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Interval-valued intuitionistic fuzzy aggregation methodology for decision making with a prioritization of criteria

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Abstract

Interval-valued intuitionistic fuzzy sets (IVIFSs), a generalization of fuzzy sets, is characterized by an interval-valued membership function, an interval-valued non-membership function. The objective of this paper is to deal with criteria aggregation problems using IVIFSs where there exists a prioritization relationship over the criteria. Based on the Łukasiewicz triangular norm, we first propose a prioritized arithmetic mean to IVIF multi-criteria decision making (MCDM) problem where there is a linear ordering among the criteria. The proposed aggregation operator overcomes the existing prioritized aggregation operator's shortcomings that it is not monotone with respect to the total order on interval-valued intuitionistic fuzzy values (IVIFVs). We also prove that it is bounded and monotone with respect to the total order on IVIFVs, and therefore is a true generalization of such operations. We finally propose an aggregation operators-based two-step procedure to IVIF MCDM in the situation that more than one criteria exist at some priority level.

Keywords: Interval-valued intuitionistic fuzzy sets (IVIFSs), prioritized arithmetic mean, monotonicity, multiple criteria decision making (MCDM), Łukasiewicz triangular norm.

1 Introduction

As we all know, uncertainty always exists in the human world. Fuzzy set theory is a perfect means for modeling uncertainty (or imprecision) arising from mental phenomena which are neither random nor stochastic [2, 4, 6, 7, 8, 20, 21, 22, 23, 24, 25, 26, 27]. The concept of an interval-valued intuitionistic fuzzy set (IVIFS) was introduced by Atanassov and Gargov [1], and it was characterized by a membership function, a non-membership function, and a hesitancy function, whose values are intervals rather than exact numbers.

Multiple criteria decision making (MCDM) has been playing an important role in modern decision science [9][28]. In general, MCDM problems require the inclusion of information about importance associated with the different criteria. Especially, there is a kind of MCDM problems in the situation in which the information regarding the importances of the individual criteria is captured by a prioritization of the criteria. According to Yager [16], when considering the situation in which air travel is selected based upon the criteria of passenger safety and economic concerns, the decision maker must not allow a saving on gasoline usage to compensate for a loss in passenger safety, that is to say, tradeoffs between saving on gasoline usage and jeopardizing passenger safety are unacceptable. This situation can be called an aggregation problem, where there exists a prioritization relationship over the criteria. In such a case, Yager [16] considerd the problem of MCDM in the situation in which there exists a prioritization of criteria, and modeled this prioritization of criteria by using importance weights in which the weights associated with the lower priority criteria are related to the satisfaction of the higher priority criteria. Yager [18] presented the prioritized aggregation operators by modeling the prioritization of attributes with respect to the weights associated with the attributes dependent upon the satisfaction of the higher priority attributes, and researched prioritized "and" and "or" operators and employed

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them to the aggregation problems when the criteria are in different priority levels. Yager [17] used importance weights to enforce this prioritization imperative and applied the developed priority-based importance weights to a case in which the scope of the criteria aggregation was an ordered weighted averaging (OWA) type of aggregation. Yu et al. [19] proposed some interval-valued intuitionistic fuzzy aggregation operators such as the interval-valued intuitionistic fuzzy prioritized weighted average (IVIFPWA) operator, the interval-valued intuitionistic fuzzy prioritized weighted geometric (IVIFPWG) operator, and applied them to group decision making under interval-valued intuitionistic fuzzy environment in which the attributes and experts are in different priority level. Li et al. [10] pointed out the unreasonable results in the application of prioritized averaging operators proposed by Yu et al. [19]. In order to overcome the shortcomings of these operators, they provided an improved method on group decision making based on interval-valued intuitionistic fuzzy prioritized operators. Chen [5] developed a prioritized aggregation operator-based approach to interval-valued intuitionistic fuzzy multiple criteria decision making in which there exists a prioritization relationship over evaluative criteria. Li et al. [10] proposed an improved method on group decision making based on interval-valued intuitionistic fuzzy prioritized operators Liu and Yang [11] proposed some prioritized aggregation operators, such as interval-valued intuitionistic uncertain linguistic prioritized weighted average operator and interval-valued intuitionistic uncertain linguistic prioritized weighted geometric operator, for dealing with the multiple attribute group decision making (MAGDM) problems under interval-valued intuitionistic uncertain linguistic environment. Wu and Su [14] proposed some prioritized hybrid weighted aggregation (PHWA) operators such as unit PHWA operator, intuitionistic fuzzy PHWA operator and interval-valued intuitionistic fuzzy PHWA (IVIF-PHWA) operator, and applied the IVIF-PHWA operator to solve the multiple attribute decision making (MADM) problem in IVIF environment.

Although these IVIF prioritized aggregation operators [5, 10, 11, 14, 19] are applied to deal with the problem of MCDM in the situation in which there exists a prioritization of criteria, there is a flaw that they do not satisfy some desirable properties, such as monotonicity. Especially, these IVIF prioritized aggregation operators are monotone with respect to the partial order instead of the total order on IVIFVs. Thus they are just the aggregation operators on the field of the lattice of nonempty IVIFVs $\tilde{L} = \{([a, b], [c, d] | [a, b], [c, d] \in D[0, 1], b + d \leqslant 1)\}$ with the partial order $\leqslant_{\tilde{L}}$ defined as $([a_1, b_1], [c_1, d_1]) \leqslant_{\tilde{L}} ([a_2, b_2], [c_2, d_2]) \Leftrightarrow [a_1, b_1] \leqslant_L [a_2, b_2] \ \& \ [c_2, d_2] \leqslant_L [c_1, d_1]$. Monotonicity is the most important properties for every aggregation operator, but the existing IVIF prioritized aggregation operators do not meet this basic property. Thus, it is natural to develop a new IVIF aggregation operator that not only models the prioritization between criteria but also maintains the desirable property - monotonicity.

The structure of this paper is as follows. Section 2 briefly reviews some basic concepts of AIFSs and IVIFSs. Section 3 analyzing the drawbacks of the existing IVIF prioritized aggregation operators. Section 4 proposes an intuitionistic fuzzy prioritized weighted arithmetic mean, whose desirable properties are studied in this section. Moreover, a prioritized aggregation operator-based two-step procedure is also proposed to the IVIF MCDM in the situation that more than one criteria exist at some priority level. Section 5 gives some concluding remarks.

2 Preliminaries

The AIFS, an extension of the notion of fuzzy set (FS), was first introduced by Atanassov [2].

Definition 2.1. [2] *Let a set X be fixed, an AIFS A in X is defined as follows.*

$$A = \{(x, \mu_A(x), \nu_A(x)) | x \in X\} \tag{1}$$

where μ_A and ν_A are mappings from X to the closed interval [0,1] such that $0 \leqslant \mu_A(x) \leqslant 1$, $0 \leqslant \nu_A(x) \leqslant 1$ and $0 \leqslant \mu_A(x) + \nu_A(x) \leqslant 1$, for all $x \in X$, and they denote the degrees of membership and non-membership of element $x \in X$ to set A, respectively. Let $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$, then it is usually called the intuitionistic fuzzy index of element $x \in X$ to set A, representing the degree of indeterminacy or hesitation of x to A. It is obvious that $0 \leqslant \pi_A(x) \leqslant 1$ for every $x \in X$.

In reality, it may not be easy to identify exact values for the membership and non-membership degrees of an element to a set. In this situation, a range of values may be a more appropriate measurement to accommodate the vagueness. As such, Atanassov and Gargov [1] introduce the notion of IVIFS.

