FUZZY HYPERVECTOR SPACES OVER VALUED FIELDS

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Abstract. In this note we first redefine the notion of a fuzzy hypervector space (see [1]) and then introduce some further concepts of fuzzy hypervector spaces, such as fuzzy convex and balance fuzzy subsets in fuzzy hypervector spaces over valued fields. Finally, we briefly discuss on the convex (balanced) hull of a given fuzzy set of a hypervector space.

1. Introduction

The notion of a fuzzy subset of a nonempty set was introduced by L. Zadeh in 1965 [31] as a function from a nonempty set $X$ to the unit real interval $I = [0, 1]$. Rosenfeld defined the concept of a fuzzy subgroup of a given group $G$ [23] and then many researchers developed it in all the fields of algebra. The concepts of a fuzzy field and a fuzzy linear space over a fuzzy field were introduced and discussed by Nanda [21]. Fuzzy vector spaces over the real or complex field were discussed by Katsaras and Liu [15].

The notion of a hypergroup was introduced by F. Marty [20]. Since then many researchers have studied this field and developed it, for example see [7, 8, 9, 28, 29]. In 1990, M. S. Tallini introduced the notion of a hypervector space [25] and studied the its basic properties.

Also fuzzy set theory has been well developed in the context of hyperalgebraic structure theory. (for example see [1-6], [10], [11], [12], [13], [16], [17], [18], [24], [32]). In [1] and [6] the author introduced the notions of fuzzy (co-) norm and fuzzy inner product in fuzzy hypervector spaces respectively. The purpose of this paper is the study of some further notions such as fuzzy convex and balanced subsets in fuzzy hypervector spaces over valued fields. Also we briefly discuss on convex (balanced) closure of a given fuzzy set of a hypervector space. Convex fuzzy sets were introduced by Zadeh [31] and subsequently such sets were analysed by in Lowen [19] in the context of fuzzy subspaces over the real or complex number fields.

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2. Preliminaries

In this section we present some definitions and simple properties of hypervector spaces and fuzzy subsets, that we shall use in later.

Let $H$ be a nonempty set. By $P^*(H)$ we mean the family of all nonempty subsets of $H$.

**Definition 2.1.** [8] A map $\cdot : H \times H \rightarrow P^*(H)$ is called a hyperoperation or join operation.

The join operation is extended to nonempty subsets of $H$ in a natural way, so that $A \cdot B$ or $AB$ is given by

$$ AB = \bigcup \{ab | a \in A \text{ and } b \in B \}.$$

**Definition 2.2.** [8] A hypergroup is a structure $(H, \cdot)$ that satisfies two axioms,

(Reproduction) $aH = H = Ha$ for all $a \in H$;

(Associativity) $a(bc) = (ab)c$ for all $a, b, c \in H$.

**Definition 2.3.** [8] Let $H$ be a hypergroup and $K$ a nonempty subset of $H$. Then $K$ is a subhypergroup of $H$ if it is a hypergroup with respect to the hyperoperation $\cdot$ restricted to $K$.

Hence it is clear that a subset $K$ of $H$ is a subhypergroup if and only if $aK = Ka = K$, under the hyperoperation on $H$.

**Definition 2.4.** [25] Let $K$ be a field and $(V, +)$ be an abelian group. We define a hypervector space over $K$ as a quadruple $(V, +, \circ, K)$, where $\circ$ is a mapping $\circ : K \times V \rightarrow P^*(V)$ such that the following conditions are satisfied:

1. $\forall a \in K, \forall x, y \in V, \ a \circ (x + y) \subseteq a \circ x + a \circ y$ (right distributivity),
2. $\forall a, b \in K, \forall x \in V, (a + b) \circ x \subseteq a \circ x + b \circ y$ (left distributivity),
3. $\forall a, b \in K, \forall x \in V, a \circ (b \circ x) = (ab) \circ x$,
4. $\forall a \in K, \forall x \in V, a \circ (-x) = -a \circ x$,
5. $\forall x \in V, x \in 1 \circ x$.

For simplicity of notation sometimes we write $ax$ instead $a \circ x$.

**Remark 2.5.** (i) In the right member of (1) and (2) the sum is meant in the sense of Frobenius, that is we consider the set of all the sums of an element of $a \circ x$ with an element of $a \circ y$. Moreover the left member of (3) means the set-theoretical union of all the sets $a \circ y$, where $y$ runs over the set $b \circ x$.

