

FUZZY SUBGROUPS AND CERTAIN EQUIVALENCE RELATIONS

A. JAIN

ABSTRACT. In this paper, we study an equivalence relation on the set of fuzzy subgroups of an arbitrary group G and give four equivalent conditions each of which characterizes this relation. We demonstrate that with this equivalence relation each equivalence class constitutes a lattice under the ordering of fuzzy set inclusion. Moreover, we study the behavior of these equivalence classes under the action of a group homomorphism.

1. Introduction

In 1981, Das [6] defined a level subgroup of a fuzzy subgroup μ of a group G as an ordinary subgroup μ_t of G , where $t \in [0, \mu(e)]$. This concept earned the distinction of being the cutoff point for many authors who formulated and proved numerous results in fuzzy algebra based on this definition of level subgroups. However, as the theory developed, it was realized that in many situations a simplification is obtained if we consider the definition of level subgroups as modified in [1]. The basic difference between the two definitions is that in the latter, the choice of t is restricted to $\text{Image } \mu$. A counterexample to an assertion in Das's paper also appeared in [14].

In the case of a finite group G , for each $t \in [0, \mu(e)]$, μ_t coincides with some μ_s , where $s \in \text{Im } \mu$. But in the case of an arbitrary group G , for any $t_0 \in [0, \mu(e)]$, two cases arise. If $t_0 \in \text{Im } \mu$, then the level subset μ_{t_0} is not equal to the strong level subset $\mu_{t_0}^>$. Otherwise we have

$$\mu_{t_0} = \mu_{t_0}^> = \bigcup_{t_0 < t \in \text{Im } \mu} \mu_t$$

The reason why the definition of level subgroup given in [1] has more suitable applications as compared to Das's definition is thoroughly discussed in [1] and this modified definition has been used very effectively for defining a new category of fuzzy subgroups [9].

In a recent paper [16], the authors have defined an equivalence relation on the set of all fuzzy subgroups $L(G)$ of a finite group G which, in fact is an intersection

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of two equivalence relations on $L(G)$. We shall discuss this in Section 3 and also provide a comparison between the equivalence relations on $L(G)$ given in [2, 15, 16]. In the same section we shall give four equivalent conditions, each of which can be taken as the definition of an equivalence relation on $L(G)$ for an arbitrary group G . Thus we delete the condition of finiteness of the given group as mandated in [2, 15, 16].

The second condition in the definition of equivalence relation in [16] says that the supports of the fuzzy subgroups μ and η are identical. The penultimate subgroup defined in [2] is an improvement over the concept of support of a fuzzy subgroup and can be more suitably applied in fuzzy algebra. Its role has already been established in [2]. Here we prove that two fuzzy subgroups in an arbitrary group G , equivalent in the sense of [2], necessarily have the same penultimate subgroups. However, their supports may be different.

In Section 4, we shall discuss the behavior of equivalence classes under homomorphism of groups. Further, we shall prove that $[\mu]$, the equivalence class consisting of the fuzzy subgroup μ , is a lattice which is closed under the operations of meet and join given by the usual intersection and union of fuzzy subgroups respectively.

2. Preliminaries

Zadeh [21] defines a fuzzy subset as a function from a nonempty set to a closed unit interval. A fuzzy subgroup of a group G is a fuzzy subset of G satisfying $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$ and $\mu(x^{-1}) = \mu(x) \forall x, y \in G$. It is known that a fuzzy subgroup attains its supremum at the identity element of the group. For a fuzzy subset μ of a set X and a real number t in the unit interval $[0, 1]$, the level subset μ_t and the strong level subset $\mu_t^>$ are defined as $\mu_t = \{x \in X \mid \mu(x) \geq t\}$ and $\mu_t^> = \{x \in X \mid \mu(x) > t\}$. A level subset μ_t for $t \leq \mu(e)$ of a fuzzy subgroup μ of a group G is an ordinary subgroup of G and is called a level subgroup of μ [6]. A better definition of level subgroup was provided in [1], where the level subset μ_t for $t \in \text{Im } \mu$ is called a level subgroup of μ . Here by $\text{Im } \mu$ we mean the range set of μ , that is $\text{Im } \mu = \{\mu(x) \mid x \in G\}$. Throughout this paper, G will denote a group, e the identity element of G and $L(G)$ the collection of all fuzzy subgroups of G .

3. A Study of Equivalence Relations on $L(G)$

Let $L_1(G)$ denote the set of all fuzzy subgroups μ of G such that $\mu(e)=1$. In [15], using the notion of level subgroups in the sense of Das [6], a binary relation denoted by ' \sim ' on $L_1(G)$ is defined for a finite group G ; this relation is shown to be an equivalence relation.

Definition 3.1. [15] Let G be a finite group and $\mu, \eta \in L_1(G)$. If $\forall x, y \in G$,

$$\mu(x) > \mu(y) \Leftrightarrow \eta(x) > \eta(y),$$

then we say $\mu \sim \eta$.

Proposition 3.2. [15] Let G be a finite group. Then $\mu \sim \eta$ in $L_1(G)$ if and only if $\Gamma_\mu(G) = \Gamma_\eta(G)$ where $\Gamma_\mu(G) = \{\mu_t \mid t \in [0, 1]\}$.

