. Section 5, evaluates the accuracy of proposed method providing some examples. Finally, Section 6 provides conclusions.

2 Preliminaries of Fuzzy Logic

In this section, the basic definitions of fuzzy sets are recalled. Let $F(\mathbb{R})$ be the set of all real fuzzy numbers (which are normal, upper semi-continuous, convex and compactly supported fuzzy sets).

Definition 2.1. [6] *The parametric form of a fuzzy number is shown by $\tilde{v} = (\underline{v}, \overline{v})$, where $\underline{v}$ and $\overline{v}$ are functions that satisfy the following requirements:*

1. *$\underline{v} : [0, 1] \to F(\mathbb{R})$ is monotonically increasing, left continuous function on $(0, 1]$ and right-continuous at 0.*
2. *$\overline{v} : [0, 1] \to F(\mathbb{R})$ is monotonically decreasing, left continuous function on $(0, 1]$ and right-continuous at 0.*
3. *$\underline{v}(r) \leq \overline{v}(r), \quad r \in [0, 1]$.*

A crisp number α, can be simply represented by $\overline{v}(r) = \underline{v}(r) = \alpha, \quad 0 \leq r \leq 1$. By considering $\widetilde{v} = (\underline{v}, \overline{v})$ and $\widetilde{u} = (\underline{u}, \overline{u})$, we have the following representation for the basic operation with fuzzy numbers ($k \in \mathbb{R}$) [6]:

$$\widetilde{u} + \widetilde{v} = (\underline{u} + \underline{v}, \overline{u} + \overline{v}), \tag{1}$$

$$\widetilde{u} - \widetilde{v} = (\underline{u} - \overline{v}, \overline{u} - \underline{v}), \tag{2}$$

$$k\widetilde{u} = \begin{cases} (k\underline{u}, k\overline{u}), & k \geq 0, \\ (k\overline{u}, k\underline{u}), & k < 0, \end{cases} \tag{3}$$

and

$$\begin{cases} \underline{u \cdot v} = \min\{\underline{u}.\underline{v}, \underline{u}.\overline{v}, \overline{u}.\underline{v}, \overline{u}.\overline{v}\}, \\ \overline{u \cdot v} = \max\{\underline{u}.\underline{v}, \underline{u}.\overline{v}, \overline{u}.\underline{v}, \overline{u}.\overline{v}\}. \end{cases} \tag{4}$$

Definition 2.2. [6] *A Trapezoidal fuzzy number, denoted by $\widetilde{v} = (v_l, v_m, v_n, v_u)$ with $v_l \leq v_m \leq v_n \leq v_u$, is:*

$$\mu_{\widetilde{v}}(x) = \begin{cases} 0, & x \leq v_l \\ \frac{x - v_l}{v_m - v_l}, & v_l \leq x \leq v_m \\ 1, & v_m \leq x \leq v_n \\ \frac{v_u - x}{v_u - v_n}, & v_n \leq x \leq v_u \\ 0, & x \geq v_u. \end{cases} \tag{5}$$

If we set $v_m = v_n$, triangular fuzzy numbers is obtained which is indicated by the form of $\widetilde{v} = (v_l, v_m, v_u)$ where v_m denotes the modal value, and the nonnegative real values v_l and v_u represent the left and right fuzziness, respectively. For example, if $\underline{u}$ and $\overline{u}$ are linear then the parametric form of triangular fuzzy number $\widetilde{v} = (v_l, v_m, v_u)$ is:

$$\widetilde{v} = (\underline{v}, \overline{v}) = (v_m + v_l(r - 1), v_m + v_u(1 - r)), \quad 0 \leq r \leq 1.$$

If $\widetilde{u}$ and $\widetilde{v}$ are two triangular fuzzy numbers and k is a real number, then $k\widetilde{u}$ and $\widetilde{u} + \widetilde{v}$ are triangular fuzzy numbers.

Definition 2.3. [8] *A continuous function $s : [0, 1] \to [0, 1]$ with the following properties is a regular reducing function:*

1. *s is an increasing function.*
2. *$s(1) = 1$.*
3. *$s(0) = 0$.*
4. *$\int_0^1 s(r)dr = \frac{1}{2}$.*

Definition 2.4. [6] *The Value and Ambiguity of a fuzzy number $\tilde{u}$ are defined as follows:*

$$Val(\tilde{u}) = \int_0^1 s(r)(\overline{u}(r) + \underline{u}(r))dr, \tag{6}$$

$$Amb(\tilde{u}) = \int_0^1 s(r)(\overline{u}(r) - \underline{u}(r))dr, \tag{7}$$

where s is a regular reducing function.

