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FUZZY HYPERVECTOR SPACES OVER VALUED FIELDS

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Abstract. In this note we first redefine the notion of a fuzzy hypervector space (see [1]) and then introduce some further concepts of fuzzy hypervector spaces, such as fuzzy convex and balance fuzzy subsets in fuzzy hypervector spaces over valued fields. Finally, we briefly discuss on the convex (balanced) hull of a given fuzzy set of a hypervector space.

1. **Introduction**

The notion of a fuzzy subset of a nonempty set was introduced by L. Zadeh in 1965 [31] as a function from a nonempty set X to the unit real interval $I = [0,1]$. Rosenfeld defined the concept of a fuzzy subgroup of a given group G [23] and then many researchers developed it in all the fields of algebra. The concepts of a fuzzy field and a fuzzy linear space over a fuzzy field were introduced and discussed by Nanda [21]. Fuzzy vector spaces over the real or complex field were discussed by Katsaras and Liu [15].

The notion of a hypergroup was introduced by F. Marty [20]. Since then many researchers have studied this field and developed it, for example see [7, 8, 9, 28, 29]. In 1990, M. S. Tallini introduced the notion of a hypervector space [25] and studied the its basic properties.

Also fuzzy set theory has been well developed in the context of hyperalgebraic structure theory.(for example see [1-6], [10], [11], [12], [13], [16], [17], [18], [24], [32]). In [1] and [6] the author introduced the notions of fuzzy (co-) norm and fuzzy inner product in fuzzy hypervector spaces respectively. The purpose of this paper is the study of some further notions such as fuzzy convex and balanced subsets in fuzzy hypervector spaces over valued fields. Also we briefly discuss on convex (balanced) closure of a given fuzzy set of a hypervector space. Convex fuzzy sets were introduced by Zadeh [31] and subsequently such sets were analysed by in Lowen [19] in the context of fuzzy subspaces over the real or complex number fields.

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2. **Preliminaries**

In this section we present some definitions and simple properties of hypervector spaces and fuzzy subsets, that we shall use in later.

Let H be a nonempty set. By $P_*(H)$ we mean the family of all nonempty subsets of H.

Definition 2.1. [8] A map $. : H \times H \longrightarrow P_*(H)$ is called a *hyperoperation* or join operation.

The join operation is extended to nonempty subsets of H in a natural way, so that $A.B$ or AB is given by

$$AB = \bigcup\{ab | a \in A \ \text{ and } \ b \in B \ \}.$$

Definition 2.2. [8] A *hypergroup* is a structure $(H, .)$ that satisfies two axioms,
(Reproduction) $aH = H = Ha$ for all $a \in H$;
(Associativity) $a(bc) = (ab)c$. for all $a, b, c \in H$.

Definition 2.3. [8] Let H be a hypergroup and K a nonempty subset of H. Then K is a *subhypergroup* of H if it is a hypergroup with respect to the hyperoperation . restricted to K.

Hence it is clear that a subset K of H is a subhypergroup if and only if $aK = Ka = K$, under the hyperoperation on H.

Definition 2.4. [25] Let K be a field and $(V, +)$ be an abelian group. We define a hypervector space over K as a quadruple $(V, +, \circ, K)$, where $\circ$ is a mapping

$$\circ : K \times V \longrightarrow P_*(V)$$

such that the following conditions are satisfied:
(1) $\forall a \in K, \forall x, y \in V, \quad a \circ (x + y) \subseteq a \circ x + a \circ y$ (right distributivity),
(2) $\forall a, b \in K, \forall x \in V, (a + b) \circ x \subseteq a \circ x + b \circ y$ (left distributivity) ,
(3) $\forall a, b \in K, \forall x \in V, a \circ (b \circ x) = (ab) \circ x$,
(4) $\forall a \in K, \forall x \in V, a \circ (-x) = -a \circ x$,
(5) $\forall x \in V, x \in 1 \circ x$.
For simplicity of notation sometimes we write ax instead $a \circ x$.

Remark 2.5. (i) In the right member of (1) and (2) the sum is meant in the sense of Frobenius, that is we consider the set of all the sums of an element of $a \circ x$ with an element of $a \circ y$. Moreover the left member of (3) means the set-theoretical union of all the sets $a \circ y$, where y runs over the set $b \circ x$.