Definition 2.2. [1] *Let a set X be fixed and $D[0, 1]$ be the set of all closed subintervals of [0,1], then an IVIFS $\tilde{A}$ in X is defined as follows.*

$$\tilde{A} = \{(x, \bar{\mu}_{\tilde{A}}(x), \bar{\nu}_{\tilde{A}}(x)) | x \in X\} \tag{2}$$

where $\bar{\mu}_{\tilde{A}}(x)$ and $\bar{\nu}_{\tilde{A}}(x)$ are mappings from X to $D[0,1]$ such that $0 \leqslant \sup\bar{\mu}_{\tilde{A}}(x) + \sup\bar{\nu}_{\tilde{A}}(x) \leqslant 1$, for all $x \in X$, and they denote the interval degrees of membership and non-membership of element $x \in X$ to set $\tilde{A}$, respectively.

Especially, if each of the intervals $\bar{\mu}_{\tilde{A}}(x)$ and $\bar{\nu}_{\tilde{A}}(x)$ contains exactly one element, then the given IVIFS $\tilde{A}$ is transformed to an AIFS.

For any given x, the pair $\bar{\mu}_{\tilde{A}}(x)$ and $\bar{\nu}_{\tilde{A}}(x)$ is called an interval-valued intuitionistic fuzzy value (IVIFV). For convenience, an IVIFV is denoted by $\tilde{\alpha} = ([a,b],[c,d])$, where $[a,b] \in D[0,1]$, $[c,d] \in D[0,1]$ and $b+d \leqslant 1$. Moreover, the set of all IVIFNs is denoted by $\tilde{\Omega}$ [15].

Beliakov et al. [3] defined the lattice of non-empty intervals as

Definition 2.3. [3] *Let $L = \{[a,b] | (a,b) \in [0,1]^2, a \leqslant b\}$ be non-empty intervals, then the lattice of L, denoted by L, with the partial order $\leqslant_L$ defined as $[a,b] \leqslant_L [c,d]$ if and only if $a \leqslant c$ and $b \leqslant d$, and the top and bottom elements are $1_L = [1,1]$ and $0_L = [0,0]$ respectively.*

Later, Wang et al. [12] defined the lattice of non-empty IVIFVs as

Definition 2.4. [12] *Let $\tilde{L} = \{([a,b],[c,d]) | \ [a,b],[c,d] \in D[0,1], b+d \leqslant 1\}$ be non-empty IVIFVs, then the lattice of $\tilde{L}$, denoted by $\tilde{L}$, with the partial order $\leqslant_{\tilde{L}}$ defined as*

$$([a_1,b_1],[c_1,d_1]) \leqslant_{\tilde{L}} ([a_2,b_2],[c_2,d_2]) \Leftrightarrow [a_1,b_1] \leqslant_L [a_2,b_2] \quad \& \quad [c_2,d_2] \leqslant_L [c_1,d_1]$$

and the top and bottom elements are $\tilde{1}_{\tilde{L}} = ([1,1],[0,0])$ and $\tilde{0}_{\tilde{L}} = ([0,0],[1,1])$, respectively.

Atanassov and Gargov [1] defined some operations on IVIFVs:

Definition 2.5. [1] *Let $\tilde{\alpha} = ([a,b],[c,d])$, $\tilde{\alpha}_1 = ([a_1,b_1],[c_1,d_1])$ and $\tilde{\alpha}_2 = ([a_2,b_2],[c_2,d_2])$ be three IVIFVs, then*

1) $\tilde{\alpha}^c = ([c,d],[a,b])$,
2) $\tilde{\alpha}_1 \subset \tilde{\alpha}_2$ *if and only if* $[a_1,b_1] \leqslant_L [a_2,b_2]$ & $[c_2,d_2] \leqslant_L [c_1,d_1]$,
3) $\tilde{\alpha}_1 \cup \tilde{\alpha}_2 = ([\max\{a_1,a_2\}, \max\{b_1,b_2\}], [\min\{c_1,c_2\}, \min\{d_1,d_2\}])$,
4) $\tilde{\alpha}_1 \cap \tilde{\alpha}_2 = ([\min\{a_1,a_2\}, \min\{b_1,b_2\}], [\max\{c_1,c_2\}, \max\{d_1,d_2\}])$,
5) $\tilde{\alpha}_1 \oplus \tilde{\alpha}_2 = ([a_1+a_2-a_1a_2, b_1+b_2-b_1b_2],[c_1c_2,d_1d_2])$,
6) $\alpha_1 \otimes \alpha_2 = ([a_1a_2,b_1b_2],[c_1+c_2-c_1c_2,d_1+d_2-d_1d_2])$,
7) $\lambda\tilde{\alpha} = \big([(1-(1-a)^\lambda,(1-(1-b)^\lambda],[c^\lambda,d^\lambda]\big)$,
8) $\tilde{\alpha}^\lambda = \big([a^\lambda,b^\lambda],[(1-(1-c)^\lambda,(1-(1-d)^\lambda]\big)$.

In order to characterize IVIFV, several indices [13, 15] were introduced

Definition 2.6. [13, 15] *Let $\tilde{\alpha} = ([a,b],[c,d])$ be an IVIFV, then the score function S, accuracy function H, membership uncertainty index T and hesitation uncertainty index G of an IVIFN $\tilde{\alpha}$ were defined by $S(\tilde{\alpha}) = (2+a+b-c-d)/4$ (S is equivalent to $S' = (a+b-c-d)/2$), $H(\tilde{\alpha}) = (a+b+c+d)/2$, $T(\tilde{\alpha}) = b+c-a-d$ and $G(\tilde{\alpha}) = b+d-a-c$, respectively.*

Based on these indices of IVIFVs, Wang et al. [13] establishes an approach to comparing any two IVIFVs by taking a prioritized sequence of score, accuracy, membership uncertainty index, and hesitation uncertainty index functions.

Definition 2.7. [13] *For any two IVIFVs $\tilde{\alpha_1}$ and $\tilde{\alpha_2}$, then*

1) *if $S(\tilde{\alpha_1}) < S(\tilde{\alpha_2})$, then $\tilde{\alpha_1} < \tilde{\alpha_2}$,*

2) *if $S(\tilde{\alpha_1}) = S(\tilde{\alpha_2})$, then*

   a) *if $H(\tilde{\alpha_1}) < H(\tilde{\alpha_2})$, then $\tilde{\alpha_1} < \tilde{\alpha_2}$,*
   b) *if $H(\tilde{\alpha_1}) = H(\tilde{\alpha_2})$, then*
      I) *if $T(\tilde{\alpha_1}) > T(\tilde{\alpha_2})$, then $\tilde{\alpha_1} < \tilde{\alpha_2}$,*
      II) *if $T(\tilde{\alpha_1}) = T(\tilde{\alpha_2})$, then*

i) if $G(\tilde{\alpha_1}) > G(\tilde{\alpha_2})$, then $\tilde{\alpha_1} < \tilde{\alpha_2}$,
ii) if $G(\tilde{\alpha_1}) = G(\tilde{\alpha_2})$, then $\tilde{\alpha_1} = \tilde{\alpha_2}$.

which is also called as the total ordering on IVIFVs.

Example 2.8. *Assume that $\tilde{\alpha_1} = ([0.05, 0.35], [0.25, 0.55])$, $\tilde{\alpha_2} = ([0.1, 0.3], [0.3, 0.5])$, $\tilde{\alpha_3} = ([0.15, 0.25], [0.35, 0.45])$ and $\tilde{\alpha_4} = ([0.2, 0.2], [0.4, 0.4])$, then, $S(\tilde{\alpha_1}) = S(\tilde{\alpha_2}) = S(\tilde{\alpha_3}) = S(\tilde{\alpha_4}) = -0.2$, $H(\tilde{\alpha_1}) = H(\tilde{\alpha_2}) = H(\tilde{\alpha_3}) = H(\tilde{\alpha_4}) = 0.6$, $T(\tilde{\alpha_1}) = T(\tilde{\alpha_2}) = T(\tilde{\alpha_3}) = T(\tilde{\alpha_4}) = 0$. Therefore, $\tilde{\alpha_1}$,$\tilde{\alpha_2}$,$\tilde{\alpha_3}$ and $\tilde{\alpha_4}$ cannot be differentiated by using S, H and T. Since $G(\tilde{\alpha_1}) = 0.6$, $G(\tilde{\alpha_2}) = 0.4$,$G(\tilde{\alpha_3}) = 0.2$, $G(\tilde{\alpha_4}) = 0$ and $G(\tilde{\alpha_1}) > G(\tilde{\alpha_2}) > G(\tilde{\alpha_3}) > G(\tilde{\alpha_4})$, then $\tilde{\alpha_1} < \tilde{\alpha_2} < \tilde{\alpha_3} < \tilde{\alpha_4}$.*

3 Analyzing the drawbacks of the existing IVIF prioritized aggregation operators

Up to now, the IVIF prioritized aggregation operators has attracted more and more scholars attention and has been applied to IVIF MCDM in which there are prioritization relationships over the criteria. Chen [5] proposed some scoring type IVIF prioritized aggregation operators as follows.