In [1], the notion of level subgroup is modified and in [2], this modified notion is used to provide a definition of equivalence relations on $L(G)$ for an arbitrary group G .

Definition 3.3. [2] For any group G , let $\mu, \eta \in L(G)$. Then μ is said to be equivalent to η denoted by $\mu \approx \eta$ if μ and η have the same chain of level subgroups. That is,

$$\{\mu_t\}_{t \in \text{Im } \mu} = \{\eta_s\}_{s \in \text{Im } \eta}$$

Clearly, $\mu \approx \eta$ is an equivalence relation on $L(G)$.

In a recent paper [16], another definition of equivalence relation on $L(G)$ is provided for a finite group G and the authors have attempted to obtain several results accordingly. The definition is as follows:

Definition 3.4. [16] Let $\mu, \eta \in L(G)$. Then μ is said to be equivalent to η ($\mu \simeq \eta$) if $\forall x, y \in G$,

- (i) $\mu(x) > \mu(y) \Leftrightarrow \eta(x) > \eta(y)$, and
- (ii) $\mu(x) = 0 \Leftrightarrow \eta(x) = 0$.

Notice that the above equivalence relation is an intersection of two equivalences. The later one, that is $\mu(x)=0 \Leftrightarrow \eta(x)=0$ avers that the supports of fuzzy subgroups μ and η are identical. The authors in [12], point out that their definition is a generalization of the notion of equality of sets. However, the same is true regarding the other two equivalences [Definitions 3.1, 3.3]. In [16], the following necessary condition has been stated with an incomplete proof. The complete proof is provided by C. Degang et al. in [7].

Proposition 3.5. [Proposition 2.5 in Murali and Makamba [16]] Suppose μ and ν are two fuzzy subsets of X such that μ is equivalent to ν . Then for each $t \in [0, 1]$ there is an $s \in [0, 1]$ such that $\mu_t = \nu_s$ or $\mu_t = \nu_s^>$. \square

Moreover, in their remark at the end of Section 2, the authors in [16] claim that they have strong evidence to suggest that $\mu_t = \nu_s$ in all cases but have not been able to prove this. They then assert that if the images of fuzzy sets are finite, then $\mu_t = \nu_s$.

However both these assertions have been very easily disproved by C. Degang et al. [7]. We recall the following example from [7] :

Example 3.6. Let $X = \{a, b, c, d\}$ and for $x \in X$, $\mu(x) = 1/2$ and $\nu(x) = 1$. Then μ is equivalent to ν in the sense of definition given by Murali and Makamba [16]. Take $t = 3/4$, then $\mu_t = \emptyset$. For any $s \in [0, 1]$, $\nu_s = X$. So the above two assertions are not true. \square

It is also demonstrated by an example in [7] that the converse implication of the above assertion is also not true. In other words, two fuzzy sets μ and ν having the same chain of level subsets may not be equivalent in the sense of Definition 3.4. Here we emphasize that by giving a little twist to the notion of level subsets or that of level subgroups, the whole scenario changes. We state here the following proposition whose proof appears later :

Proposition 3.7. *For an arbitrary group G , $\mu \sim \eta$ in $L(G)$ if and only if μ and η have the same chain of level subgroups. That is :*

$$\{\mu_t\}_{t \in \text{Im } \mu} = \{\eta_s\}_{s \in \text{Im } \eta}.$$

Notice that in the definition of equivalence " \sim ", it is not assumed that $\text{supp}(\mu) = \text{supp}(\eta)$. Motivated by the above proposition, we adopt the following definition of level subgroups [1].

Definition 3.8. Let μ be a fuzzy subgroup of a group G . Then each level subset μ_t for $t \in \text{Im } \mu$ is a subgroup of G and is called a level subgroup of μ .

The problem arising in the proof of Proposition 2.5 in [16] can be taken care of by the fact that $t \in \text{Im } \mu$ if and only if $\mu_t \neq \mu_t^>$. We prove in Theorem 3.14 that for an arbitrary group G ,

$$\mu \sim \eta \text{ if and only if } \mu \approx \eta \text{ [Definitions 3.1 and 3.3].} \quad (\text{I})$$

Notice that our definition of equivalence (Definition 3.3) and all further investigation is for an arbitrary group G whereas Makamba and Murali [16] defined equivalence of two fuzzy subgroups of a finite group G and with the extra condition that $\mu(x)=0$ if and only if $\eta(x)=0$, that is supports of μ and η are same. The attempt of Makamba and Murali was to obtain a level subset characterization of their notion of equivalence; however, as exhibited in Example 3.6 due to Degang et al.[7], this is not possible, but with the changed definition of level subgroups, we are able to obtain (I) immediately and without the assumption that supports of fuzzy subgroups are identical (see Definition 3.8 and Theorem 3.14). On the other hand, as discussed in [2], the notion of support can be replaced more fruitfully by the concept of the penultimate subgroup $P(\mu)$ of fuzzy subgroup μ . This is basically due to the fact

that the range set $\text{Im } \mu$ of the fuzzy subgroup μ may not contain the least element 0 of the evaluation lattice $[0, 1]$. It follows as a consequence of Definition 3.3 that the penultimate subgroups of two equivalent fuzzy subgroups of a group G are identical (See Proposition 3.12). We do not assume that the supports of two fuzzy subgroups μ and η are identical as Murali and Makamba have done. However, in case the images of the fuzzy subgroups μ and η contain the least element 0, then the supports of μ and η coincide with their penultimate subgroups and hence are equal by Proposition 3.12.