(ii) We say that $(V, +, \circ, K)$ is *strongly left distributive* iff

$$\forall a \in K, \forall x, y \in V, \quad a \circ (x + y) = a \circ x + a \circ y,$$

and *anti distributive*, iff

$$\forall a \in K, \forall x, y \in V, \quad a \circ (x + y) \supseteq a \circ x + a \circ y,$$

In a similar way we define the strongly right distributive law.

(iii) Let $\Omega = 0 \circ \underline{0}$, where $\underline{0}$ is the zero of $(V, +)$. In [25] it is shown that if V is either strongly right or left, then Ω is a subgroup of $(V, +)$.

Proposition 2.6. *(See page 169 [25]). Let V be a strongly left distributive hypervector space over the field K. Then the following statements hold:*

(1) *Ω is a subgroup of $(V, +)$;*

(2) $\forall a \in K, a \circ 0 = \Omega = a \circ \Omega$;

(3) $\forall x \in V, 0 \circ x \supseteq \Omega$

(4) *$\forall x \in V, 0 \circ x$ is a subhypergroup of $(V, +)$.*

Definition 2.7. (i) For a fuzzy subset μ of X, the level subset μ_t is defined by

$$\mu_t = \{x \in X | \mu(x) \geq t\}, \quad t \in [0, 1].$$

(ii) The image of μ is denoted by $Im(\mu)$ and is defined by

$$Im(\mu) = \{\mu(x) | x \in X\}.$$

(iii) Let $f : X \longrightarrow Y$ be a mapping and $\mu \in FS(X)$ and $\nu \in FS(Y)$. Then we define $f(\mu) \in FS(Y)$ and $f^{-1}(\nu) \in FS(X)$ respectively as follows:

$$f(\mu)(y) = \begin{cases} \bigvee\limits_{x \in f^{-1}(y)} \mu(x) & \text{if } f^{-1}(y) \neq \emptyset, \\ 0 & \text{otherwise} \end{cases}$$

and

$$f^{-1}(\nu)(x) = \nu(f(x)), \quad \forall x \in X.$$

Definition 2.8. [30] Let K be a field and $\nu \in FS(K)$. Suppose the following conditions hold:

(i) $\nu(a + b) \geq \nu(a) \wedge \nu(b) \quad \forall a, b \in K$;

(ii) $\nu(-a) \geq \nu(a), \quad \forall a \in K$;

(iii) $\nu(ab) \geq \nu(a) \wedge \nu(b) \quad \forall a, b \in K$;

(iv) $\nu(a^{-1}) \geq \nu(a) \quad \forall a (\neq 0) \in K$;

Then we call ν a *fuzzy field* in K and denote it by ν_K. Also ν_K is called a fuzzy field of K.

Obviously, Definition 2.8 is a generalization of the classical field notion.

Proposition 2.9. *(Proposition 2.1 [25]). If ν_K is a fuzzy field of K, then*

(i) $\nu(0) \geq \nu(x) \quad \forall a \in K$.

(ii) $\nu(1) \geq \nu(a), \forall a \in K \setminus \{0\}$.

(iii) $\nu(0) \geq \nu(1)$.

In the following we state a well-known result in fuzzy field theory.

Proposition 2.10. *Let K be a field and $\nu \in FS(K)$. Then $\nu \in FS(K)$ is a fuzzy subfield of K if and only if ν_α is a subfield of K for all $\alpha \in Im(\nu)$.*

3. **Fuzzy Hypervector Spaces**

In [1] and [6] the author introduced the definition of a fuzzy hyperspace over a fuzzy field. In the following we first recall the definition of a hypervector space and then give some related results.

Definition 3.1. (Modification Version of Definition 3.3 [1]). Let V be a hypervector space over a field K and ν be a fuzzy subfield of K. A fuzzy set μ_V of V is said to be a *fuzzy hypervector space* of V over fuzzy field ν_K if, for all $x, y \in V$, and all $a \in K$, the following conditions are satisfied:

(i) $\mu_V(x+y) \geq \mu_V(x) \wedge \mu_V(y)$;

(ii) $\mu_V(-x) \geq \mu_V(x)$;

(iii) $\bigwedge_{y \in a \circ x} \mu_V(y) \geq \mu_V(x) \wedge \nu_K(a)$.