Definition 3.1. [5] *Suppose a collection of criteria partitioned into q distinct categories $H_1, H_2, \cdots, H_q$, such that $H_i = \{C_{i1}, C_{i2}, \cdots, C_{in_i}\}$, Here C_{ij} are the criteria in category H_i, and there are a prioritization between these categories $H_1 \succ H_2 \succ \cdots \succ H_q$. The criteria in the class H_i have a higher priority than those in H_k if $i < k$. The total set of criteria is $C = \cup_{i=1}^{q} H_i$. the total number of criteria is $n = \sum_{i=1}^{q} n_i$. For any alternative $x \in X$, an IVIFV $\tilde{\alpha}_{ij} = \tilde{\alpha}_{ij}(x) \in \Omega$ indicates its satisfaction to criteria C_{ij}. Two kinds of scoring type IVIF prioritized aggregation (IVIFPA) operator $IVIFPA: \tilde{L}^n \to \tilde{L}$ were proposed to calculate overall $\tilde{\alpha}(x)$ for any alternative such that*

$$IVIFPA_1\left((\tilde{\alpha}_{11}, \cdots, \tilde{\alpha}_{1n_1}), \cdots, (\tilde{\alpha}_{q1}, \cdots, \tilde{\alpha}_{qn_q})\right) = \cup_{i=1}^{q}\left(\cup_{j=1}^{n_i}(\tilde{\beta}_i \cap \tilde{\alpha}_{ij}(x))\right) \tag{3}$$

where the weights $\tilde{\beta}_i$ are functions of x and are used to reflect the priority relationship, such as $\tilde{\beta}_i = \otimes_{k=1}^{i}\tilde{\lambda}_{k-1}$, $\tilde{\lambda}_0 = ([1, 1], [0, 0])$, $\tilde{\lambda}_i = \cap_j\{\tilde{\alpha}_{ij}(x)\}$, if $1 \leqslant i \leqslant q$.

$$IVIFPA_2\left((\tilde{\alpha}_{11}, \cdots, \tilde{\alpha}_{1n_1}), \cdots, (\tilde{\alpha}_{q1}, \cdots, \tilde{\alpha}_{qn_q})\right) = \oplus_{i=1}^{q}\left(\oplus_{j=1}^{n_i}(\tilde{\beta}_i \otimes \tilde{\alpha}_{ij}(x))\right) \tag{4}$$

where the weights $\tilde{\beta}_i$ are functions of x and are used to reflect the priority relationship, such as $\tilde{\beta}_i = \otimes_{k=1}^{i}\tilde{\lambda}_{k-1}$, $\tilde{\lambda}_0 = ([1, 1], [0, 0])$, $\tilde{\lambda}_i = \otimes_j\{\tilde{\alpha}_{ij}(x)\}$, if $1 \leqslant i \leqslant q$.

If the priority relationship between the criteria is a linear ordering, no ties allowed, then Yu et al. [19] defined an IVIF prioritized weighted average (IFPWA) operator with a normalized weights as follows.

Definition 3.2. [19] *Let $C_1, C_2, \cdots, C_n$ be a collection of criteria with a prioritization between the criteria expressed by the linear ordering $C_1 \succ C_2 \succ \cdots \succ C_n$. Thus here there is a linear ordering among the criteria. The criteria C_j have a higher priority than C_k if $j < k$. For each criteria C_j, an IVIFV $\tilde{\alpha}_j = ([a_j, b_j], [c_j, d_j]) \in \Omega$ indicating its satisfaction to criteria C_j, then The IVIF prioritized weighted averaging (IVIFPWA) operator and IVIF prioritized weighted geometric (IVIFPWG) operator are two mappings $\tilde{L}^n \to \tilde{L}$ to calculate $\tilde{\alpha}(x) = ([a, b], [c, d])$ for any alternative such that*

$$IVIFPWA\left(\tilde{\alpha}_1, \tilde{\alpha}_2, \cdots, \tilde{\alpha}_n\right) = \oplus_{j=1}^{n} w_j\tilde{\alpha}_j = \left([1 - \prod_{j=1}^{n}(1 - a_j)^{w_j}, 1 - \prod_{j=1}^{n}(1 - b_j)^{w_j}], [\prod_{j=1}^{n} {c_j}^{w_j}, \prod_{j=1}^{n} {d_j}^{w_j}]\right) \tag{5}$$

and

$$IVIFPWG\left(\tilde{\alpha}_1, \tilde{\alpha}_2, \cdots, \tilde{\alpha}_n\right) = \otimes_{j=1}^{n}\tilde{\alpha}_j^{w_j} = \left([\prod_{j=1}^{n} {a_j}^{w_j}, \prod_{j=1}^{n} {b_j}^{w_j}], [1 - \prod_{j=1}^{n}(1 - c_j)^{w_j}, 1 - \prod_{j=1}^{n}(1 - d_j)^{w_j}]\right) \tag{6}$$

where $w_j = T_j/T$, $T_1 = 1$, and for $j > 1$, $T_j = \prod_{k=1}^{j-1} s(\tilde{\alpha}_k)$, $T = \sum_{j=1}^{n} T_j$ and $s(\tilde{\alpha}_k) = (2 + a_k + b_k - c_k - d_k)/4$.

Since these existing IVIF prioritized aggregation operators (3)-(6) are based on the operations 3)-8) on IVIFVs, then we first analyze the properties of the operations 3)-8) on IVIFVs.

Corollary 3.3. [5, 10, 11, 14, 19] *The operations 3)-8) on IVIFVs satisfy monotonicity in the lattice $\tilde{L}$, that is to say, these operations are monotone with respect to $\leqslant_{\tilde{L}}$.*

One undesirable property of the total ordering is mentioned, i.e., it is not preserved under the operations 3)-8), for example,

a) Union: $\hat{\alpha}_1 < \hat{\alpha}_2$ does not necessarily imply $\tilde{\alpha}_1 \cup \tilde{\alpha} < \tilde{\alpha}_2 \cup \tilde{\alpha}$ where $\tilde{\alpha}$ is IVIFV;

b) Algebraic sum: $\tilde{\alpha}_1 < \tilde{\alpha}_2$ does not necessarily imply $\tilde{\alpha}_1 \oplus \tilde{\alpha} < \tilde{\alpha}_2 \oplus \tilde{\alpha}$ where $\tilde{\alpha}$ is IVIFV;

c) Multiplication by a scalar: $\tilde{\alpha}_1 < \tilde{\alpha}_2$ does not necessarily imply $k\tilde{\alpha}_1 < k\tilde{\alpha}_2$ where k is a scalar, as can be seen from the Examples 1-3 below.