Definition 3.9. [2] Let $\mu \in L(G)$. Then, $P(\mu) = \{x \in G \mid \mu(x) > \text{Inf } \mu\}$ is a subgroup of G , called the penultimate subgroup of μ .

In the above definition, $\text{Inf } \mu$ means $\text{Inf } \{\mu(x) \mid x \in G\}$. In case $\text{Inf } \mu \notin \text{Im } \mu$, we have $P(\mu) = G$. Otherwise $P(\mu)$ is a proper subgroup of G . Now, in order to establish an interesting fact regarding Definition 3.3, we recall the following results whose proofs are straightforward and hence are omitted. Here G is an arbitrary group.

Lemma 3.10. Let $\mu \in L(G)$. Then, $\mu_{r_o}^> = \bigcup_{\substack{r > r_o \\ r \in \text{Im } \mu}} \mu_r$ for any $r_o \in [0, 1]$. □

The following lemma has application at later stages of this section.

Lemma 3.11. Let $\mu \in L(G)$. Then, $\mu_{t_o} = \bigcap_{\substack{t < t_o \\ t \in \text{Im } \mu}} \mu_t^>$ for each $t_o \in [0, 1]$. □

Proposition 3.12. Let $\mu, \eta \in L(G)$ such that $\mu \approx \eta$. Then, $P(\mu) = P(\eta)$.

Proof. It is easy to verify that either both μ and η attain their infimum or neither does. In the latter case we have $P(\mu) = P(\eta) = G$. Now let us assume that both μ and η attain their infimum. Let $\text{Inf } \mu = r_o \in \text{Im } \mu$ and $\text{Inf } \eta = t_o \in \text{Im } \eta$. Since $\mu \approx \eta$, we have

$$\bigcup_{r \in \text{Im } \mu} \mu_r = \bigcup_{t \in \text{Im } \eta} \eta_t$$

Since $\mu_{r_o} = \eta_{t_o} = G$, it follows that

$$\bigcup_{\substack{r \in \text{Im } \mu \\ r > r_o}} \mu_r = \bigcup_{\substack{t \in \text{Im } \eta \\ t > t_o}} \eta_t$$

By Lemma 3.10, this implies that $\mu_{r_o}^> = \eta_{t_o}^>$. We thus have $P(\mu) = P(\eta)$. □

The following proposition is an important consequence of equivalence of two fuzzy subgroups μ and η as given by Definition 3.3.

Proposition 3.13. *Let $\mu, \eta \in L(G)$ such that $\mu \approx \eta$. Then, there exists an order preserving bijection from $\text{Im } \mu$ to $\text{Im } \eta$.*

Proof. Since $\mu \approx \eta$, by Definition 3.3 for each $t \in \text{Im } \mu$, there exists an $s \in \text{Im } \eta$ such that $\mu_t = \eta_s$. Since $\eta_{s_1} = \eta_{s_2}$ if and only if $s_1 = s_2$ for $s_1, s_2 \in \text{Im } \eta$, it is clear that $s \in \text{Im } \eta$ is unique. We thus get an obvious association say α for the elements of $\text{Im } \mu$ to the elements of $\text{Im } \eta$ wherein :

$$\alpha(t) = s \quad \text{and} \quad \mu_t = \eta_{\alpha(t)}.$$

Similarly we get an association say β for the elements of $\text{Im } \eta$ with the elements of $\text{Im } \mu$ whereby:

$$\eta_s = \mu_{\beta(s)}.$$

The explicit structure of the map α is more clear from the following:

Since $t \in \text{Im } \mu$, $t = \mu(x_0)$ for some $x_0 \in G$. We claim here that $\alpha(t) = \eta(x_0)$. That is

$$\alpha(\mu(x)) = \eta(x) \quad \text{for each } x \in G.$$

Suppose that $\eta(x_0) = t'$. Then it is enough to show that $\mu_t = \eta_{t'}$, because $\mu_t = \eta_{t'}$ implies $\eta_{t'} = \eta_{\alpha(t)}$, which further implies $\alpha(t) = t' = \eta(x_0)$ since both $\alpha(t)$ and t' belong to $\text{Im } \eta$. Now to prove $\mu_t = \eta_{t'}$, let $y \in \mu_t$.

If $\mu(y) = \mu(x_0)$, then since $\eta(x_0) = t'$, we have $x_0 \in \eta_{t'} = \mu_{\beta(t')}$. This implies

$$\mu(y) = \mu(x_0) \geq \beta(t').$$

i.e. $y \in \mu_{\beta(t')} = \eta_{t'}$.

If $\mu(y) \neq \mu(x_0)$, then $y \in \mu_t$ implies

$$\mu(y) > t = \mu(x_0).$$

Let $\mu(y) = t_0$. Then

$$y \in \mu_{t_0} = \eta_{\alpha(t_0)} \quad \text{whereas} \quad x_0 \notin \mu_{t_0} = \eta_{\alpha(t_0)}.$$

Thus $\eta(y) \geq \alpha(t_0)$ and $t' = \eta(x_0) < \alpha(t_0)$. Hence $\eta(y) > t'$. This gives $y \in \eta_{t'}$. Therefore $\mu_t \subseteq \eta_{t'}$. We get $\eta_{t'} \subseteq \mu_t$ by a similar argument, thus implying $\mu_t = \eta_{t'}$.