(iv) $\nu_K(1) \geq \mu_V(0)$.

We say that μ_V is a fuzzy hypervector space over the fuzzy field ν_K. Hereafter except for ambiguous cases we shall drop the subscripts on μ and ν.

Obviously, Definition 3.1 is a generalization of the concept of a fuzzy vector space and also of the classical notion of a hypervector space (in the Tallini's sense [25]).

Remark 3.2. (i) In Definition 3.1 if we consider $\nu = \chi_K$, the characteristic function of K, then μ is a fuzzy hypervector space.

(ii) In the sequel of this note, unless otherwise specified, we assume that V is a hypervector space over the field K.

Proposition 3.3. *If μ_V is a fuzzy hypervector space over the fuzzy field ν_K, then*

(i) $\nu(0) \geq \mu(0)$;

(ii) $\mu(0) \geq \mu(x), \quad \forall x \in V$;

(iii) $\nu(0) \geq \mu(x), \quad \forall x \in V$.

Proof. The proof is an immediate consequence of Proposition 2.9 and Definition 3.1. □

Proposition 3.4. *Let V be a strongly left distributive hypervector space over the field K and ν_K be a fuzzy field. Let $\mu \in FS(V)$. Then μ_V is a fuzzy hypervector space over ν_K iff*

(i) $\bigwedge_{z \in \alpha \circ x + \beta \circ y} \mu(z) \geq ((\nu(\alpha) \wedge \mu(x)) \bigwedge (\nu(\beta) \wedge \mu(y)), \quad \forall x, y \in V, \forall \alpha, \beta \in K$;

(ii) $\nu(1) \geq \mu(x), \quad \forall x \in V$.

Proof. First suppose that μ_V is a fuzzy hypervector space over fuzzy field ν_K. Then for $\alpha, \beta \in K$ and $x, y \in V$ we have

$$\begin{aligned} \bigwedge_{z \in \alpha \circ x + \beta \circ y} \mu(z) &= \bigwedge_{z \in u+v, u \in \alpha \circ x, v \in \beta \circ y} \mu(z) \\ &\geq ((\nu(\alpha) \wedge \mu(x)) \bigwedge (\nu(\beta) \wedge \mu(y)). \end{aligned}$$

By Proposition 3.3 (ii) and Definition 3.1 (iv),

$$\nu(1) \geq \mu(x), \quad \forall x \in V.$$

Conversely, suppose the inequalities of Proposition 3.4 hold for all $x, y \in V$. Then

$$\begin{aligned}
\bigwedge_{z \in 1 \circ x + 1 \circ y} \mu(z) &= \bigwedge_{z \in u+v, u \in 1 \circ x, v \in 1 \circ y} \mu(z) \\
&\geq ((\nu(1) \wedge \mu(x)) \bigwedge (\nu(1) \wedge \mu(y)) \\
&\geq \mu(x) \wedge \mu(y),
\end{aligned}$$

whence we get (i) of the Definition 3.1. On the other hand, since ν_K is a fuzzy field, then

$$\nu(0) \geq \nu(1) \geq \mu(a) \quad \text{and} \quad \nu(-1) \geq \mu(1) \geq \nu(a) \quad \forall a \in K.$$

Thus

$$\begin{aligned}
\mu(-x) &\geq \bigwedge_{z \in 0 \circ x + (-1) \circ x} \mu(z) \quad (0 \in 0 \circ x, \quad \text{by Proposition 2.6 (4)}) \\
&\geq (\nu(0) \wedge \mu(x)) \bigwedge (\nu(-1) \wedge \mu(x)) \\
&= \mu(x) \wedge \mu(x) \\
&= \mu(x),
\end{aligned}$$

and hence condition (ii) of the Definition 3.1 holds.

For all $a \in K$ and $x \in V$, we have

$$\begin{aligned}
\bigwedge_{z \in a \circ x} \mu(z) &\geq \bigwedge_{z \in u+v, u \in 0 \circ x, v \in a \circ x} \mu(z) \\
&\geq ((\nu(0) \wedge \mu(x)) \bigwedge (\nu(a) \wedge \mu(x)) \\
&= \mu(x) \wedge (\nu(a) \wedge \mu(x)) \\
&= \nu(a) \wedge \mu(x).
\end{aligned}$$

This means that the condition (iii) of the Definition 3.1 holds.