Definition 2.5. [6, 23] *Suppose that B is a linear norm space and A is subset of B. If $f \in B$, then we introduce the distance of f from the set A by, $dist(f, A) = \inf_{a\in A} \parallel f - a \parallel$.*

Definition 2.6. [6] *Let A and B are two subsets of a linear norm space. The Housdorf metric of this subsets is defined as bellow:*

$$d_H(A, B) = \max\{\sup_{a\in A} dist(a, B), \sup_{b\in B} dist(b, A)\}. \tag{8}$$

Definition 2.7. [6] *Let $\tilde{u}, \tilde{v} \in F(\mathbb{R})$. The source distance between $\tilde{u}$ and $\tilde{v}$ is defined as follows:*

$$D(\tilde{u}, \tilde{v}) = \frac{1}{2}\{(| Val(\tilde{u}) - Val(\tilde{v}) | + | Amb(\tilde{u}) - Amb(\tilde{v}) | + d_H([\tilde{u}]^1, [\tilde{v}]^1))\}, \tag{9}$$

where d_H is the Hausdorf metric, and $[.]^1$ is the 1-cut representation of a fuzzy number.

Definition 2.8. [8] *We name two fuzzy numbers $\tilde{x}$ and $\tilde{y}$ are called to be distinct, if $D(\tilde{x}, \tilde{y}) \neq 0$, where D is a distance.*

In this paper we used source distance.

3 Multiquadric Quasi-Interpolation

In this subsection, the Multiquadric (RBF) is described as an interpolating basis. Moreover quasi-interpolation and its advantages are introduced. The RBF method is one of the primary tools for the interpolation of multidimensional scattered data.

The method's abilities to handle arbitrary scattered data, to easily generalize to several spacial dimensions, and to provide spectral accuracy have made it particularly popular in several different types of applications, such as cartography, neural networks, medical imaging and numerical solution of partial differential equations [25]. Multi-dimensional interpolations with fuzzy sets are discussed in some papers [3, 5, 12, 20].

Approximation of continuous fuzzy functions analysied by multivariate fuzzy polynomials in [18] and they used Bernstein polynomials. Interpolation of one-dimensional data has been considered by many authors. The main procedure of these methods are: for a given set of N data points $x_j,\ j = 1, ..., N$, and corresponding data values $f_j,\ j = 1, ..., N$, a set of basis functions $H_j,\ j = 1, ..., N$, is chosen such that a linear combination of these functions as indicated below satisfies the interpolation conditions $s(x) = \sum_{j=1}^{N} \lambda_j H_j(x)$. The unknown coefficient λ_j are obtained by exerting the interpolation conditions $s(x_j) = f_j,\ j = 1, ..., N,$.

In 1968, Hardy [15] used for the first time RBF as the basis for interpolation space. He used a set of N scattered data $\{(x_j, f_j)\}_{j=1}^{N}$ and proposed the following form of interpolation function, $s(x) = \sum_{j=1}^{N} \lambda_j \phi(c\|x - x_j\|)$, where $\lambda_j,\ j = 1, ..., N$, are determined by the collocation method i.e. $s(x_j) = f_j,\ j = 1, ..., N$. The commonly used RBFs are Gaussian $e^{-(cr)^2}$(for infinitely smooth with a free parameter), Thin-Plate Spline $r^2 \ln(r)$ (this radial basis function is an example of a smoothing spline, as popularized by Grace Wahba) and Multiquadric $\sqrt{c^2 + r^2}$ [25].

The condition number of coefficient matrix depends on the shape parameter. Increasing shape parameter is lead to increase the convergence rate at the expense of increasing condition number. If the condition number is large then numerical stability can not be guaranteed mathematically. Although some software packages such as "Multiprecision computing Toolbox" can be averted ill-conditioning by using extended precision. It is available in http://www.advanpix.com. There is a section in this site on MQ-RBFs in which the author described the situation. Computing optimal shape parameter is still an open problem [7, 22]. Depending upon the applications some authors proposed some regularization methods to solve such ill-conditioned systems [3]. Quasi-interpolation is usually used to overcome such problems as follows [26]:

$$(L_D f) = \sum_{j=1}^{N} f_j \psi_j, \tag{10}$$

where ψ_j is a linear combination of RBF:

$$\psi_1(x) = \frac{1}{2} + \frac{\phi_2(x) - (x - x_1)}{2(x_2 - x_1)}, \tag{11}$$

$$\psi_N(x) = \frac{1}{2} + \frac{\phi_{N-1}(x) - (x - x_N)}{2(x_2 - x_1)}, \tag{12}$$

$$\psi_j(x) = \frac{\phi_{j+1}(x) - \phi_j(x)}{2(x_{j+1} - x_j)} - \frac{\phi_j(x) - \phi_{j-1}(x)}{2(x_j - x_{j-1})}, \quad j = 2, ..., N-1. \tag{13}$$