It is easy to verify that this correspondence α from $\text{Im } \mu$ onto $\text{Im } \eta$ is one to one . In the case that the bijection α is not order preserving, there exist $t_i, t_j \in \text{Im } \mu$ such that

$$t_i < t_j \text{ but } \alpha(t_i) > \alpha(t_j). \quad (1)$$

Now $t_i < t_j$ implies $\mu_{t_j} \subsetneq \mu_{t_i}$ and hence $\eta_{\alpha(t_j)} \subsetneq \eta_{\alpha(t_i)}$. This gives $\alpha(t_i) < \alpha(t_j)$ which contradicts (1). Hence α is the required order preserving bijection from $\text{Im } \mu$ to $\text{Im } \eta$. \square

In the above proposition, the process of shifting of image sets of fuzzy subgroups [2] is used. The next theorem gives four equivalent conditions for the equivalence relation $\mu \approx \eta$ of two fuzzy subgroups μ and η of an arbitrary group G.

Theorem 3.14. *Let $\mu, \eta \in L(G)$. Then the following are equivalent:*

- (i) $\mu \approx \eta$
- (ii) $\forall x, y \in G, \mu(x) > \mu(y) \Leftrightarrow \eta(x) > \eta(y)$
- (iii) $\forall x, y \in G, \mu(x) \geq \mu(y) \Leftrightarrow \eta(x) \geq \eta(y)$
- (iv) $\{\mu_t^>\}_{t \in \text{Im } \mu} = \{\eta_s^>\}_{s \in \text{Im } \eta}$.

Proof. (i) \Rightarrow (ii) Let $x, y \in G$ such that $\mu(x) > \mu(y)$. Suppose that $\mu(x) = t_1$ and $\mu(y) = t_2$. Then $x \in \mu_{t_1}, y \in \mu_{t_2}$ and $y \notin \mu_{t_1}$. By (i) we have $\mu_{t_1} = \eta_{r_1}$ for some $r_1 \in \text{Im } \eta$. Now, $x \in \mu_{t_1} = \eta_{r_1}$ and $y \notin \mu_{t_1} = \eta_{r_1}$ imply that $\eta(x) \geq r_1$ and $\eta(y) < r_1$. Thus, $\eta(x) > \eta(y)$. Similarly it can be proved that $\eta(x) > \eta(y)$ implies $\mu(x) > \mu(y)$. This proves (ii).

(ii) \Rightarrow (i) To prove $\{\mu_t^>\}_{t \in \text{Im } \mu} = \{\eta_r^>\}_{r \in \text{Im } \eta}$, let $\mu_{t_0} \in \{\mu_t^>\}_{t \in \text{Im } \mu}$.

If $\mu_{t_0} = G$, then clearly $t_0 = \text{Inf}(\text{Im } \mu)$. Since $t_0 \in \text{Im } \mu$, μ attains its infimum. Also $t_0 = \mu(x_0)$ for some $x_0 \in G$. We first prove that η also attains its infimum. Suppose η does not attain its infimum. Let $\text{Inf}(\text{Im } \eta) = t_1$. Then $\eta(x) > t_1$ for each $x \in G$. In particular $\eta(x_0) > t_1$. Thus there exists some $x_1 \in G$ such that $\eta(x_0) > \eta(x_1) > t_1$. By the given condition, it follows that $\mu(x_0) > \mu(x_1)$. This contradicts $\mu(x_0) = \text{Inf}(\text{Im } \mu)$ thereby implying that η also attains its infimum. Let $\text{Inf}(\text{Im } \eta) = r_0$. Then $r_0 \in \text{Im } \eta$ and thus

$$\mu_{t_0} = G = \eta_{r_0} \in \{\eta_r^>\}_{r \in \text{Im } \eta}.$$

On the other hand if $\mu_{t_0} \neq G$, then for $t_0 \in \text{Im } \mu$, there exists $x_0 \in G$ such that $t_0 = \mu(x_0)$. We take $r_0 = \eta(x_0)$. Now, $y \in G \sim \mu_{t_0} \Leftrightarrow \mu(y) < t_0 = \mu(x_0) \Leftrightarrow \eta(y) < \eta(x_0) = r_0 \Leftrightarrow y \in G \sim \eta_{r_0}$. Thus, $\mu_{t_0} = \eta_{r_0} \in \{\eta_r\}_{r \in \text{Im } \eta}$. This proves that

$$\{\mu_t\}_{t \in \text{Im } \mu} \subseteq \{\eta_r\}_{r \in \text{Im } \eta}.$$

The other side can be proved similarly. Hence $\mu \approx \eta$.

(ii) \Rightarrow (iii) Let $x, y \in G$ such that $\mu(x) \geq \mu(y)$. Suppose, if possible, that $\eta(x) < \eta(y)$. Then by (ii), $\mu(x) < \mu(y)$. This contradiction implies $\eta(x) \geq \eta(y)$. Similarly, $\eta(x) \geq \eta(y)$ implies $\mu(x) \geq \mu(y)$. This proves (iii).