Example 3.4. *Take* $\tilde{\alpha} = ([0.25, 0.35], [0.05, 0.15])$, $\tilde{\alpha}_1 = ([0.35, 0.45], [0.15, 0.25])$ *and* $\tilde{\alpha}_2 = ([0.45, 0.55], [0.35, 0.45])$. *Since* $S(\tilde{\alpha}_1) = 0.6$ *and* $S(\tilde{\alpha}_2) = 0.55$, *then* $\tilde{\alpha}_2 < \tilde{\alpha}_1$. *However,* $\tilde{\alpha} \cup \tilde{\alpha}_1 = ([\max\{0.25, 0.35\}, \max\{0.35, 0.45\}], [\min\{0.05, 0.15\}, \min\{0.15, 0.25\}]) = ([0.35, 0.45], [0.05, 0.15])$, $\tilde{\alpha} \cup \tilde{\alpha}_2 = ([\max\{0.25, 0.45\}, \max\{0.35,\ 0.55\}], [\min\{0.05, 0.35\}, \min\{0.15, 0.45\}]) = ([0.45, 0.55], [0.05, 0.15])$ *and* $S(\tilde{\alpha}\cup\tilde{\alpha}_1) = 0.65, S(\tilde{\alpha}\cup\tilde{\alpha}_2) = 0.7$, *so* $\tilde{\alpha}\cup\tilde{\alpha}_1 < \tilde{\alpha}\cup\tilde{\alpha}_2$. *Thus* $\tilde{\alpha}_2 < \tilde{\alpha}_1$ *does not imply* $\tilde{\alpha}\cup\tilde{\alpha}_2 < \tilde{\alpha}\cup\tilde{\alpha}_1$.

Example 3.5. *Take* $\tilde{\alpha} = ([0.25, 0.35], [0.05, 0.15])$, $\tilde{\alpha}_1 = ([0.35, 0.45], [0.15, 0.25])$ *and* $\tilde{\alpha}_2 = ([0.45, 0.55], [0.35, 0.45])$. *Since* $S(\tilde{\alpha}_1) = 0.6$ *and* $S(\tilde{\alpha}_2) = 0.55$, *then* $\tilde{\alpha}_2 < \tilde{\alpha}_1$. *However,* $\tilde{\alpha} \oplus \tilde{\alpha}_1 = ([0.25 + 0.35 - 0.25 \times 0.35, 0.35 + 0.45 - 0.35 \times 0.45], [0.05 \times 0.15, 0.15 \times 0.25]) = ([0.5125, 0.6425], [0.0075, 0.0375])$, $\tilde{\alpha} \oplus \tilde{\alpha}_2 = ([0.25 + 0.45 - 0.25 \times 0.45, 0.35 + 0.55 - 0.35 \times 0.55], [0.05 \times 0.35, 0.15 \times 0.45]) = ([0.5875, 0.7075], [0.0175,\ 0.0675])$ *and* $S(\tilde{\alpha} \oplus \tilde{\alpha}_1) = 0.7775$, $S(\tilde{\alpha} \oplus \tilde{\alpha}_2) = 0.8025$, *so* $\tilde{\alpha} \oplus \tilde{\alpha}_1 < \tilde{\alpha} \oplus \tilde{\alpha}_2$. *Thus* $\tilde{\alpha}_2 < \tilde{\alpha}_1$ *does not imply* $\tilde{\alpha} \oplus \tilde{\alpha}_2 < \tilde{\alpha} \oplus \tilde{\alpha}_1$.

Example 3.6. *Take* $\tilde{\alpha}_1 = ([0.05, 0.35], [0.25, 0.55])$, $\tilde{\alpha}_2 = ([0.15, 0.25], [0.35, 0.45])$ *and* $k = 0.4$. *Since* $S(\tilde{\alpha}_1) = S(\tilde{\alpha}_2) = 0.4$, $H(\tilde{\alpha}_1) = H(\tilde{\alpha}_2) = 0.6$, $T(\tilde{\alpha}_1) = T(\tilde{\alpha}_2) = 0$, $G(\alpha_1) = 0.6$ *and* $G(\alpha_2) = 0.2$, *then* $\tilde{\alpha}_1 < \tilde{\alpha}_2$. *However,* $k\tilde{\alpha}_1 = ([1 - (1 - 0.05)^{0.4}, 1 - (1 - 0.35)^{0.4}], [0.25^{0.4}, 0.55^{0.4}]) \cong ([0.0203, 0.1583], [0.5743, 0.7873])$, $k\tilde{\alpha}_2 = ([1 - (1 - 0.15)^{0.4}, 1 - (1 - 0.25)^{0.4}], [0.35^{0.4}, 0.45^{0.4}]) \cong ([0.0629, 0.1087],\ [0.6571, 0.7266])$, *and* $S(k\tilde{\alpha}_1) = -0.5915$, $S(k\tilde{\alpha}_2) = -0.6060$, *so* $k\tilde{\alpha}_2 < k\tilde{\alpha}_1$. *Thus* $\tilde{\alpha}_1 < \tilde{\alpha}_2$ *does not imply* $k\tilde{\alpha}_1 < k\tilde{\alpha}_2$.

Corollary 3.7. *The operations 3)-8) on IVIFVs are not monotone with respect to the total order, that's to say, the total order is not preserved under the operations 3)-8).*

This has a profound implication on the lack of monotonicity of aggregation functions for IVIFV defined by (3)-(6) with respect to the chosen total ordering, as shown in Proposition 3.8.

Proposition 3.8. *Aggregation operators for IVIFV defined by (3)-(6) are not monotone with respect to the total ordering.*

Another undesirable feature of (6) is that whenever one of the arguments $\tilde{\alpha}_j = ([0, 0], [1, 1])$ and the corresponding weight is not zero, we have $IVIFPWG\,(\tilde{\alpha}_1(x),\ \ \tilde{\alpha}_2(x), \cdots, \tilde{\alpha}_n(x)) = ([0, 0], [1, 1])$, which is rather counterintuitive. When one of the arguments is $([1, 1], [0, 0])$ and its weight is not zero, then it is not accounted for at all, which is again counterintuitive. The aggregation operator (5) also has similar undesirable feature.

Obviously, the IVIF prioritized aggregation operators (3)-(6) are just monotone with respect to the partial order $\leqslant_{L^*}$. In next section, we will deal with an question of whether there is alternative aggregation operators which are monotone with respect to the total order.

4 Interval-valued intuitionistic fuzzy prioritized weighted arithmetic mean (IVIFPWAM) operator

Generally, the operation $\tilde{\alpha}_1 + \tilde{\alpha}_2$ on IVIFVs is written as follows $\tilde{\alpha}_1 + \tilde{\alpha}_2 = ([S(a_1, a_2), S(b_1, b_2)], [T(c_1, c_2), T(d_1, d_2)])$, where T is any t-norm and S is its dual t-conorm, defined by $S(x, y) = 1 - T(1 - x, 1 - y)$ [12].

Taking into account that S_L, T_L, as the Łukasiewicz t-conorm and t-norm respectively, $S_L(x, y) = \min\{1, x + y\}$, $T_L(x, y) = \max\{0, x + y - 1\}$ [27], then for $\lambda \in [0, 1]$ we have

9) $\tilde{\alpha}_1 + \tilde{\alpha}_2 = ([\min\{1, a_1 + a_2\}, \min\{1, b_1 + b_2\}], [\max\{0, c_1 + c_2 - 1\}, \max\{0, \quad d_1 + d_2 - 1\}])$,

10) $\lambda\tilde{\alpha} = ([\lambda a, \lambda b], [1 - \lambda(1 - c), 1 - \lambda(1 - d)])$.