It has been proved that if L_D possessed the linear polynomial reproduction property, the maximum convergence rate is $O(h^2|\ln h|)$ as $c = O(h)$. ψ_j can be considered as a linear combination of any type of radial basis functions and always it can be proved [21]:

$$\sum_{j=1}^{N} \psi_j(x) = 1. \tag{14}$$

In quasi-interpolation, there is no need to solve a linear system of equations due to the substitution of $f_j, \quad j = 1, ..., N$ instead of λ_j in a RBF interpolation.

4 Quasi-Interpolation of Fuzzy Function

Approximation of a fuzzy function on some given points in the form of $\{(x_i, \tilde{f}_i)\}_{i=1}^{n}$ or $\{(\tilde{x}_i, f_i)\}_{i=1}^{n}$ is considered by some authors. The interpolation of fuzzy data by using spline functions of odd degree was considered, with natural splines in [4], and with complete splines in [1]. Abbasbandy and Amirfakhrian [3] considered data of the form $\{(x_i, \tilde{f}_i)\}_{i=1}^{n}$ and found a fuzzy polynomial $\tilde{p^*}(x)$ as the best approximation of $\tilde{f}(x)$. They proved that the best approximation exists. Adibitabar [5] found an interpolation function $\tilde{f} : F(\mathbb{R}) \to F(\mathbb{R})$ on the set of points $(\tilde{x}_i, \tilde{f}_i)$, $i = 1, 2, ..., n$ by applying the recursions method which is based on signed distance of fuzzy numbers and Neville's algorithm for interpolating real data. Amirfakhrian et.al [7] construct a fuzzy quasi-interpolation $\tilde{f} : \mathbb{R} \to F(\mathbb{R})$ and used it to solve the Fredholm fuzzy integral equations. Similar to every interpolation scheme, we have to solve a system of equations to obtain fuzzy coefficients in fuzzy interpolation.

In this paper, we consider $n+1$ distinct fuzzy numbers $\tilde{x}_1, ..., \tilde{x}_n$. We will find a fuzzy quasi-interpolation $\tilde{f}^*$: $F(\mathbb{R}) \to F(\mathbb{R})$ such that $\tilde{f}(\tilde{x}_i) = \tilde{f}^*(\tilde{x}_i), \quad i = 1, 2, ..., n$. The first step of constructing such a fuzzy quasi-interpolating function is defining a fuzzy multiquadric radial basis function. To extend Equation (10) to fuzzy case one can use an arbitrary distance, and in this paper we use the source distance.

Here some of the necessary definitions and theorems are given by authors.

Definition 4.1. *We define a fuzzy multiquadric radial basis function as follow:*

$$\phi_j(\tilde{x}) = \sqrt{D(\tilde{x}, \tilde{x}_j)^2 + c^2}, \tag{15}$$

where c is a shape parameter and D is the source distance.

Another step to construct a quasi-interpolation is considering a linear combination of radial basis function as a basis. In Equations (11)-(13) $\psi_j : \mathbb{R} \to \mathbb{R}$ is described. We are going to make $\psi_j : F(\mathbb{R}) \to \mathbb{R}$ as a defuzzification function.

Definition 4.2. *We define a linear combination of fuzzy multiquadric radial basis function $\psi_j : F(\mathbb{R}) \to \mathbb{R}$ as below:*

$$\psi_0(\tilde{x}) = \frac{1}{2} + \frac{\phi_1(\tilde{x}) - \phi_0(\tilde{x})}{2D(\tilde{x}_1, \tilde{x}_0)}, \tag{16}$$

$$\psi_n(\tilde{x}) = \frac{1}{2} - \frac{\phi_n(\tilde{x}) - \phi_{n-1}(\tilde{x})}{2D(\tilde{x}_n, \tilde{x}_{n-1})}, \tag{17}$$

$$\psi_j(\tilde{x}) = \frac{\phi_{j+1}(\tilde{x}) - \phi_j(\tilde{x})}{2D(\tilde{x}_{j+1}, \tilde{x}_j)} - \frac{\phi_j(\tilde{x}) - \phi_{j-1}(\tilde{x})}{2D(\tilde{x}_j, \tilde{x}_{j-1})}, \quad j = 1, ..., n-1. \tag{18}$$

Now, according to Equation (10), we can define fuzzy quasi-interpolation.