(iii) \Rightarrow (ii) Let $x, y \in G$ such that $\mu(x) > \mu(y)$. Suppose, if possible, that $\eta(x) \leq \eta(y)$. Then by (iii), $\mu(x) \leq \mu(y)$. This contradiction implies $\eta(x) > \eta(y)$. Similarly, $\eta(x) > \eta(y)$ implies $\mu(x) > \mu(y)$. This proves (ii).

(i) \Rightarrow (iv) Since $\mu \approx \eta$, by Proposition 3.12, there exists a bijective order preserving map $\alpha : \text{Im } \mu \rightarrow \text{Im } \eta$ such that $\mu_t = \eta_{\alpha(t)} \forall t \in \text{Im } \mu$. Let $t_0 \in \text{Im } \mu$. Then, by Lemma 3.9 we have

$$\mu_{t_0}^> = \bigcup_{\substack{t > t_0 \\ t \in \text{Im } \mu}} \mu_t. \quad (1)$$

If $s_0 = \alpha(t_0)$, then $s_0 \in \text{Im } \eta$ and $t > t_0$ implies $s = \alpha(t) > \alpha(t_0) = s_0$. We then have $\mu_t = \eta_{\alpha(t)} = \eta_s$. This special correspondence induced by α , when applied to (1), gives

$$\mu_{t_0}^> = \bigcup_{\substack{t > t_0 \\ t \in \text{Im } \mu}} \mu_t = \bigcup_{\substack{s > s_0 \\ s \in \text{Im } \eta}} \eta_s = \eta_{s_0}^>.$$

Thus, $\{\mu_t^>\}_{t \in \text{Im } \mu} \subseteq \{\eta_s^>\}_{s \in \text{Im } \eta}$. Similarly, we have $\{\eta_s^>\}_{s \in \text{Im } \eta} \subseteq \{\mu_t^>\}_{t \in \text{Im } \mu}$ and thus (iv) is verified.

(iv) \Rightarrow (i) Here, it is assumed that $\{\mu_t^>\}_{t \in \text{Im } \mu} = \{\eta_s^>\}_{s \in \text{Im } \eta}$. As in Proposition 3.13, it can be established that there exists a bijective order preserving map $\beta : \text{Im } \mu \rightarrow \text{Im } \eta$ such that $\mu_t^> = \eta_{\beta(t)}^>$. If $t_0 \in \text{Im } \mu$, then by Lemma 3.11,

$$\mu_{t_0} = \bigcap_{\substack{t < t_0 \\ t \in \text{Im } \mu}} \mu_t^>. \quad (2)$$

Taking $s_0 = \beta(t_0)$, $s_0 \in \text{Im } \eta$ and $t < t_0$ implies $s = \beta(t) < \beta(t_0) = s_0$. Therefore

$$\mu_t^> = \eta_{\beta(t)}^> = \eta_s^>.$$

This correspondence induced by β , when applied to (2) gives

$$\mu_{t_0} = \bigcap_{\substack{t < t_0 \\ t \in \text{Im } \mu}} \mu_t^> = \bigcap_{\substack{s < s_0 \\ s \in \text{Im } \eta}} \eta_s^> = \eta_{s_0}.$$

We thus have $\{\mu_t\}_{t \in \text{Im } \mu} \subseteq \{\eta_s\}_{s \in \text{Im } \eta}$. A similar verification gives $\{\eta_s\}_{s \in \text{Im } \eta} \subseteq \{\mu_t\}_{t \in \text{Im } \mu}$. Hence (i) holds. \square

In [16], the authors have tried to compare the concepts of fuzzy isomorphism [12,13] and fuzzy equivalence [16] and observed that equivalence is finer than isomorphism. A more concrete comparison appears in [7].

4. Action of Group Homomorphisms on Equivalence Classes

For any group G , $[\mu]$ denotes the equivalence class consisting of the fuzzy subgroup μ of G . In this section, we study the behavior of equivalence classes of fuzzy subgroups under group homomorphisms. These equivalence classes are the ones obtained under the action of equivalence relation as given by definition 3.3. As pointed out by one of the referees, such studies were initiated by Murali and Makamba in [16] where the definition of equivalence relation is different from that of ours. Moreover, their studies were carried out for finite groups and hence the fuzzy subgroups which were considered in their work are all with finite range sets, whereas, in this paper, we study the action of homomorphisms on equivalence classes of fuzzy subgroups of arbitrary groups under the condition of the sup property, which is a generalization of fuzzy subgroups with finite range sets. Fuzzy subgroups with finite images surely have the sup property but, as is shown in Example 4.2, the converse is not true.

Definition 4.1. A fuzzy subgroup μ of G is said to have the sup property if, for each nonempty subset A of G , $\sup_{x \in A} \mu(x) = \mu(a)$ for some $a \in A$.

Here we provide an example of a fuzzy subgroup with the sup property with infinite range set.