Based on operations 9) and 10), Wang et al. [12] consequently obtained an IVIF weighted arithmetic mean (IVIFWAM).

$$IVIFWAM\left(\tilde{\alpha}_1, \tilde{\alpha}_2, \cdots, \tilde{\alpha}_n\right) = \sum_{j=1}^{n} w_j \tilde{\alpha}_j = \left(\left[\sum_{j=1}^{n} w_j a_j, \sum_{j=1}^{n} w_j b_j\right], \left[\sum_{j=1}^{n} w_j c_j, \sum_{j=1}^{n} w_j d_j\right]\right) \tag{7}$$

Based on the IVIFWAM (7), an IVIF prioritized arithmetic mean (IVIFPAM) is defined as follows:

Definition 4.1. *Let $C_1, C_2, \cdots, C_n$ be a collection of criteria with a prioritization between the criteria expressed by the linear ordering $C_1 \succ C_2 \succ \cdots \succ C_n$. Thus here there is a linear ordering among the criteria. The criteria C_j have a higher priority than C_k if $j < k$. For each criteria C_j, an IVIFV $\tilde{\alpha}_j \in \Omega$ indicating its satisfaction to criteria C_j, then the IVIFPAM operator is a mapping $\Omega^n \to \Omega$ for any alternative such that*

$$IVIFPAM\left(\tilde{\alpha}_1(x), \tilde{\alpha}_2(x), \cdots, \tilde{\alpha}_n(x)\right) = \sum_{j=1}^{n} w_j \tilde{\alpha}_j = \left(\left[\sum_{j=1}^{n} w_j a_j, \sum_{j=1}^{n} w_j b_j\right], \left[\sum_{j=1}^{n} w_j c_j, \sum_{j=1}^{n} w_j d_j\right]\right) \tag{8}$$

where $w_j = T_j/T$, $T_1 = 1$, and for $j > 1$, $T_j = \prod_{k=1}^{j-1} s(\tilde{\alpha}_k)$, $s(\tilde{\alpha}_k) = (2 + a_k + b_k - c_k - d_k)/4$ and $T = \sum_{j=1}^{n} T_j$.

Example 4.2. *Consider a collection of criteria $\{C_1, C_2, C_3, C_4\}$ with a prioritization between the criteria expressed by the following linear ordering: $C_1 \succ C_2 \succ C_3 \succ C_4$.*
Assume for alternative x we have
$\tilde{\alpha}_1 = ([0.5, 0.6], [0.3, 0.4])$, $\tilde{\alpha}_2 = ([0.4, 0.6], [0.2, 0.4])$,$\tilde{\alpha}_3 = ([0.6, 0.7], [0.1, 0.3])$, $\tilde{\alpha}_4 = ([0.5, 0.8], [0.1, 0.2])$.
We first calculate
$S(\tilde{\alpha}_1) = (2 + 0.5 + 0.6 - 0.3 - 0.4)/4 = 0.6$,
$S(\tilde{\alpha}_2) = (2 + 0.4 + 0.6 - 0.2 - 0.4)/4 = 0.6$,
$S(\tilde{\alpha}_3) = (2 + 0.6 + 0.7 - 0.1 - 0.3)/4 = 0.725$.
Using this we get
$T_1 = 1,\ T_2 = S(\tilde{\alpha}_1) = 0.6,\ T_3 = S(\tilde{\alpha}_1)S(\tilde{\alpha}_2) = 0.36,\ T_4 = S(\tilde{\alpha}_1)S(\tilde{\alpha}_2)S(\tilde{\alpha}_3) = 0.261,\ T = \Sigma_{k=1}^{4} T_k = 2.221$.
From this we obtain
$w_1 = T_1/T = 0.4502,\ w_2 = T_2/T = 0.2702,\ w_3 = T_3/T = 0.1621,\ w_4 = T_4/T = 0.1175$.
We now calculate

$$\tilde{\alpha}(x) = \left(\left[\sum_{j=1}^{4} w_j a_j, \sum_{j=1}^{4} w_j b_j\right], \left[\sum_{j=1}^{4} w_j c_j, \sum_{j=1}^{4} w_j d_j\right]\right) = ([0.4892, 0.6397], [0.2171, 0.3603])$$

Note that the prioritized aggregation operators proposed by Yu et al. [19] and Li et al. [10] are not suitable to be utilized in Example 4.2, for the parameters $\tilde{\alpha}_j$ $(j = 1, 2, 3, 4)$ are not the elements of the lattice $\tilde{L}$ but the ones of the set $\tilde{\Omega}$.

We now look at some further properties of the proposed IVIFPAM (8). We recall $C_1, C_2, \cdots, C_n$ be a collection of criteria with a prioritization between the criteria expressed by the linear ordering $C_1 \succ C_2 \succ \cdots \succ C_n$, and the criteria C_j have a higher priority than C_k if $j < k$. For each criteria C_j, an IVIFV $\tilde{\alpha}_j = ([a_j, b_j], [c_j, d_j]) \in \tilde{\Omega}$ indicating its satisfaction to criteria C_j.

Proposition 4.3. *(Idempotency) If all $\tilde{\alpha}_j$ $(j = 1, 2, \cdots, n)$ are equal, i.e. $\tilde{\alpha}_j = \tilde{\alpha} = ([a, b], [c, d])$, for all j, then*

$$IVIFPAM\left(\tilde{\alpha}_1, \tilde{\alpha}_2, \cdots, \tilde{\alpha}_n\right) = \tilde{\alpha}, \tag{9}$$

where $w_j = T_j/T$, $T_1 = 1$, and for $j > 1$, $T_j = \prod_{k=1}^{j-1} s(\tilde{\alpha}_k)$, $s(\tilde{\alpha}_k) = (2 + a_k + b_k - c_k - d_k)/4$ and $T = \sum_{j=1}^{n} T_j$.

Proposition 4.4. *(Boundary) If $\tilde{\beta} = \min\{\tilde{\alpha}_1, \tilde{\alpha}_2, \cdots, \tilde{\alpha}_n\}$ and $\tilde{\gamma} = \max\{\tilde{\alpha}_1, \tilde{\alpha}_2, \cdots, \tilde{\alpha}_n\}$, then*

$$\tilde{\beta} \leqslant IVIFPAM\left(\tilde{\alpha}_1, \tilde{\alpha}_2, \cdots, \tilde{\alpha}_n\right) \leqslant \tilde{\gamma}, \tag{10}$$

where $w_j = T_j/T$, $T_1 = 1$, and for $j > 1$, $T_j = \prod_{k=1}^{j-1} S(\tilde{\alpha}_k)$, $S(\tilde{\alpha}_k) = (2 + a_k + b_k - c_k - d_k)/4$ and $T = \sum_{j=1}^{n} T_j$.

Proof. Let $IVIFPAM\,(\tilde{\alpha}_1, \tilde{\alpha}_2, \cdots, \tilde{\alpha}_n) = \left(\left[\sum_{j=1}^n w_j a_j, \sum_{j=1}^n w_j b_j\right], \left[\sum_{j=1}^n w_j c_j, \quad \sum_{j=1}^n w_j d_j\right]\right) = \tilde{\alpha}$ Since $\tilde{\beta} = \min\{\tilde{\alpha}_1, \tilde{\alpha}_2, \cdots, \tilde{\alpha}_n\}$, then $\tilde{\beta} \leqslant \tilde{\alpha}_j$,

Case 1. For all j, $S(\tilde{\beta}) \leqslant S(\tilde{\alpha}_j)$ and at least one of these inequalities holds strictly, i.e., $S(\tilde{\beta}) \leqslant (2 + a_j + b_j - c_j - d_j)/4$, then

$$S(\tilde{\beta}) < \sum_{j=1}^n w_j \frac{(2 + a_j + b_j - c_j - d_j)}{4} \text{ i.e., } S(\tilde{\beta}) < S(\tilde{\alpha}). \tag{11}$$

Case 2. For all j, $S(\tilde{\beta}) = S(\tilde{\alpha})$, $H(\tilde{\beta}) \leqslant H(\tilde{\alpha}_j)$ and at least one of these inequalities holds strictly, i.e., $S(\tilde{\beta}) = (2 + a_j + b_j - c_j - d_j)/4$ and $H(\tilde{\beta}) \leqslant (a_j + b_j + c_j + d_j)/2$, then

$$S(\tilde{\beta}) = \sum_{j=1}^n w_j \frac{(2 + a_j + b_j - c_j - d_j)}{4} \text{ and } H(\tilde{\beta}) < \sum_{j=1}^n w_j \frac{(a_j + b_j + c_j + d_j)}{2}. \text{ i.e., } S(\tilde{\beta}) = S(\tilde{\alpha}) \;\&\; H(\tilde{\beta}) < H(\tilde{\alpha}). \tag{12}$$