(ii) We say that $(V, +, \circ, K)$ is strongly left distributive iff

$$ \forall a \in K, \forall x, y \in V, \ a \circ (x + y) = a \circ x + a \circ y,$$

and anti distributive, iff

$$ \forall a \in K, \forall x, y \in V, \ a \circ (x + y) \supseteq a \circ x + a \circ y,$$
In a similar way we define the strongly right distributive law.

(iii) Let $\Omega = 0 \circ 0$, where $0$ is the zero of $(V, +)$. In [25] it is shown that if $V$ is either strongly right or left, then $\Omega$ is a subgroup of $(V, +)$.

**Proposition 2.6.** (See page 169 [25].) Let $V$ be a strongly left distributive hypervector space over the field $K$. Then the following statements hold:

1. $\Omega$ is a subgroup of $(V, +)$;
2. $\forall a \in K, a \circ 0 = \Omega = a \circ \Omega$;
3. $\forall x \in V, 0 \circ x \supseteq \Omega$;
4. $\forall x \in V, 0 \circ x$ is a subhypergroup of $(V, +)$.

**Definition 2.7.**

(i) For a fuzzy subset $\mu$ of $X$, the level subset $\mu_t$ is defined by

\[ \mu_t = \{ x \in X | \mu(x) \geq t \}, \quad t \in [0, 1]. \]

(ii) The image of $\mu$ is denoted by $\text{Im}(\mu)$ and is defined by

\[ \text{Im}(\mu) = \{ \mu(x) | x \in X \}. \]

(iii) Let $f : X \rightarrow Y$ be a mapping and $\mu \in \text{FS}(X)$ and $\nu \in \text{FS}(Y)$. Then we define $f(\mu) \in \text{FS}(Y)$ and $f^{-1}(\nu) \in \text{FS}(X)$ respectively as follows:

\[ f(\mu)(y) = \begin{cases} \bigvee_{x \in f^{-1}(y)} \mu(x) & \text{if } f^{-1}(y) \neq \emptyset, \\ 0 & \text{otherwise} \end{cases} \]

and

\[ f^{-1}(\nu)(x) = \nu(f(x)), \quad \forall x \in X. \]

**Definition 2.8.** [30] Let $K$ be a field and $\nu \in \text{FS}(K)$. Suppose the following conditions hold:

(i) $\nu(a + b) \geq \nu(a) \land \nu(b)$, $\forall a, b \in K$;
(ii) $\nu(-a) \geq \nu(a)$, $\forall a \in K$;
(iii) $\nu(ab) \geq \nu(a) \land \nu(b)$, $\forall a, b \in K$;
(iv) $\nu(a^{-1}) \geq \nu(a)$, $\forall a(\neq 0) \in K$;

Then we call $\nu$ a fuzzy field in $K$ and denote it by $\nu_K$. Also $\nu_K$ is called a fuzzy field of $K$.

Obviously, Definition 2.8 is a generalization of the classical field notion.

**Proposition 2.9.** (Proposition 2.1 [25]). If $\nu_K$ is a fuzzy field of $K$, then

(i) $\nu(0) \geq \nu(a)$, $\forall a \in K$.
(ii) $\nu(1) \geq \nu(a)$, $\forall a \in K \setminus \{0\}$.
(iii) $\nu(0) \geq \nu(1)$.

In the following we state a well-known result in fuzzy field theory.

**Proposition 2.10.** Let $K$ be a field and $\nu \in \text{FS}(K)$. Then $\nu \in \text{FS}(K)$ is a fuzzy subfield of $K$ if and only if $\nu_\alpha$ is a subfield of $K$ for all $\alpha \in \text{Im}(\nu)$. 
3. Fuzzy Hypervector Spaces

In [1] and [6] the author introduced the definition of a fuzzy hyperspace over a fuzzy field. In the following we first recall the definition of a hypervector space and then give some related results.

**Definition 3.1.** (Modification Version of Definition 3.3 [1]). Let \( V \) be a hypervector space over a field \( K \) and \( \nu \) be a fuzzy subfield of \( K \). A fuzzy set \( \mu \) of \( V \) is said to be a fuzzy hypervector space of \( V \) over fuzzy field \( \nu_K \) if, for all \( x, y \in V \), and all \( a \in K \), the following conditions are satisfied:

(i) \( \mu(x + y) \geq \mu(x) \land \mu(y) \);

(ii) \( \mu(-x) \geq \mu(x) \);

(iii) \( \bigwedge_{y \in a \circ x} \mu(y) \geq \mu(x) \land \nu_K(a) \).