Example 4.2. Let G be the group of positive real numbers under the composition of multiplication. Let us denote by $\langle 3^{1/2^n} \rangle$ the subgroup generated by $3^{1/2^n}$ where n is a fixed positive integer. Define a fuzzy set μ in G as follows:

$$\mu(x) = \begin{cases} 1 & \text{if } x \in \langle 3^{1/2} \rangle, \\ \frac{1}{2}(1 + 1/2^n) & \text{if } x \in \langle 3^{1/2^{n+1}} \rangle - \langle 3^{1/2^n} \rangle, \\ & n = 1, 2, 3, \dots, \\ \infty & \\ \frac{1}{2} & \text{if } x \in G - \bigcup_{n=1} \langle 3^{1/2^n} \rangle. \end{cases}$$

Here, $\text{Im } \mu = \{1, 3/4, 5/8, \dots, 1/2\}$. Clearly for each $t \in \text{Im } \mu$, μ_t is a subgroup of G . Therefore μ is a fuzzy subgroup of G . Now it can be seen that any subset of $\text{Im } \mu$ contains its supremum. Thus μ has the sup property. Also, $1/2$ is a cluster point of $\text{Im } \mu$. \square

The following facts about the sup property are worth noticing. In particular, we note that if the equivalence relation is as defined in Definition 3.3, the sup property is invariant.

Lemma 4.3. *If $\mu \in L(G)$ and μ has the sup property, then each $\eta \in [\mu]$ also has the sup property.*

Proof. Let μ be a fuzzy subgroup of G with the sup property and let $\eta \in [\mu]$. If A is any nonempty subset of G , then since μ has the sup property, there exists an $x_0 \in A$ such that

$$\sup_{x \in A} \mu(x) = \mu(x_0).$$

Let

$$\sup_{x \in A} \eta(x) = s.$$

Since $\mu(x) \leq \mu(x_0) \forall x \in A$, therefore $\eta(x) \leq \eta(x_0) \forall x \in A$ as $\mu \approx \eta$ (By Theorem 3.14 (iii)). This implies that $s \leq \eta(x_0)$. Also,

$$\sup_{x \in A} \eta(x) = s, \text{ and } x_0 \in A.$$

Therefore $\eta(x_0) \leq s$. We thus get that there exists an $x_0 \in A$ such that

$$\sup_{x \in A} \eta(x) = \eta(x_0),$$

thereby implying that η has the sup property. \square

For the sake of completeness we recall here that the sup property is also preserved under the action of a group homomorphism and under the inverse of a group homomorphism.

Proposition 4.4. *If f is a group homomorphism from a group G to a group H and if $\mu \in L(G)$ has the sup property, then $f(\mu) \in L(H)$ also has the sup property. \square*

Proposition 4.5. *If f is a group homomorphism from a group G to a group H and if $\eta \in L(H)$ has the sup property, then $f^{-1}(\eta) \in L(G)$ also has the sup property.*

Proof. Suppose $\eta \in L(H)$ has the sup property. Let A be a nonempty subset of G . Consider

$$\begin{aligned} \sup_{g \in A} f^{-1}(\eta)(g) &= \sup_{g \in A} \eta(f(g)) \\ &= \sup_{h \in A^*} \eta(h). \end{aligned}$$

where $A^* = f(A)$. A^* is clearly a nonempty subset of H . Therefore, since η has the sup property,

$$\sup_{h \in A^*} \eta(h) = \eta(h_o) \text{ for some } h_o \in A^*.$$

Now $h_o \in A^*$ implies that $h_o = f(g_o)$ for some $g_o \in A$. Thus we have

$$\begin{aligned} \sup_{g \in A} f^{-1}(\eta)(g) &= \sup_{h \in A^*} \eta(h) \\ &= \eta(h_o) \\ &= \eta(f(g_o)) \\ &= f^{-1}(\eta)(g_o) \text{ where } g_o \in A. \end{aligned}$$

Hence $f^{-1}(\eta)$ has the sup property. \square

The following is the main result of this section. In order to arrive at this result, we make use Lemma 4.3. We remark here that Makamba and Murali did not use the sup property since their results are derived for a restricted class of fuzzy subgroups of finite groups, i.e. fuzzy subgroups having finite range sets.

Proposition 4.6. *Let f be a surjective group homomorphism from a group G to a group H and let $\mu \in L(G)$ have the sup property. Then $f([\mu]) \subseteq [f(\mu)]$.*

Proof. Let $\mu \in L(G)$ have the sup property. It is enough to prove that if $\mu \approx \eta$ in $L(G)$, then $f(\mu) \approx f(\eta)$ in $L(H)$. $\mu \approx \eta$ in $L(G)$ implies that $\{\mu_t\}_{t \in \text{Im } \mu} = \{\eta_s\}_{s \in \text{Im } \eta}$ and

by Lemma 4.3, both μ and η have the sup property. Let $t \in \text{Im } f(\mu)$. Since μ has the sup property and f is surjective, we have the following:

$$\begin{aligned} t &= f(\mu)(h) \text{ for some } h \in H. \\ &= \sup_{g_i \in f^{-1}(h)} \mu(g_i) \\ &= \sup_{i \in \Lambda} t_i \text{ where } t_i = \mu(g_i) \forall i \in \Lambda \text{ and } \Lambda = \{i / f(g_i) = h\} \\ &= \mu(g_j) \text{ for some } j \in \Lambda. \end{aligned}$$