The condition (iv) of the definition obviously holds. Therefore μ_V is a fuzzy hypervector space over ν_K. □

Theorem 3.5. *Let V be a hypervector space over field K. Let $\mu \in FS(V)$ and $\nu \in FS(K)$. Then μ_V is a fuzzy hypervector space over ν_K if and only if μ_α is a hypervector space over field ν_α, and $\nu(1) \geq \mu(0)$ for all $\alpha \in Im(\mu)$.*

Proof. Suppose μ_V is a fuzzy hypervector space over ν_K and $\alpha \in Im(\mu)$. By Proposition 3.3 (iii) it follows that ν_α is nonempty and moreover, according to Proposition 2.10, it is a subfield of K. It is easy to verify that μ_α is a subgroup of $(V, +)$. Let $a \in \nu_\alpha, x \in \mu_\alpha$ and $z \in a \circ x$. Then

$$\mu(z) \geq \bigwedge_{y \in a \circ x} \mu(y) \geq \nu(a) \wedge \mu(x) \geq \alpha,$$

i.e. $a \circ x \subseteq \mu_\alpha$, and hence μ_α is a hypervector space over ν_α. Obviously, by Definition 3.1 (iv), the condition $\nu(1) \geq \mu(0)$ holds.

Conversely, let $a \in K$ and for $x, y \in V$ set α =min$(\mu(x), \mu(y))$. Then by hypothesis, μ_α is a fuzzy hypervector space over ν_α. Thus $x + y \in \mu_\alpha, -x \in \mu_\alpha$, i.e.

$$\mu(x + y) \geq \alpha = \mu(x) \wedge \mu(y),$$

and $\mu(-x) \geq \mu(x)$.

For $z \in a \circ x$, set $\alpha = \mu(x) \wedge \nu(a)$. Then $a \circ x \subseteq \mu_\alpha$, i.e $\mu(z) \geq \alpha$. Thus

$$\bigwedge_{z \in a \circ x} \mu(z) \geq \mu(x) \wedge \nu(a).$$

Therefore μ_V is a fuzzy hypervector space over ν_K.

□

Proposition 3.6. *The intersection of a family of fuzzy hypervector spaces is a fuzzy hypervector space.*

Proof. The proof is straightforward. □

Definition 3.7. Let V and W be hypervector spaces over K. A mapping $T : V \longrightarrow W$ is called

(i) *linear (transformation)* iff

$$T(x + y) = T(x) + T(y), \quad T(k \circ x) \subseteq k \circ T(x), \quad \forall x, y \in V, k \in K,$$

(ii) *anti linear (transformation)* iff

$$T(x + y) = T(x) + T(y), \quad T(k \circ x) \supseteq k \circ T(x), \quad \forall x, y \in V, k \in K,$$

(iii) *strong linear (transformation)* iff

$$T(x + y) = T(x) + T(y), \quad T(k \circ x) = k \circ T(x), \quad \forall x, y \in V, k \in K.$$

Proposition 3.8. *Let V and W be strongly left distributive hypervector spaces over the field K, and $T : V \longrightarrow W$ be a linear transformation. Let ν_K be a fuzzy field, and μ_W be a fuzzy hypervector space over ν_K. Then $T^{-1}(\mu)$ is a fuzzy hypervector space of V over ν_K.*

Proof. Let $a, b \in K$ and $x, y \in V$. Then

$$\begin{aligned}
\bigwedge_{z\in\alpha x+\beta y} T^{-1}\mu(z) &= \bigwedge_{z\in\alpha x+\beta y} \mu(T(z)) \\
&= \bigwedge_{u\in\alpha x, v\in\beta y} \mu(T(u+v)) \\
&\geq \bigwedge_{T(u)\in\alpha T(x), T(v)\in\beta T(y)} \mu(T(u+v)) \quad (\text{ since } \quad T \quad \text{is linear}). \\
&\geq ((\mu(T(x)) \wedge \nu(\alpha)) \bigwedge (\mu(T(y)) \wedge \nu(\beta)) \\
&= (T^{-1}(\mu)(x) \wedge \nu(\alpha)) \bigwedge (T^{-1}(\mu)(y) \wedge \nu(\beta)).
\end{aligned}$$

Evidently, for any $x \in V$, we have

$$\nu(1) \geq T^{-1}(\mu)(x).$$

Thus $T^{-1}(\mu)$ is a fuzzy hypervector space of V over ν_K.