Case 3. For all j, $S(\tilde{\beta}) = S(\tilde{\alpha})$, $H(\tilde{\beta}) = H(\tilde{\alpha})$, $T(\tilde{\beta}) \geqslant T(\tilde{\alpha}_j)$ and at least one of these inequalities holds strictly, i.e., $S(\tilde{\beta}) = (2 + a_j + b_j - c_j - d_j)/4$, $H(\tilde{\beta}) = (a_j + b_j + c_j + d_j)/2$ and $T(\tilde{\beta}) \geqslant b_j + c_j - a_j - d_j$, then

$$S(\tilde{\beta}) = \sum_{j=1}^n w_j \frac{(2 + a_j + b_j - c_j - d_j)}{4}, \quad H(\tilde{\beta}) = \sum_{j=1}^n w_j \frac{(a_j + b_j + c_j + d_j)}{2}$$

and

$$T(\tilde{\beta}) > \sum_{j=1}^n w_j (b_j + c_j - a_j - d_j), \text{ i.e., } S(\tilde{\beta}) = S(\tilde{\alpha}) \;\&\; H(\tilde{\beta}) = H(\tilde{\alpha}) \;\&\; T(\tilde{\beta}) > T(\tilde{\alpha}). \tag{13}$$

Case 4. For all j, $S(\tilde{\beta}) = S(\tilde{\alpha})$, $H(\tilde{\beta}) = H(\tilde{\alpha})$, $H(\tilde{\beta}) = H(\tilde{\alpha})$ and $G(\tilde{\beta}) \geqslant G(\tilde{\alpha}_j)$, i.e., $S(\tilde{\beta}) = (2 + a_j + b_j - c_j - d_j)/4$, $H(\tilde{\beta}) = (a_j + b_j + c_j + d_j)/2$ and $T(\tilde{\beta}) = b_j + c_j - a_j - d_j$ and $G(\tilde{\beta}) \geqslant b_j + d_j - a_j - c_j$, then

$$S(\tilde{\beta}) = \sum_{j=1}^n w_j \frac{(2 + a_j + b_j - c_j - d_j)}{4}, \quad H(\tilde{\beta}) = \sum_{j=1}^n w_j \frac{(a_j + b_j + c_j + d_j)}{2},$$

$$T(\tilde{\beta}) = \sum_{j=1}^n w_j (b_j + c_j - a_j - d_j) \text{ and } G(\tilde{\beta}) \geqslant \sum_{j=1}^n w_j (b_j + d_j - a_j - c_j).$$

i.e.,

$$S(\tilde{\beta}) = S(\tilde{\alpha}) \;\&\; H(\tilde{\beta}) = H(\tilde{\alpha}) \;\&\; T(\tilde{\beta}) = T(\tilde{\alpha}) \;\&\; G(\tilde{\beta}) \geqslant T(\tilde{\alpha}). \tag{14}$$

Based on (11) - (14), we get $\hat{\beta} \leqslant IVIFPAM\,(\tilde{\alpha}_1, \tilde{\alpha}_2, \cdots, \tilde{\alpha}_n)$. Similarly, we have $IVIFPAM\,(\tilde{\alpha}_1, \tilde{\alpha}_2, \cdots, \tilde{\alpha}_n) \leqslant \tilde{\gamma}$. □

For linear ordered criteria, Yager [18] obtained a prioritized averaging aggregation operator, and showed that this aggregation method was monotonic.

Lemma 4.5. *[18] Suppose $C_1, C_2, \cdots, C_n$ be a collection of criteria with a prioritization between the criteria expressed by the linear ordering $C_1 \succ C_2 \succ \cdots \succ C_n$, and the criteria C_j have a higher priority than C_k if $j < k$, $\{a_1, a_2, \cdots, a_n\}$ and $\{b_1, b_2, \cdots, b_n\}$ be two collection of satisfaction to criteria belong to $[0, 1]$, for all j, $a_j \leqslant b_j$, then*

$$\sum_{j=1}^n w_j a_j \leqslant \sum_{j=1}^n w'_j b_j \tag{15}$$

where $w_j = T_j/T$, $T_1 = 1$, for $j > 1$, $T_j = \prod_{k=1}^{j-1} a_k$ and $T = \sum_{j=1}^n T_j$; $w'_j = \frac{T'_j}{T'}$, $T'_1 = 1$, for $j > 1$, $T'_j = \prod_{k=1}^{j-1} b_k$ and $T' = \sum_{j=1}^n T'_j$.

Proposition 4.6. *(Monotonicity) If* $\tilde{\alpha}_j = ([a_j, b_j], [c_j, d_j])$, $\tilde{\beta}_j = ([a'_j, b'_j], [c'_j, d'_j])$ *and* $\tilde{\alpha}_j \leqslant \tilde{\beta}_j$ *for all* j*, then*

$$IVIFPAM(\tilde{\alpha}_1, \tilde{\alpha}_2, \cdots, \tilde{\alpha}_n) \leqslant IVIFPAM\left(\tilde{\beta}_1, \tilde{\beta}_2, \cdots, \tilde{\beta}_n\right), \tag{16}$$

where $w_j = T_j/T$, $T_1 = 1$*, and for* $j > 1$, $T_j = \prod_{k=1}^{j-1} S(\tilde{\alpha}_k)$, $T = \sum_{j=1}^{n} T_j$ *and* $S(\tilde{\alpha}_k) = (2 + a_k + b_k - c_k - d_k)/4$*;* $w'_j = \frac{T'_j}{T'}$, $T'_1 = 1$*, for* $j > 1$, $T'_j = \prod_{k=1}^{j-1} S(\tilde{\beta}_k)$, $T' = \sum_{j=1}^{n} T'_j$ *and* $S(\tilde{\beta}_k) = (2 + a'_k + b'_k - c'_k - d'_k)/4$.

Proof. Let $IVIFPAM(\tilde{\alpha}_1, \tilde{\alpha}_2, \cdots, \tilde{\alpha}_n) = \left(\left[\sum_{j=1}^{n} w_j a_j, \sum_{j=1}^{n} w_j b_j\right], \left[\sum_{j=1}^{n} w_j c_j, \ \sum_{j=1}^{n} w_j d_j\right]\right) = \tilde{\alpha}$ and $IVIFPAM$ $\left(\tilde{\beta}_1, \tilde{\beta}_2, \cdots, \tilde{\beta}_n\right) = \left(\left[\sum_{j=1}^{n} w'_j a'_j, \sum_{j=1}^{n} w'_j b'_j\right], \ \left[\sum_{j=1}^{n} w'_j c'_j, \sum_{j=1}^{n} w'_j d'_j\right]\right) = \tilde{\beta}$. Since for all j, $\tilde{\alpha}_j \leqslant \tilde{\beta}_j$, then