Definition 4.3. *We define a fuzzy quasi-interpolation as below:*

$$\tilde{f}^*(\tilde{x}) = \sum_{j=0}^{n} \tilde{f}(\tilde{x}_j)\psi_j(\tilde{x}). \tag{19}$$

In this kind of quasi-interpolation the coefficients ψ_j are not always positive, so we consider:

$$a_j(\tilde{x}) = \begin{cases} \psi_j(\tilde{x}), & \psi_j(\tilde{x}) \geq 0 \\ 0, & \psi_j(\tilde{x}) < 0, \end{cases}, \tag{20}$$

and

$$b_j(\tilde{x}) = \begin{cases} 0, & \psi_j(\tilde{x}) \geq 0, \\ -\psi_j(\tilde{x}), & \psi_j(\tilde{x}) < 0. \end{cases} \tag{21}$$

Therefore the parametric form of f^* in (19) can be considered as:

$$\underline{f^*(\tilde{x})} = \sum_{j=0}^{n} (\underline{f(\tilde{x}_j)} a_j(\tilde{x}) + \overline{f(\tilde{x}_j)} b_j(\tilde{x})), \tag{22}$$

and

$$\overline{f^*(\tilde{x})} = \sum_{j=0}^{n} (\overline{f(\tilde{x}_j)} a_j(\tilde{x}) + \underline{f(\tilde{x}_j)} b_j(\tilde{x})). \tag{23}$$

The error functions can be defined as $E = (\underline{E}, \overline{E})$, in which $\underline{E} = \underline{f^*(\tilde{x}_i)} - \underline{f(\tilde{x}_i)}$, and $\overline{E} = \overline{f^*(\tilde{x}_i)} - \overline{f(\tilde{x}_i)}$.

Theorem 4.4. *If $\tilde{f}(\tilde{x}_i) = \tilde{y}_i$, $i = 0, 1, ..., n$ is a triangular (trapezoidal) fuzzy number, then each of $\tilde{f}^*(\tilde{x}_i)$ is a triangular (trapezoidal) fuzzy number, too.*

Proof. Since for a fuzzy number $\tilde{x}$, $\psi_j(\tilde{x})$ is a crisp number if $\tilde{f}(\tilde{x}_i)$ is a triangular (trapezoidal) fuzzy number then each of $\{\psi_j(\tilde{x})\tilde{f}(\tilde{x}_i)\}_{i=0}^{n}$ is a triangular (trapezoidal) fuzzy number. It is obvious that summation of some triangular (trapezoidal) fuzzy numbers is also a triangular (trapezoidal) fuzzy number and it completes the proof. ☐

Theorem 4.5. *Let $\underline{f}_j(r) = \alpha_j r + \beta_j$ and $\overline{f}_j(r) = \gamma_j r + \theta_j$, $j = 0, ..., n$. If the following two conditions are set for $\tilde{x}$, $\sum_{j=0}^{n}((\alpha_j a_j(\tilde{x}) + \gamma_j b_j(\tilde{x})) \geq 0$, and $\sum_{j=0}^{n}((\gamma_j a_j(\tilde{x}) + \alpha_j b_j(\tilde{x})) \leq 0$, then $\tilde{f}^*(\tilde{x})$ is a fuzzy number.*

Proof. In order to show that with these assumptions we have, $\underline{f^*(\tilde{x})} = \sum_{j=0}^{n}(\underline{f(\tilde{x}_j)} a_j(\tilde{x}) + \overline{f(\tilde{x}_j)} b_j(\tilde{x}))$, by considering $\underline{f}_j(r) = \alpha_j r + \beta_j$ and $\overline{f}_j(r) = \gamma_j r + \theta_j$, it can be written: $\underline{f^*(\tilde{x})} = \sum_{j=0}^{n}((\alpha_j a_j(\tilde{x}) + \gamma_j b_j(\tilde{x}))r + \beta_j a_j(\tilde{x}) + \theta_j b_j(\tilde{x}))$, since the coefficient of r is positive, $\underline{f^*(\tilde{x})}$ is increasing. Similarly, it can be seen that $\underline{f^*(\tilde{x})}$ is decreasing. ☐