(iv) \( \nu_K(1) \geq \mu(0) \).

We say that \( \mu \) is a fuzzy hypervector space over the fuzzy field \( \nu_K \). Hereafter except for ambiguous cases we shall drop the subscripts on \( \mu \) and \( \nu \).

Obviously, Definition 3.1 is a generalization of the concept of a fuzzy vector space and also of the classical notion of a hypervector space (in the Tallini’s sense [25]).

**Remark 3.2.**

(i) In Definition 3.1 if we consider \( \nu = \chi_K \), the characteristic function of \( K \), then \( \mu \) is a fuzzy hypervector space.

(ii) In the sequel of this note, unless otherwise specified, we assume that \( V \) is a hypervector space over the field \( K \).

**Proposition 3.3.** If \( \mu_V \) is a fuzzy hypervector space over the fuzzy field \( \nu_K \), then

(i) \( \nu(0) \geq \mu(0) \);

(ii) \( \mu(0) \geq \mu(x) \), \( \forall x \in V \);

(iii) \( \nu(0) \geq \mu(x) \), \( \forall x \in V \).

**Proof.** The proof is an immediate consequence of Proposition 2.9 and Definition 3.1. \( \square \)

**Proposition 3.4.** Let \( V \) be a strongly left distributive hypervector space over the field \( K \) and \( \nu_K \) be a fuzzy field. Let \( \mu \in FS(V) \). Then \( \mu_V \) is a fuzzy hypervector space over \( \nu_K \) iff

(i) \( \bigwedge_{z \in a \circ x + \beta \circ y} \mu(z) \geq ((\nu(\alpha) \land \mu(x)) \bigwedge (\nu(\beta) \land \mu(y)), \forall x, y \in V, \forall \alpha, \beta \in K \);

(ii) \( \nu(1) \geq \mu(x), \forall x \in V \).

**Proof.** First suppose that \( \mu_V \) is a fuzzy hypervector space over fuzzy field \( \nu_K \). Then for \( \alpha, \beta \in K \) and \( x, y \in V \) we have

\[
\bigwedge_{z \in a \circ x + \beta \circ y} \mu(z) = \bigwedge_{z \in \alpha \circ x + \beta \circ y} \mu(z) \geq ((\nu(\alpha) \land \mu(x)) \bigwedge (\nu(\beta) \land \mu(y)).
\]

By Proposition 3.3 (ii) and Definition 3.1 (iv),
\[ \nu(1) \geq \mu(x), \ \forall x \in V. \]

Conversely, suppose the inequalities of Proposition 3.4 hold for all \( x, y \in V \).
Then
\[
\bigwedge_{z \in 1 \circ x + 1 \circ y} \mu(z) = \bigwedge_{z \in u + v, u \in 1 \circ x, v \in 1 \circ y} \mu(z) \\
\geq ((\nu(1) \land \mu(x)) \land (\nu(1) \land \mu(y))) \\
\geq \mu(x) \land \mu(y),
\]
whence we get (i) of the Definition 3.1. On the other hand, since \( \nu_K \) is a fuzzy field, then
\[
\nu(0) \geq \nu(1) \geq \mu(a) \quad \text{and} \quad \nu(-1) \geq \nu(1) \geq \nu(a) \quad \forall a \in K.
\]
Thus
\[
\mu(-x) \geq \bigwedge_{z \in 0 \circ x + (-1) \circ x} \mu(z) \quad (0 \in 0 \circ x, \ \text{by Proposition 2.6 (4)}) \\
\geq (\nu(0) \land \mu(x)) \land (\nu(-1) \land \mu(x)) \\
= \mu(x) \land \mu(x) \\
= \mu(x),
\]
and hence condition (ii) of the Definition 3.1 holds.
For all \( a \in K \) and \( x \in V \), we have
\[
\bigwedge_{z \in a \circ x} \mu(z) = \bigwedge_{z \in u + v, u \in a \circ x, v \in a \circ x} \mu(z) \\
\geq ((\nu(0) \land \mu(x)) \land (\nu(a) \land \mu(x))) \\
= \mu(x) \land (\nu(a) \land \mu(x)) \\
= \nu(a) \land \mu(x).
\]
This means that the condition (iii) of the Definition 3.1 holds.