Case 1. $S(\tilde{\alpha}_j) \leqslant S(\tilde{\beta}_j)$ for all j and at least one of these inequalities holds strictly, i.e., $\{\tilde{s}(\tilde{\alpha}_1), S(\tilde{\alpha}_2), \cdots, S(\tilde{\alpha}_n)\}$ and $\{S(\tilde{\beta}_1), S(\tilde{\beta}_2), \cdots, S(\tilde{\beta}_n)\}$ be two collection of real numbers belong to $[0, 1]$. Based on Lemma 4.5, we have

$$\sum_{j=1}^{n} w_j S(\tilde{\alpha}_j) < \sum_{j=1}^{n} w'_j S(\tilde{\beta}_j) \tag{17}$$

where $w_j = T_j/T$, $T_1 = 1$, and for $j > 1$,$T_j = \prod_{k=1}^{j-1} S(\tilde{\alpha}_k)$ and $T = \sum_{j=1}^{n} T_j$; $w'_j = \frac{T'_j}{T'}$, $T'_1 = 1$, for $j > 1$, $T'_j = \prod_{k=1}^{j-1} S(\tilde{\beta}_k)$ and $T' = \sum_{j=1}^{n} T'_j$. Inequality (17) is equivalent to

$$\sum_{j=1}^{n} w_j \frac{2 + a_j + b_j - c_j - d_j}{4} < \sum_{j=1}^{n} w'_j \frac{2 + a'_j + b'_j - c'_j - d'_j}{4} \quad \text{i.e.,} \quad S(\tilde{\alpha}) < S(\tilde{\beta}); \tag{18}$$

Case 2. $S(\tilde{\alpha}_j) = S(\tilde{\beta}_j)$ and $H(\tilde{\alpha}_j) \leqslant H(\tilde{\beta}_j)$ for all j and at least one of these inequalities holds strictly.
On one hand, since for all j, $S(\tilde{\alpha}_j) = S(\tilde{\beta}_j)$ (here $\{S(\tilde{\alpha}_1), S(\tilde{\alpha}_2), \cdots, S(\tilde{\alpha}_n)\}$ and $\{S(\tilde{\beta}_1), S(\tilde{\beta}_2), \cdots, S(\tilde{\beta}_n)\}$ be two collection of real numbers belong to $[0, 1]$), based on Lemma 4.5, we have

$$\sum_{j=1}^{n} w_j S(\tilde{\alpha}_j) = \sum_{j=1}^{n} w'_j S(\tilde{\beta}_j) \tag{19}$$

where $w_j = w'_j = T_j/T$, $T_1 = 1$, and for $j > 1$,$T_j = \prod_{k=1}^{j-1} S(\tilde{\alpha}_k)$ and $T = \sum_{j=1}^{n} T_j$.
Equality (19) is equivalent to

$$\sum_{j=1}^{n} w_j \frac{2 + a_j + b_j - c_j - d_j}{4} = \sum_{j=1}^{n} w'_j \frac{2 + a'_j + b'_j - c'_j - d'_j}{4} \quad \text{i.e.,} \quad S(\tilde{\alpha}) = S(\tilde{\beta}); \tag{20}$$

On the other hand, since for all j, $H(\tilde{\alpha}_j) \leqslant H(\tilde{\beta}_j)$ and at least one of these inequalities holds strictly, i.e.,

$$\frac{a_j + b_j + c_j + d_j}{2} \leqslant \frac{a'_j + b'_j + c'_j + d'_j}{2}, \quad j = 1, 2, \cdots, n,$$

we can get

$$\sum_{j=1}^{n} w_j \frac{a_j + b_j + c_j + d_j}{2} < \sum_{j=1}^{n} w'_j \frac{a'_j + b'_j + c'_j + d'_j}{2} \tag{21}$$

where $w_j = w'_j = T_j/T$, $T_1 = 1$, and for $j > 1$,$T_j = \prod_{k=1}^{j-1} S(\tilde{\alpha}_k)$ and $T = \sum_{j=1}^{n} T_j$.
Inequality (21) is equivalent to

$$H(\tilde{\alpha}) < H(\tilde{\beta}). \tag{22}$$

Case 3. For all j, $S(\tilde{\alpha_j}) = S(\tilde{\beta}_j)$, $H(\tilde{\alpha_j}) = H(\tilde{\beta}_j)$, $T(\tilde{\alpha_j}) \geqslant T(\tilde{\beta}_j)$ and at least one of these inequalities holds strictly.
On one hand, since for all j, $S(\tilde{\alpha_j}) = S(\tilde{\beta}_j)$, $H(\tilde{\alpha_j}) = H(\tilde{\beta}_j)$, then we have

$$\sum_{j=1}^{n} w_j S(\tilde{\alpha}_j) = \sum_{j=1}^{n} w'_j S(\tilde{\beta}_j), \quad \text{and} \quad \sum_{j=1}^{n} w_j H(\tilde{\alpha}_j) = \sum_{j=1}^{n} w'_j H(\tilde{\beta}_j) \tag{23}$$

where $w_j = w'_j = T_j/T$, $T_1 = 1$, and for $j > 1$,$T_j = \prod_{k=1}^{j-1} S(\tilde{\alpha}_k)$ and $T = \sum_{j=1}^{n} T_j$.
Inequalities (23) are equivalent to

$$S(\tilde{\alpha}) = S(\tilde{\beta}), \quad H(\tilde{\alpha}) = H(\tilde{\beta}). \tag{24}$$

On the other hand, since for all j, $T(\tilde{\alpha}_j) \geqslant T(\tilde{\beta}_j)$ and at least one of these inequalities holds strictly, i.e.,

$$b_j + c_j - a_j - d_j \geqslant b'_j + c'_j - a'_j - d'_j, \quad j = 1, 2, \cdots, n,$$

we can get

$$\sum_{j=1}^{n} w_j (b_j + c_j - a_j - d_j) > \sum_{j=1}^{n} w'_j (b'_j + c'_j - a'_j - d'_j) \tag{25}$$

where $w_j = w'_j = T_j/T$, $T_1 = 1$, and for $j > 1$,$T_j = \prod_{k=1}^{j-1} S(\tilde{\alpha}_k)$ and $T = \sum_{j=1}^{n} T_j$.
Inequality (25) is equivalent to

$$T(\tilde{\alpha}) > T(\tilde{\beta}). \tag{26}$$

Case 4. For all j, $S(\tilde{\alpha_j}) = S(\tilde{\beta}_j)$, $H(\tilde{\alpha_j}) = H(\tilde{\beta}_j)$, $T(\tilde{\alpha_j}) = T(\tilde{\beta}_j)$, $G(\tilde{\alpha_j}) \geqslant G(\tilde{\beta}_j)$ and at least one of these inequalities holds strictly.
On one hand, since for all j, $S(\tilde{\alpha_j}) = S(\tilde{\beta}_j)$, $H(\tilde{\alpha_j}) = H(\tilde{\beta}_j)$ and $T(\tilde{\alpha_j}) = T(\tilde{\beta}_j)$, then we have

$$\sum_{j=1}^{n} w_j S(\tilde{\alpha}_j) = \sum_{j=1}^{n} w'_j S(\tilde{\beta}_j); \quad \sum_{j=1}^{n} w_j H(\tilde{\alpha}_j) = \sum_{j=1}^{n} w'_j H(\tilde{\beta}_j); \quad \sum_{j=1}^{n} w_j T(\tilde{\alpha}_j) = \sum_{j=1}^{n} w'_j T(\tilde{\beta}_j). \tag{27}$$

where $w_j = w'_j = T_j/T$, $T_1 = 1$, and for $j > 1$,$T_j = \prod_{k=1}^{j-1} S(\tilde{\alpha}_k)$ and $T = \sum_{j=1}^{n} T_j$.
Inequalities (27) are equivalent to

$$S(\tilde{\alpha}) = S(\tilde{\beta}), \quad H(\tilde{\alpha}) = H(\tilde{\beta}), \quad T(\tilde{\alpha}) = T(\tilde{\beta}). \tag{28}$$

On the other hand, since for all j, $G(\tilde{\alpha}_j) \geqslant G(\tilde{\beta}_j)$ and at least one of these inequalities holds strictly, i.e.,

$$b_j + d_j - a_j - c_j \geqslant b'_j + d'_j - a'_j - c'_j, \quad j = 1, 2, \cdots, n,$$

we can get

$$\sum_{j=1}^{n} w_j (b_j + d_j - a_j - c_j) < \sum_{j=1}^{n} w'_j (b'_j + d'_j - a'_j - c'_j) \tag{29}$$

where $w_j = w'_j = T_j/T$, $T_1 = 1$, and for $j > 1$,$T_j = \prod_{k=1}^{j-1} S(\tilde{\alpha}_k)$ and $T = \sum_{j=1}^{n} T_j$.
Inequality (29) is equivalent to

$$G(\tilde{\alpha}) > G(\tilde{\beta}). \tag{30}$$

According to the results of the four cases, we can obtain $\tilde{\alpha} \leqslant \tilde{\beta}$, i.e.,

$$IVIFPAM\left(\tilde{\alpha}_1, \tilde{\alpha}_2, \cdots, \tilde{\alpha}_n\right) \leqslant IVIFPAM\left(\tilde{\beta}_1, \tilde{\beta}_2, \cdots, \tilde{\beta}_n\right),$$

□

5 Illustrative example

In this section, we take an example that a panel of decision makers of a Chinese university recruit overseas outstanding teachers (from the reference ([19])) so as to illustrate the practicability of the IVIFPAM).