Theorem 4.6. *The fuzzy quasi-interpolation which is introduced in Equation (19) possesses the constant preserving property.*

Proof. Let $\underline{f}_j(r) = \alpha_j r + \beta_j$ and $\overline{f}_j(r) = \gamma_j r + \theta_j$, $j = 0, ..., n$. If we get $\alpha_j = \gamma_j = 0$ and $\theta_j = \beta_j = C$, $j = 0, ..., n$, then we have: $\underline{f^*(\tilde{x}_i)} = \sum_{j=0}^{n}(\underline{f(\tilde{x}_j)} a_j(\tilde{x}_i) + \overline{f(\tilde{x}_j)} b_j(\tilde{x}_i)) = \sum_{j=0}^{n}((\alpha_j r + \beta_j) a_j(\tilde{x}_i) + (\gamma_j r + \theta_j) b_j(\tilde{x}_i))$. According to the assumption of theorem and Equation (14) we have: $\underline{f^*(\tilde{x}_i)} = \sum_{j=0}^{n} C(a_j(\tilde{x}_i) + b_j(\tilde{x}_i)) = \sum_{j=0}^{n} C\psi_j(\tilde{x}_i) = C$. ☐

In the last theorem of this section the error analysis of our method is studied. For the proof of this theorem, we need to a definition of fully fuzzy modulus of continuity.

Definition 4.7. *[8] We define the modulus of continuity with respect to the source distance of a fuzzy function $\tilde{f}(\tilde{x})$ as below:*

$$w_\delta(\tilde{f}) = \sup_{D(\tilde{x}_i, \tilde{x}_j) \leq \delta} D(\tilde{f}(\tilde{x}_i), \tilde{f}(\tilde{x}_j)). \tag{24}$$

The next theorem presents the error analysis of our method.

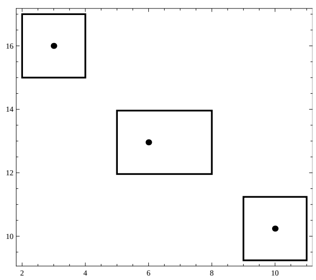
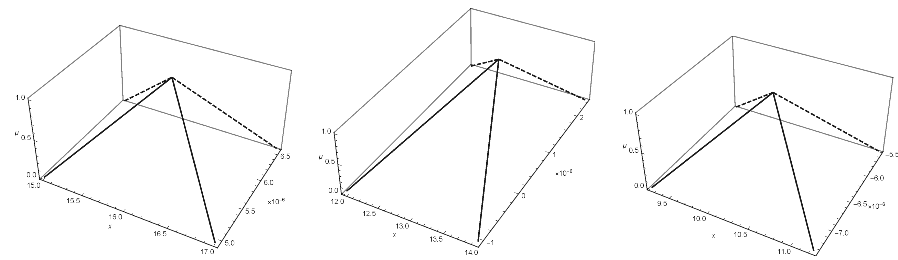


Figure 1: The position of $\{(\tilde{x}_i, \tilde{f}_i^*)\}_{i=0}^{2}$.



(a) Error in $\tilde{x}_1$ (b) Error in $\tilde{x}_2$ (c) Error in $\tilde{x}_3$

Figure 2: The errors of approximation in $\tilde{x}_i$ for Example 1.

Theorem 4.8. *(Error Analysis) If $\tilde{x}_i$, $i = 0, ..., n$ are set such that $\psi_j(\tilde{x}_i)$, $i, j = 0, \cdots, n$ are non-negative then the fuzzy quasi-interpolation $\tilde{f}^*$ which is introduced in Equation (19), has the following upper bound:*

$$D(\tilde{f}^*(\tilde{x}_i), \tilde{f}(\tilde{x}_i)) \leq w_\delta(\tilde{f}). \tag{25}$$