Thus $t \in \text{Im } \mu$. If α is the order preserving bijection from $\text{Im } \mu$ to $\text{Im } \eta$, then

$$\alpha(t) = \eta(g_j) \quad \text{and} \quad \mu_t = \eta_{\alpha(t)}.$$

Also for each $i \in \Lambda$, $\alpha(t_i) = \eta(g_i)$. Since η has the sup property, therefore we have

$$\begin{aligned} \sup_{i \in \Lambda} \alpha(t_i) &= \sup_{i \in \Lambda} \eta(g_i) \\ &= \eta(g_k) \text{ for some } k \in \Lambda. \end{aligned}$$

We prove here that $\eta(g_k) = \eta(g_j)$. Since $\sup_{i \in \Lambda} \mu(g_i) \leq \mu(g_j)$, we have

$$\mu(g_i) \leq \mu(g_j) \text{ for each } i \in \Lambda.$$

This implies $\eta(g_i) \leq \eta(g_j)$ for each $i \in \Lambda$ as $\mu \approx \eta$. Therefore

$$\sup_{i \in \Lambda} \eta(g_i) \leq \eta(g_j).$$

Hence $\eta(g_k) \leq \eta(g_j)$. Also, $j \in \Lambda$ implies that $\eta(g_j) \leq \eta(g_k)$. We thus have $\eta(g_k) = \eta(g_j)$. Therefore,

$$\begin{aligned} \sup_{i \in \Lambda} \alpha(t_i) &= \sup_{i \in \Lambda} \eta(g_i) \\ &= \eta(g_j) \\ &= \alpha(t). \end{aligned}$$

Furthermore,

$$\sup_{i \in \Lambda} \eta(g_i) = f(\eta)(h) \in \text{Im } f(\eta).$$

This implies that $\alpha(t) \in \text{Im } f(\eta)$. We shall now prove that $f(\mu)_t = f(\eta)_{\alpha(t)}$.

If $x \in f(\mu)_t$, then

$$f(\mu)(x) \geq t = \mu(g_j).$$

This implies

$$\sup_{g \in f^{-1}(x)} \mu(g) \geq \mu(g_j).$$

However, since μ has the sup property, $\sup \mu(g) = \mu(g_o)$ for some $g_o \in f^{-1}(x)$.

Therefore we have $g \in f^{-1}(x)$.

Now $\mu(g_o) \geq \mu(g_j)$, which gives $\eta(g_o) \geq \eta(g_j)$ as $\mu \approx \eta$. By an argument similar to that used to prove $\alpha(t) \in \text{Im } f(\eta)$, it can be shown that

$$\eta(g_o) = \sup_{g \in f^{-1}(x)} \eta(g).$$

Thus

$$\sup_{g \in f^{-1}(x)} \eta(g) \geq \eta(g_j).$$

That is

$$f(\eta)(x) \geq \eta(g_j) = \alpha(t).$$

This proves that $x \in f(\eta)_{\alpha(t)}$. We thus have the containment $f(\mu)_t \subseteq f(\eta)_{\alpha(t)}$. It can be shown similarly that $f(\eta)_{\alpha(t)} \subseteq f(\mu)_t$. Therefore, for each $t \in \text{Im } f(\mu)$, there exists $\alpha(t) \in \text{Im } f(\eta)$, such that $f(\mu)_t = f(\eta)_{\alpha(t)}$. This gives

$$\{f(\mu)_t\}_{t \in \text{Im } f(\mu)} \subseteq \{f(\eta)_s\}_{s \in \text{Im } f(\eta)}.$$

The proof of the other side of the containment is similar. Hence $f(\mu) \approx f(\eta)$. \square

For our definition of equivalence relation (Definition 3.3), we state the following proposition (Proposition 4.7) without proof. This proposition is true in general and without any restriction on fuzzy subgroups; Makamba and Murali have used their definition to establish a similar result for finite groups.

Proposition 4.7. *Let f be a group homomorphism from a group G to a group H and let $\eta \in L(H)$. Then $f^{-1}([\eta]) \subseteq [f^{-1}(\eta)]$.* \square

Proposition 4.8. *Let f be a bijective group homomorphism from a group G to a group H and let $\mu \in L(G)$ have the sup property. Then $f([\mu]) = [f(\mu)]$. \square*

Proposition 4.9. *Let f be a bijective group homomorphism from a group G to a group H and let $\eta \in L(H)$. Then $f^{-1}([\eta]) = [f^{-1}(\eta)]$. \square*

5. The Lattice Structure of an Equivalence Class

The construction of new lattices of fuzzy subgroups of a group is a peculiarity of fuzzy setting [3, 4, 5, 11]. In this section we shall show that the fuzzy subgroups of a group belonging to same equivalence class obtained under Definition 3.3 constitute a lattice under a join operation which is simply fuzzy set theoretic union. This makes the study of lattice structure different from that of earlier authors. The equivalence class $[\mu]$ is clearly a poset with respect to the order relation of fuzzy set inclusion. In the following theorem, the meet \wedge and the join \vee are given by minimum and maximum operators respectively.