□

Proposition 3.9. *Let V and W be strongly left distributive hypervector spaces over the field K, and $T : V \longrightarrow W$ be a linear transformation. Let ν_K be a fuzzy field and μ_V be a fuzzy hypervector space over ν_K. Then $T(\mu)_W$ is a fuzzy hypervector space over ν_K.*

Proof. Let $a, b \in K$ and $x, y \in W$. If either $T^{-1}(x)$ or $T^{-1}(y)$ is empty, then the identity (i) of Proposition 3.4 is satisfied. Thus we assume that $T^{-1}(x)$, $T^{-1}(y)$ are nonempty, which implies that $T^{-1}(ax+by)$ is nonempty, $Tu = x$ and $Tv = y$ imply that $ax+by = aTu+bTv \supseteq T(au+bv)$, since T is linear. Thus $ax+by \supseteq T(au+bv)$ and hence $T^{-1}(ax+by) \supseteq au+bv$. Then

$$\begin{aligned}
\bigwedge_{w\in ax+by} T(\mu)(w) &= \bigwedge_{w\in ax+by, T(z)=w} \mu(z) \\
&\geq \bigwedge_{u\in T^{-1}(x), v\in T^{-1}(y), z=z_1+z_2, z_1\in au, z_2\in bv} \mu(z_1) \wedge \mu(z_2) \\
&\geq ((\nu(a) \wedge \mu(u)) \bigwedge ((\nu(b) \wedge \mu(v))) \\
&= ((\nu(a) \wedge T(\mu)(x)) \bigwedge (\nu(b) \wedge T(\mu)(y)).
\end{aligned}$$

Obviously, for any $x \in W$, we have

$$\nu(1) \geq T(\mu)(x).$$

Therefore $T(\mu)_W$ is a fuzzy hypervector space over ν_K.

□

Now from Remark 2.5, Definition 3.7 and Propositions 3.8, 3.9 we get the following result.

Corollary 3.10. *Let V and W be strongly left distributive hypervector spaces over the field K, and $T : V \longrightarrow W$ be a strong linear transformation. Let μ_V and θ_W be fuzzy hypervector spaces over ν_K. Then $T^{-1}(\theta)_V$ and $T(\mu)_W$ are fuzzy hypervector spaces over ν_K.*

4. **Balanced and Convex Fuzzy Subsets**

Throughout this section by V we mean a hypervector space over the field K.

Definition 4.1. Let K be a field. The map $|\ \ |: K \longrightarrow R$ (where R is the real numbers) is called a *valuation* on K if for all $\alpha, \beta \in K$,
(i) $| \alpha | \geq 0$ and we have equality iff $\alpha = 0$,
(ii) $| \alpha\beta | = | \alpha || \beta |$,
(iii) $| \alpha + \beta | \leq | \alpha | + | \beta |$.

A field K together with a valuation is called a *valuation field*. The valuation is said to be *non-Archimedean* if the condition (iii) is replaced by
$(iii)'$ $| \alpha + \beta | \leq max(| \alpha |, | \beta |)$,
otherwise it is called *Archimedean*. The set

$$V = \{\alpha \ \mid \ |\alpha| \leq 1\}$$

is called a *valuation ring* of K, or the *ring of integers* of K if the valuation is non-Archimedean.

Definition 4.2. Let V be a hypervector space over a non-Archimedean valued field K and let A be a fuzzy hypervector space in V. Let $S \in FS(V)$. Then S is said to be a fuzzy *balanced* subset if

$$S(x) \geq \bigwedge_{y \in k \circ x} S(y) \quad \forall k \in K, \forall x \in V.$$

Also S is said to be fuzzy *convex* iff

$$\bigwedge_{y \in k_1 \circ x_1 + k_2 \circ x_2} S(y) \geq S(x_1) \wedge S(x_2) \quad \forall k_1, k_2 \in K, \forall x_1, x_2 \in V.$$

Definition 4.3. Let V be a hypervector space over a field K. For $\mu \in FS(V)$ and $\lambda \in K$ we define $\lambda\mu$ by

$$\lambda\mu(x) = \bigwedge_{y \in \lambda^{-1} \circ x} \mu(y).$$

Theorem 4.4. *Let V and W be hypervector spaces over K and $T : V \longrightarrow W$ be a linear map. Let A and B be fuzzy hypervector spaces of V and W respectively. Then*
(i) if A is a fuzzy convex (balanced) subset in V, then $T(A)$ is a fuzzy convex (balanced) subset in W.
(ii) if B is a fuzzy convex (balanced) in W, then $T^{-1}(B)$ is a fuzzy convex (balanced) subset in V.