Proof. $D(\tilde{f}^*(\tilde{x}_i), \tilde{f}(\tilde{x}_i)) = D(\sum_{j=0}^{n} \tilde{f}(\tilde{x}_j)\psi_j(\tilde{x}_i), \tilde{f}(\tilde{x}_i))$. According to Equation (14) and the property of the source distance we have: $D(\tilde{f}^*(\tilde{x}_i), \tilde{f}(\tilde{x}_i)) \leq \sum_{j=0}^{n} D(\tilde{f}(\tilde{x}_j), \tilde{f}(\tilde{x}_i))\psi_j(\tilde{x}_i)$. So we have:

$$D(\tilde{f}^*(\tilde{x}_i), \tilde{f}(\tilde{x}_i)) \leq \sum_{j=0}^{n} \sup D(\tilde{f}(\tilde{x}_j), \tilde{f}(\tilde{x}_i))\psi_j(\tilde{x}_i),$$

by considering Equations (14) and (24) we can write; $D(\tilde{f}^*(\tilde{x}_i), \tilde{f}(\tilde{x}_i)) \leq \sum_{j=0}^{n} w_\delta(\tilde{f})\psi_j(\tilde{x}_i) = w_\delta(\tilde{f})$, and the proof is completed. □

Definition 4.9. [8] *The fuzzy function $\tilde{f} : F(\mathbb{R}) \to F(\mathbb{R})$ satisfies an α-Lipschitz condition of order α on the interval $[a, b]$, if there exists two positive constants $L \geq 0$ and $\alpha > 0$ such that $D(\tilde{f}(\tilde{x}), \tilde{f}(\tilde{y})) \leq LD(\tilde{x}, \tilde{y})^\alpha$, and we denote it as $\tilde{f} \in Lip_L^\alpha$.*

Corollary 4.10. *In particular if $\tilde{x}_i$, $i = 0, ..., n$ are constructed such that $\psi_j(\tilde{x}_i)$, $i, j = 0, \cdots, n$ are non-negative and if we let $\tilde{f} \in Lip_L^\alpha$, then the distance between the approximate and exact solution has the following upper bound:*

$$D(\tilde{f}^*(\tilde{x}_i), \tilde{f}(\tilde{x}_i)) \leq L\delta^\alpha. \tag{26}$$

Proof. By considering Equation (24) we have, $D(\tilde{f}^*(\tilde{x}_i), \tilde{f}(\tilde{x}_i)) \leq w_\delta(\tilde{f}) = \sup\limits_{D(\tilde{x}_i, \tilde{x}_j) \leq \delta} D(\tilde{f}(\tilde{x}_i), \tilde{f}(\tilde{x}_j))$. According to the definition of the Lipschitz condition we get, $D(\tilde{f}^*(\tilde{x}_i), \tilde{f}(\tilde{x}_i)) \leq LD(\tilde{x}_i, \tilde{x}_j)^\alpha \leq L\delta^\alpha$.□ If we let $h = \max D(\tilde{x}_i, \tilde{x}_j)$, $i, j = 0, 1, ..., n$, then to achieve an adequate error we can select $h < \sqrt[\alpha]{\frac{\epsilon}{L}}$. □

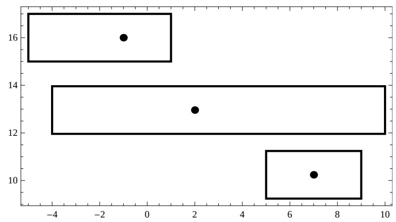


Figure 3: The position of $\{(\tilde{x}_i, \tilde{f}_i^*)\}_{i=0}^2$.

5 Examples

In this section, we apply our method to approximate some fuzzy functions. The results are obtained from commercial software Mathematica 10. In Figures 1, 3 and 5, observe that the dark circles represent the crisp numbers, and the lines represent the extent of the fuzzyness for each number. In Figures 2, 4 and 6; every subfigures shows a membership function of error in a specified point.

Example 5.1. *Consider the following fuzzy data,* $\tilde{x}_0 = (15 + r, 17 - r),\ \tilde{x}_1 = (11.96 + r, 13.96 - r),$ $\tilde{x}_2 = (9.24 + r, 11.24 - r)$.

In this example we set $f_0 = (2 + r, 4 - r),\ f_1 = (5 + r, 8 - 2r),\ f_2 = (9 + r, 11 - r)$. According to equation. (19), the quasi-interpolating function can be written as follow:

$$\tilde{f}^*(\tilde{x}) = \tilde{f}_0\psi_0(\tilde{x}) + \tilde{f}_1\psi_1(\tilde{x}) + \tilde{f}_2\psi_2(\tilde{x}).$$

where $\psi_j(\tilde{x})$, $j = 0, 1, 2$ is introduced in Equations (16)-(18). In Figure 1, the positions of $\{(\tilde{x}_i, \tilde{f}_i^*)\}_{i=0}^2$ are presented. In Figure 2, the errors of approximating function in $\tilde{x}_i$, for $i = 0, 1, 2$ is presented. In Figures 2(a)-2(c) the coordinate of the maximum value of the membership function in $\tilde{x}_0$, $\tilde{x}_1$ and $\tilde{x}_2$ are $(16, 5.93421E{-}6)$, $(12.96, 1.41873E{-}6)$ and $(10.24, -6.35293E{-}6)$, respectively.