Proposition 5.1. *If μ is a fuzzy subgroup of a group G , then $[\mu]$ forms a lattice under the ordering of fuzzy set inclusion.*

Proof. We shall first prove that if $\eta \in [\mu]$, then $\mu \wedge \eta \in [\mu]$. Since $\eta \in [\mu]$, we have $\mu \approx \eta$ thereby implying that $\{\mu_t\}_{t \in \text{Im } \mu} = \{\eta_s\}_{s \in \text{Im } \eta}$. We shall use the following obvious statements:

$$(\mu \wedge \eta)_t = \mu_t \cap \eta_t \quad \forall t \in [0, 1], \text{ and } t \in \text{Im}(\mu \wedge \eta) \Rightarrow t \in \text{Im } \mu \text{ or } t \in \text{Im } \eta.$$

Now let $t \in \text{Im}(\mu \wedge \eta)$.

Case 1: $t \in \text{Im } \mu$ and $t \in \text{Im } \eta$.

Since $t \in \text{Im } \mu$, there exists $s \in \text{Im } \eta$ such that $\mu_t = \eta_s$. Now, if $t \leq s$ then $\eta_s \subseteq \eta_t$.

Therefore $\eta_s \cap \eta_t = \eta_s$. As $s \in \text{Im } \eta$, this gives

$$\mu_t \cap \eta_t = \eta_s \in \{\mu_r\}_{r \in \text{Im } \mu}.$$

And if $t > s$ then $\eta_t \subseteq \eta_s$ implies that $\eta_t \cap \eta_s = \eta_t$. Therefore, as $t \in \text{Im } \eta$,

$$\mu_t \cap \eta_t = \eta_t \in \{\mu_r\}_{r \in \text{Im } \mu}.$$

Thus $\mu_t \cap \eta_t \in \{\mu_r\}_{r \in \text{Im } \mu}$.

Case 2: $t \in \text{Im } \mu$ and $t \notin \text{Im } \eta$.

Since $t \in \text{Im}(\mu \wedge \eta)$, therefore for some $x \in G$, $t = \min\{\mu(x), \eta(x)\}$. Now $t \notin \text{Im} \eta$ implies that $t = \mu(x) < \eta(x)$. Let $\eta(x) = t'$. Since $\mu \approx \eta$, there exists an order preserving bijection say α from $\text{Im} \mu$ to $\text{Im} \eta$ and hence

$$\alpha(t) = \alpha(\mu(x)) = \eta(x) = t', \text{ and } \mu_t = \eta_{\alpha(t)} = \eta_{t'}.$$

Now $t < t'$ implies that $\eta_{t'} \subseteq \eta_t$. Thus, $\eta_{t'} \cap \eta_t = \eta_{t'} \in \{\mu_r\}_{r \in \text{Im} \mu}$ as $t' \in \text{Im} \eta$. Hence we have

$$\mu_t \cap \eta_t \in \{\mu_r\}_{r \in \text{Im} \mu}.$$

Case 3: $t \in \text{Im} \eta$ and $t \notin \text{Im} \mu$.

This case is similar to Case 2.

We thus get that

$$\{(\mu \wedge \eta)_t\}_{t \in \text{Im}(\mu \wedge \eta)} \subseteq \{\mu_r\}_{r \in \text{Im} \mu}. \quad (1)$$

On the other hand, if $t \in \text{Im} \mu$, then $t = \mu(x)$ for some $x \in G$. Then $\alpha(t) = \eta(x)$ where α is the order preserving bijection from $\text{Im} \mu$ to $\text{Im} \eta$ as $\mu \approx \eta$ and $\mu_t = \eta_{\alpha(t)}$. Let $\min\{\mu(x), \eta(x)\} = r$. Then $r \in \text{Im}(\mu \wedge \eta)$. We claim that

$$\mu_t = (\mu \wedge \eta)_r = \mu_r \cap \eta_r.$$

If $\mu(x) \leq \eta(x)$, then we have

$$r = \mu(x) = t \leq \eta(x) = \alpha(t).$$

Now $t \leq \alpha(t)$ implies that $\eta_{\alpha(t)} \subseteq \eta_t$. This gives

$$\eta_{\alpha(t)} = \eta_{\alpha(t)} \cap \eta_t,$$

and hence

$$\mu_t = \mu_r \cap \eta_r.$$

If $\mu(x) > \eta(x)$, then

$$r = \alpha(t) = \eta(x) < \mu(x) = t.$$

This implies $\mu_t \subseteq \mu_r$. Now

$$\mu_t = \eta_{\alpha(t)} = \eta_r.$$

Therefore $\eta_r \subseteq \mu_r$, which implies

$$\mu_t = \eta_{\alpha(t)} = \eta_r = \mu_r \cap \eta_r.$$

We thus get that for each $t \in \text{Im } \mu$, there exists an $r \in \text{Im } (\mu \wedge \eta)$ such that $\mu_t = (\mu \wedge \eta)_r$. Therefore,

$$\{\mu_t\}_{t \in \text{Im } \mu} \subseteq \{(\mu \wedge \eta)_r\}_{r \in \text{Im } (\mu \wedge \eta)}. \quad (2)$$

From (1) and (2), it is evident that $\mu \wedge \eta \approx \mu$. A similar procedure shows that if $\eta \in [\mu]$, then $\mu \vee \eta \in [\mu]$. We thus have the lattice structure of $[\mu]$, the equivalence class of the fuzzy subgroup μ of G . \square

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APARNA JAIN, DEPARTMENT OF MATHEMATICS, SHIVAJI COLLEGE, UNIVERSITY OF DELHI, NEW DELHI,
INDIA

E-mail address: jainaparna@yahoo.com