The condition (iv) of the definition obviously holds. Therefore \( \mu_V \) is a fuzzy hypervector space over \( \nu_K \). □

**Theorem 3.5.** Let \( V \) be a hypervector space over field \( K \). Let \( \mu \in FS(V) \) and \( \nu \in FS(K) \). Then \( \mu_V \) is a fuzzy hypervector space over \( \nu_K \) if and only if \( \mu_\alpha \) is a hypervector space over field \( \nu_\alpha \), and \( \nu(1) \geq \mu(0) \) for all \( \alpha \in Im(\mu) \).

**Proof.** Suppose \( \mu_V \) is a fuzzy hypervector space over \( \nu_K \) and \( \alpha \in Im(\mu) \). By Proposition 3.3 (iii) it follows that \( \nu_\alpha \) is nonempty and moreover, according to Proposition 2.10, it is a subfield of \( K \). It is easy to verify that \( \mu_\alpha \) is a subgroup of \( (V, +) \). Let \( a \in \nu_\alpha, x \in \mu_\alpha \) and \( z \in a \circ x \). Then
\[ \mu(z) \geq \bigwedge_{y \in a \circ x} \mu(y) \geq \nu(a) \land \mu(x) \geq \alpha, \]

i.e. \( a \circ x \subseteq \mu_\alpha \), and hence \( \mu_\alpha \) is a hypervector space over \( \nu_\alpha \). Obviously, by Definition 3.1 (iv), the condition \( \nu(1) \geq \mu(0) \) holds.

Conversely, let \( a \in K \) and for \( x, y \in V \) set \( \alpha = \min(\mu(x), \mu(y)) \). Then by hypothesis, \( \mu_\alpha \) is a fuzzy hypervector space over \( \nu_\alpha \). Thus \( x + y \in \mu_\alpha, -x \in \mu_\alpha \), i.e.

\[ \mu(x + y) \geq \alpha = \mu(x) \land \mu(y), \]

and \( \mu(-x) \geq \mu(x) \).

For \( z \in a \circ x \), set \( \alpha = \mu(x) \land \nu(a) \). Then \( a \circ x \subseteq \mu_\alpha \), i.e \( \mu(z) \geq \alpha \). Thus

\[ \bigwedge_{z \in a \circ x} \mu(z) \geq \mu(x) \land \nu(a). \]

Therefore \( \mu_V \) is a fuzzy hypervector space over \( \nu_K \).

\[ \square \]

**Proposition 3.6.** The intersection of a family of fuzzy hypervector spaces is a fuzzy hypervector space.

*Proof. The proof is straightforward.\[ \square \]*

**Definition 3.7.** Let \( V \) and \( W \) be hypervector spaces over \( K \). A mapping \( T : V \rightarrow W \) is called

(i) **linear (transformation)** iff

\[ T(x + y) = T(x) + T(y), \quad T(k \circ x) \subseteq k \circ T(x), \quad \forall x, y \in V, k \in K, \]

(ii) **anti linear (transformation)** iff

\[ T(x + y) = T(x) + T(y), \quad T(k \circ x) \supseteq k \circ T(x), \quad \forall x, y \in V, k \in K, \]

(iii) **strong linear (transformation)** iff

\[ T(x + y) = T(x) + T(y), \quad T(k \circ x) = k \circ T(x), \quad \forall x, y \in V, k \in K. \]

**Proposition 3.8.** Let \( V \) and \( W \) be strongly left distributive hypervector spaces over the field \( K \), and \( T : V \rightarrow W \) be a linear transformation. Let \( \nu_K \) be a fuzzy field, and \( \mu_W \) be a fuzzy hypervector space over \( \nu_K \). Then \( T^{-1}(\mu) \) is a fuzzy hypervector space of \( V \) over \( \nu_K \).

*Proof. Let \( a, b \in K \) and \( x, y \in V \). Then
\[
\bigwedge_{z \in \alpha x + \beta y} T^{-1}(\mu(z)) = \bigwedge_{z \in \alpha x + \beta y} \mu(T(z)) \\
= \bigwedge_{u \in \alpha x, v \in \beta y} \mu(T(u + v)) \\
\geq \bigwedge_{T(u) \in \alpha T(x), T(v) \in \beta T(y)} \mu(T(u + v)) \quad \text{(since } T \text{ is linear)}.
\]

\[
= \bigwedge_{u \in \alpha x, v \in \beta y} \mu(T(u + v)) \quad \text{(since } T \text{ is linear)}.
\]

Evidently, for any \(x \in V\), we have

\[
\nu(1) \geq T^{-1}(\mu)(x).
\]

Thus \(T^{-1}(\mu)\) is a fuzzy hypervector space of \(V\) over \(\nu_K\).