Example 5.2. *Consider the following fuzzy data:*

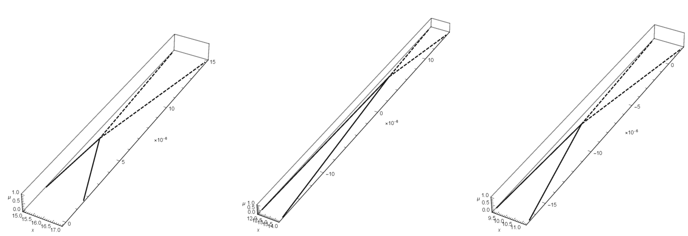
$$\tilde{x}_0 = (15 + r, 17 - r),\ \tilde{x}_1 = (11.96 + r, 13.96 - r),\ \tilde{x}_2 = (9.24 + r, 11.24 - r).$$

In this example we set $f_0 = (4r - 5, -2r + 1),\ f_1 = (6r - 4, -8r + 10),\ f_2 = (5 + 2r, 9 - 2r)$. According to equation. (19), the quasi-interpolating function can be written by, $\tilde{f}^*(\tilde{x}) = \tilde{f}_0\psi_0(\tilde{x}) + \tilde{f}_1\psi_1(\tilde{x}) + \tilde{f}_2\psi_2(\tilde{x})$. where $\psi_j(\tilde{x})$, $j = 0, 1, 2$ is introduced in Equations (16)-(18). In Figure 3, the positions of $\{(\tilde{x}_i, \tilde{f}_i^*)\}_{i=0}^2$ are presented.

In Figure 4, the errors of $\tilde{x}_i$, for $i = 0, 1, 2$ is presented. In Figures 4(a)-4(c) the coordinate of the maximum value of the membership function in $\tilde{x}_0$, $\tilde{x}_1$ and $\tilde{x}_2$ are $(16, 4.93421E{-}6)$, $(12.96, 4.25696E{-}6)$ and $(10.24, -9, 19116E{-}6)$, respectively.

Example 5.3. *Consider the following fuzzy data:* $\tilde{x}_0 = (r - 4, -2 - r),\ \tilde{x}_1 = (1 + r, 3 - r),\ \tilde{x}_2 = (4 + 2r, 7 - r)$.

In this example we set $f_0 = (-2 + r, 1 - 2r),\ f_1 = (2 + r, 4 - r),\ f_2 = (6 + r, 10 - 3r)$. According to Equation (19), quasi-interpolating function can be written as follow, $\tilde{f}^*(\tilde{x}) = \tilde{f}_0\psi_0(\tilde{x}) + \tilde{f}_1\psi_1(\tilde{x}) + \tilde{f}_2\psi_2(\tilde{x})$. where $\psi_j(\tilde{x})$, $j = 0, 1, 2$ is introduced in Equations (16)-(18). In Figure 5, the positions of $\{(\tilde{x}_i, \tilde{f}_i^*)\}_{i=0}^2$ are presented. In Figure 6, the errors of $\tilde{x}_i$, for $i = 0, 1, 2$ is presented. In Figures 6(a)-6(c) the coordinate of the maximum value of the membership function in $\tilde{x}_0$, $\tilde{x}_1$ and $\tilde{x}_2$ are $(-3, 4E{-}6)$, $(2, 1.1E{-}6)$ and $(6, -6E{-}6)$, respectively.



(a) Error in $\tilde{x}_1$ (b) Error in $\tilde{x}_2$ (c) Error in $\tilde{x}_3$

Figure 4: The errors of approximation in $\tilde{x}_i$ for Example 2.

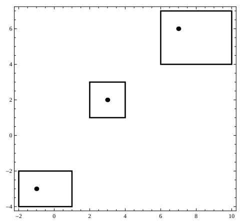
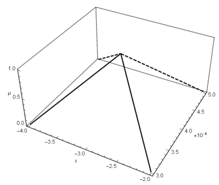
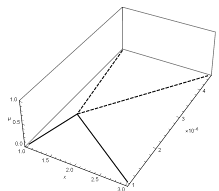


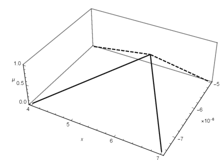
Figure 5: The position of $\{(\tilde{x}_i, \tilde{f}_i^*)\}_{i=0}^2$.