Proof. We shall prove the results only for the convex case. (i) Let $k_1, k_2 \in K$ and $y_1, y_2 \in V$. Then it is easy to verify that

$$T(A)(k_1y_1 + k_2y_2) \geq T(A)(y_1) + T(A)(y_2).$$

Thus $T(A)$ is a fuzzy convex subset.

(ii) Suppose B is a fuzzy convex subset in W and $k_1, k_2 \in K$. Set

$$M = k_1T^{-1}(B) + k_2T^{-1}(B).$$

Then

$$T(M) = k_1T(T^{-1}(B)) + k_2T(T^{-1}(B)) \subseteq k_1B + k_2B \subseteq B.$$

Thus $M \subseteq T^{-1}(B)$ and this completes the proof. □

Theorem 4.5. *Let $\{A_i|I \in I\}$ be a family of convex (resp. balanced) fuzzy subsets in a hypervector space V. Then $\bigcap_{i\in I} A_i$ is a fuzzy convex (resp. balanced) subset in V.*

Proof. Let $k_1, k_2 \in K, x, y \in V$. Then $\forall z \in k_1 \circ x + k_2 \circ y$ we have

$$\begin{aligned} A(z) &= \bigwedge_{i\in I} A_i(z) \\ &\geq \bigwedge(A_i(x) \wedge A_i(y)) \quad (\text{ by Definition 4.2}) \\ &\geq ((\wedge_{i\in I} A_i(x)) \bigwedge (\wedge_{i\in I} A_i(y)) \\ &= A(x) \wedge A(y). \end{aligned}$$

Moreover, we conclude that

$$\begin{aligned} \bigwedge_{t\in k\circ x} A(t) &= \bigwedge_{t\in k\circ x} \bigwedge_{i\in I} A_i(t) \\ &= \bigwedge_{i\in I} \bigwedge_{t\in k\circ x} A_i(t) \\ &\leq \bigwedge_{i\in I} A_i(x) \\ &= A(x). \end{aligned}$$

This completes the proof.

□

Definition 4.6. Let A be a fuzzy set in a hypervector space V over K. The fuzzy *convex (balanced) closure* of A is the smallest fuzzy convex (balanced) subset in V, which contains A.

It follows from Theorem 4.5 that the fuzzy convex (balanced) closure of A is the
intersection of all fuzzy convex (balanced) subsets in V which contain A.

Theorem 4.7. *Let A be a fuzzy subset in a hypervector space V over K. Then the fuzzy balanced closure of A is the fuzzy subset* $\bigcup_{\lambda \in K} \lambda A$.

Proof. Let $B = \bigcup_{\lambda \in K} \lambda A$ and let S be an arbitrary fuzzy balanced subset in V containing A. Then for any $\lambda \in K$, we have

$$\begin{aligned} \lambda A(y) &= \bigwedge_{x \in \lambda^{-1} y} A(x) \\ &\leq \bigwedge_{x \in \lambda^{-1} y} S(x) \\ &\leq S(y) \ \ \text{(since } S \text{ is fuzzy balanced).} \end{aligned}$$

Thus B is included in any fuzzy balanced subset, which contains A. On the other hand, B is a fuzzy balanced subset of V. For $a \in K$ and $x \in V$ we have

$$\begin{aligned} B(x) &= \bigvee_{\lambda \in K} \lambda A(x) \\ &\leq \bigvee_{\lambda \in K} \lambda a A(x) \\ &= \bigvee_{\lambda \in K} \lambda (\bigwedge_{y \in a \circ x} A(y)) \\ &= \bigwedge_{y \in a \circ x} (\bigvee_{\lambda \in K} \lambda A(y)) = aB(x) \end{aligned}$$

Thus $aB \subseteq B$ and this completes the proof. □

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