**Proposition 3.9.** Let \(V\) and \(W\) be strongly left distributive hypervector spaces over the field \(K\), and \(T : V \rightarrow W\) be a linear transformation. Let \(\nu_K\) be a fuzzy field and \(\mu_V\) be a fuzzy hypervector space over \(\nu_K\). Then \(T(\mu)_W\) is a fuzzy hypervector space over \(\nu_K\).

**Proof.** Let \(a, b \in K\) and \(x, y \in W\). If either \(T^{-1}(x)\) or \(T^{-1}(y)\) is empty, then the identity (i) of Proposition 3.4 is satisfied. Thus we assume that \(T^{-1}(x), T^{-1}(y)\) are nonempty, which implies that \(T^{-1}(ax + by)\) is nonempty, \(Tu = x\) and \(Tv = y\) imply that \(ax + by = aTu + bTv \supseteq T(au + bv)\), since \(T\) is linear. Thus \(ax + by \supseteq T(au + bv)\) and hence \(T^{-1}(ax + by) \supseteq T(au + bv)\). Then

\[
\bigwedge_{w \in \alpha x + \beta y} T(\mu)(w) = \bigwedge_{w \in \alpha x + \beta y, T(z) = w} \mu(z) \\
\geq \bigwedge_{u \in T^{-1}(x), v \in T^{-1}(y), z = z_1 + z_2, z_1 \in \alpha u, z_2 \in \beta v} \mu(z_1) \wedge \mu(z_2) \\
\geq ((\nu(a) \wedge \mu(u)) \bigwedge (\nu(b) \wedge \mu(v))) \\
= ((\nu(a) \wedge T(\mu)(x)) \bigwedge (\nu(b) \wedge T(\mu)(y))).
\]

Obviously, for any \(x \in W\), we have

\[
\nu(1) \geq T(\mu)(x).
\]

Thus \(T(\mu)_W\) is a fuzzy hypervector space over \(\nu_K\).

Now from Remark 2.5, Definition 3.7 and Propositions 3.8, 3.9 we get the following result.
Corollary 3.10. Let $V$ and $W$ be strongly left distributive hypervector spaces over the field $K$, and $T : V \rightarrow W$ be a strong linear transformation. Let $\mu_V$ and $\theta_W$ be fuzzy hypervector spaces over $\nu_K$. Then $T^{-1}(\theta)_V$ and $T(\mu)_W$ are fuzzy hypervector spaces over $\nu_K$.

4. Balanced and Convex Fuzzy Subsets

Throughout this section by $V$ we mean a hypervector space over the field $K$.

Definition 4.1. Let $K$ be a field. The map $| | : K \rightarrow R$ (where $R$ is the real numbers) is called a valuation on $K$ if for all $\alpha, \beta \in K$,

(i) $| \alpha | \geq 0$ and we have equality iff $\alpha = 0$,

(ii) $| \alpha \beta | = | \alpha | | \beta |$,

(iii) $| \alpha + \beta | \leq | \alpha | + | \beta |$.

A field $K$ together with a valuation is called a valuation field. The valuation is said to be non-Archimedean if the condition (iii) is replaced by

(iii)$' | \alpha + \beta | \leq \max(| \alpha |, | \beta |)$,

otherwise it is called Archimedean. The set

$$V = \{ \alpha \mid |\alpha| \leq 1 \}$$

is called a valuation ring of $K$, or the ring of integers of $K$ if the valuation is non-Archimedean.

Definition 4.2. Let $V$ be a hypervector space over a non-Archimedean valued field $K$ and let $A$ be a fuzzy hypervector space in $V$. Let $S \in FS(V)$. Then $S$ is said to be a fuzzy balanced subset if

$$S(x) \geq \bigwedge_{y \in k \circ x} S(y) \quad \forall k \in K, \forall x \in V.$$  

Also $S$ is said to be fuzzy convex iff

$$\bigwedge_{y \in k_1 \circ x_1 + k_2 \circ x_2} S(y) \geq S(x_1) \wedge S(x_2) \quad \forall k_1, k_2 \in K, \forall x_1, x_2 \in V.$$  

Definition 4.3. Let $V$ be a hypervector space over a field $K$. For $\mu \in FS(V)$ and $\lambda \in K$ we define $\lambda \mu$ by

$$\lambda \mu(x) = \bigwedge_{y \in \lambda^{-1} \circ x} \mu(y).$$

Theorem 4.4. Let $V$ and $W$ be hypervector spaces over $K$ and $T : V \rightarrow W$ be a linear map. Let $A$ and $B$ be fuzzy hypervector spaces of $V$ and $W$ respectively. Then

(i) if $A$ is a fuzzy convex (balanced) subset in $V$, then $T(A)$ is a fuzzy convex (balanced) subset in $W$.