(a) Error in $\tilde{x}_1$



(b) Error in $\tilde{x}_2$



(c) Error in $\tilde{x}_3$

Figure 6: The errors of approximation in $\tilde{x}_i$ for Example 3.

Example 5.4. *By a linguistic variable we mean a variable whose values are words or sentences in a natural or artificial language. For example, temperature is a linguistic variable if its values are linguistic rather than numerical, i.e.,cold, mild, hot and etc., rather than 5,20,40. In particular, a linguistic variable with values leads to what is called fuzzy logic. In fuzzy logic, a rule base is constructed to control the output variable.*

In Table 1, sample fuzzy rules for the temperature control system in a chiller are listed:

	Fuzzy rules
1	If temperature is cold then increase the chiller temperature.
2	If temperature is mild then moderate the chiller temperature.
3	If temperature is warm then reduce the chiller temperature.

Table 1: Sample fuzzy rules for the temperature control system.

Corresponding to each of linguistic variable we define a fuzzy number as bellow:

linguistic variables	Corresponding fuzzy numbers
Cold	$(3 + 4r, 10 - 3r)$
Mild	$(15 + 5r, 23 - 3r)$
Warm	$(32 + 3r, 37 - 2r)$

Table 2: Corresponding fuzzy numbers.

We set the optimum temperature as the fuzzy set $(20 + 6r, 30 - 4r)$. To achieve this temperature, each of the temperatures in Table 2 has the following temperature changes:

linguistic variables	Corresponding fuzzy numbers
Amount of increasing	$(17 + 2r, 20 - r)$
Amount of increasing	$(5 + r, 7 - r)$
Amount of increasing	$(-12 + 3r, -7 - 2r)$

Table 3: Corresponding fuzzy numbers.

According to Equation (19), quasi-interpolating function can be written as follow: $\tilde{f}^*(\tilde{x}) = \tilde{f}_0\psi_0(\tilde{x}) + \tilde{f}_1\psi_1(\tilde{x}) + \tilde{f}_2\psi_2(\tilde{x})$. where $\psi_j(\tilde{x})$, $j = 0, 1, 2$ is introduced in Equations (16)-(18). In Figure 7, the positions of $\{(\tilde{x}_i, \tilde{f}_i^*)\}_{i=0}^2$ are presented. In Figure 8, the errors of $\tilde{x}_i$, for $i = 0, 1, 2$ is presented.

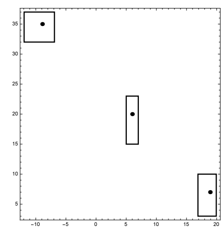
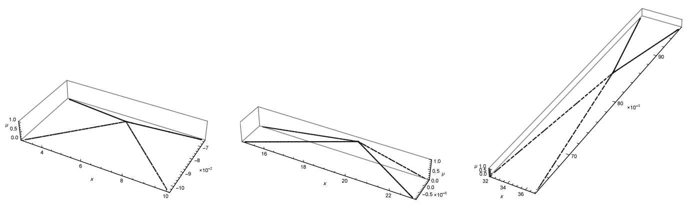


Figure 7: The position of $\{(\tilde{x}_i, \widetilde{f}_i^*)\}_{i=0}^{2}$.



(a) Error in $\tilde{x}_1$ (b) Error in $\tilde{x}_2$ (c) Error in $\tilde{x}_3$

Figure 8: The errors of approximation in $\tilde{x}_i$ for Example 4.

6 Conclusions

Fuzzy quasi-interpolations help to reduce the complexity of solving a linear system of equations compared with fuzzy interpolations. Almost all fuzzy quasi-interpolations are focused on the form of $\widetilde{f}^* : \mathbb{R} \to F(\mathbb{R})$ or $\widetilde{f}^* : F(\mathbb{R}) \to \mathbb{R}$. In this paper, we intend to offer a novel fuzzy radial basis function by the concept of source distance. Then, we will construct a fuzzy linear combination of such basis functions in order to introduce a fully fuzzy quasi-interpolation in the form of $\widetilde{f}^* : F(\mathbb{R}) \to F(\mathbb{R})$. Also the error estimation of the proposed method is proved in terms of the fully fuzzy modulus of continuity which will be introduced in this paper. Finally some examples have been given to emphasize the acceptable accuracy of our method.

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