(ii) if $B$ is a fuzzy convex (balanced) subset in $W$, then $T^{-1}(B)$ is a fuzzy convex (balanced) subset in $V$. 
Proof. We shall prove the results only for the convex case. (i) Let \( k_1, k_2 \in K \) and \( y_1, y_2 \in V \). Then it is easy to verify that

\[
T(A)(k_1 y_1 + k_2 y_2) \geq T(A)(y_1) + T(A)(y_2).
\]

Thus \( T(A) \) is a fuzzy convex subset.

(ii) Suppose \( B \) is a fuzzy convex subset in \( W \) and \( k_1, k_2 \in K \). Set

\[
M = k_1 T^{-1}(B) + k_2 T^{-1}(B).
\]

Then

\[
T(M) = k_1 T(T^{-1}(B)) + k_2 T(T^{-1}(B)) \subseteq k_1 B + k_2 B \subseteq B.
\]

Thus \( M \subseteq T^{-1}(B) \) and this completes the proof.

\[ \square \]

Theorem 4.5. Let \( \{ A_i | I \in I \} \) be a family of convex (resp. balanced) fuzzy subsets in a hypervector space \( V \). Then \( \bigcap_{i \in I} A_i \) is a fuzzy convex (resp. balanced) subset in \( V \).

Proof. Let \( k_1, k_2 \in K, x, y \in V \). Then \( \forall z \in k_1 \circ x + k_2 \circ y \) we have

\[
A(z) = \bigwedge_{i \in I} A_i(z)
\]

\[
\geq \bigwedge_{i \in I} (A_i(x) \land A_i(y)) \quad \text{(by Definition 4.2)}
\]

\[
\geq (\bigwedge_{i \in I} A_i(x)) \land (\bigwedge_{i \in I} A_i(y))
\]

\[
= A(x) \land A(y).
\]

Moreover, we conclude that

\[
\bigwedge_{t \in k_0 z} A(t) = \bigwedge_{t \in k_0 z} \bigwedge_{i \in I} A_i(t)
\]

\[
= \bigwedge_{i \in I} \bigwedge_{t \in k_0 z} A_i(t)
\]

\[
\leq \bigwedge_{i \in I} A_i(x)
\]

\[
= A(x).
\]

This completes the proof.

\[ \square \]

Definition 4.6. Let \( A \) be a fuzzy set in a hypervector space \( V \) over \( K \). The fuzzy convex (balanced) closure of \( A \) is the smallest fuzzy convex (balanced) subset in \( V \), which contains \( A \).

It follows from Theorem 4.5 that the fuzzy convex (balanced) closure of \( A \) is the intersection of all fuzzy convex (balanced) subsets in \( V \) which contain \( A \).
Theorem 4.7. Let \( A \) be a fuzzy subset in a hypervector space \( V \) over \( K \). Then the fuzzy balanced closure of \( A \) is the fuzzy subset \( \bigcup_{\lambda \in K} \lambda A \).

Proof. Let \( B = \bigcup_{\lambda \in K} \lambda A \) and let \( S \) be an arbitrary fuzzy balanced subset in \( V \) containing \( A \). Then for any \( \lambda \in K \), we have

\[
\lambda A(y) = \bigwedge_{x \in \lambda^{-1} y} A(x) \\
\leq \bigwedge_{x \in \lambda^{-1} y} S(x) \\
\leq S(y) \quad \text{(since \( S \) is fuzzy balanced)}.
\]

Thus \( B \) is included in any fuzzy balanced subset, which contains \( A \). On the other hand, \( B \) is a fuzzy balanced subset of \( V \). For \( a \in K \) and \( x \in V \) we have

\[
B(x) = \bigvee_{\lambda \in K} \lambda A(x) \\
\leq \bigvee_{\lambda \in K} \lambda a A(x) \\
= \bigvee_{\lambda \in K} \lambda (\bigwedge_{y \in a \circ x} A(y)) \\
= \bigwedge_{y \in a \circ x} (\bigvee_{\lambda \in K} \lambda A(y)) = aB(x)
\]

Thus \( aB \subseteq B \) and this completes the proof.

\[\square\]

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Fuzzy Hypervector Spaces Over Valued Fields


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