In order to strengthen academic education and promote the building of teaching body, the school of management in a Chinese university wants to recruit overseas outstanding teachers. This recruitment has been raised great attention from the school, the university president d_1, the dean of management school d_2, and the human resource officer d_3 sets up the panel of decision makers which will take the whole responsibility for this recruitment. They made strict evaluation for five candidates x_i $(i = 1, 2, \cdots, 5)$ from four aspects, namely morality C_1, research capability C_2, teaching skill C_3 and education background C_4. The university president have the absolute priority for decision making, the dean of the management school comes next, i.e., the prioritization (important) relationship for the decision makers is $d_1 \succ d_2 \succ d_3$. Besides, this recruitment will be in strict accordance with the principle of combine ability with political integrity. The prioritization relationship for the criteria is as below, $C_1 \succ C_2 \succ C_3 \succ C_4$. Three decision makers d_i $(i = 1, 2, 3)$ evaluate the candidates x_i $(i = 1, 2, \cdots, 5)$ with respect to the criteria C_j $(j = 1, 2, 3, 4)$ and construct three interval-valued intuitionistic fuzzy decision $\tilde{R}_l = (\tilde{r}_{ij}^l)_{5\times 4}$ $(l = 1, 2, 3)$ (see Tables 1-3).

Table 1: Interval-valued intuitionistic fuzzy decision matrix $\tilde{R}_1$.

	C_1	C_2	C_3	C_4
x_1	([0.6, 0.8], [0.1, 0.2])	([0.2, 0.4], [0.4, 0.5])	([0.6, 0.7], [0.2, 0.3])	([0.4, 0.5], [0.2, 0.4])
x_2	([0.4, 0.7], [0.0, 0.1])	([0.5, 0.7], [0.1, 0.2])	([0.7, 0.8], [0.1, 0.2])	([0.7, 0.8], [0.1, 0.2])
x_3	([0.3, 0.7], [0.2, 0.3])	([0.2, 0.4], [0.4, 0.5])	([0.1, 0.4], [0.4, 0.5])	([0.3, 0.4], [0.4, 0.6])
x_4	([0.7, 0.8], [0.1, 0.2])	([0.2, 0.3], [0.4, 0.6])	([0.6, 0.8], [0.0, 0.2])	([0.6, 0.8], [0.0, 0.2])
x_5	([0.5, 0.6], [0.3, 0.4])	([0.7, 0.8], [0.0, 0.1])	([0.2, 0.4], [0.4, 0.5])	([0.1, 0.3], [0.4, 0.6])

Table 2: Interval-valued intuitionistic fuzzy decision matrix $\tilde{R}_2$.

	C_1	C_2	C_3	C_4
x_1	([0.2, 0.4], [0.4, 0.5])	([0.6, 0.7], [0.1, 0.2])	([0.5, 0.7], [0.1, 0.2])	([0.5, 0.7], [0.1, 0.2])
x_2	([0.6, 0.8], [0.0, 0.2])	([0.2, 0.3], [0.4, 0.6])	([0.7, 0.8], [0.1, 0.2])	([0.2, 0.4], [0.4, 0.5])
x_3	([0.1, 0.4], [0.4, 0.5])	([0.8, 0.9], [0.0, 0.1])	([0.1, 0.4], [0.2, 0.5])	([0.4, 0.7], [0.2, 0.3])
x_4	([0.6, 0.8], [0.0, 0.2])	([0.3, 0.8], [0.0, 0.1])	([0.2, 0.3], [0.4, 0.6])	([0.6, 0.7], [0.2, 0.3])
x_5	([0.2, 0.4], [0.5, 0.6])	([0.6, 0.7], [0.2, 0.3])	([0.6, 0.8], [0.0, 0.2])	([0.1, 0.4], [0.3, 0.5])

Table 3: Interval-valued intuitionistic fuzzy decision matrix $\tilde{R}_3$.

	C_1	C_2	C_3	C_4
x_1	([0.2, 0.4], [0.4, 0.5])	([0.2, 0.4], [0.4, 0.5])	([0.4, 0.7], [0.0, 0.1])	([0.7, 0.9], [0.0, 0.1])
x_2	([0.2, 0.3], [0.4, 0.6])	([0.2, 0.3], [0.4, 0.6])	([0.6, 0.7], [0.2, 0.3])	([0.5, 0.7], [0.1, 0.2])
x_3	([0.7, 0.9], [0.0, 0.1])	([0.3, 0.4], [0.4, 0.5])	([0.1, 0.3], [0.3, 0.5])	([0.2, 0.4], [0.4, 0.5])
x_4	([0.3, 0.8], [0.1, 0.2])	([0.1, 0.2], [0.4, 0.6])	([0.2, 0.3], [0.4, 0.5])	([0.3, 0.4], [0.4, 0.6])
x_5	([0.7, 0.8], [0.0, 0.2])	([0.3, 0.8], [0.0, 0.1])	([0.4, 0.7], [0.2, 0.3])	([0.6, 0.8], [0.0, 0.2])

Since all the criteria C_j $(j = 1, 2, 3, 4)$ are of the benefit type, then the criterion values do not need normalization, therefore, Based on the IVIFPAM operator, the main steps are as follows:

Step 1. Utilize the IVIFPAM operator (8) to aggregate the individual interval-valued intuitionistic fuzzy decision matrix $\tilde{R}_l = (r_{ij}^l)_{5\times 4}$ (l = 1,2,3) into the collective intuitionistic fuzzy decision matrix $\tilde{R} = (\tilde{r}_{ij})_{5\times 4}$ (see Table 4).

$$\tilde{r}_{ij} = IVIFPAM\left(\tilde{r}_{ij}^l, \tilde{r}_{ij}^2, \tilde{r}_{ij}^3\right) = \sum_{l=1}^{3} w_{ij}^l \tilde{r}_{ij}^l = \left(\left[\sum_{l=1}^{3} w_{ij}^l a_{ij}^l, \sum_{l=1}^{3} w_{ij}^l b_{ij}^l\right], \left[\sum_{l=1}^{3} w_{ij}^l c_{ij}^l, \sum_{l=1}^{3} w_{ij}^l d_{ij}^l\right]\right) = [a_{ij}, b_{ij}], [c_{ij}, d_{ij}] \tag{31}$$

where $w_{ij}^l = T_{ij}^l / T_{ij}, (l = 1, 2, 3)$; $T_{ij}^1 = 1$, $T_{ij}^2 = s(\tilde{r}_{ij}^1)$, $T_{ij}^3 = s(\tilde{r}_{ij}^1) \times s(\tilde{r}_{ij}^2)$ and $T_{ij} = \sum_{l=1}^{3} T_{ij}^l$, $i = 1, 2, \cdots, 5; j = 1, 2, 3, 4$.

Table 4: The collective IVIF decision matrix $\tilde{R}$.

	C_1	C_2
x_1	([0.3901, 0.5901], [0.2574, 0.3574])	([0.2975, 0.4731], [0.3269, 0.4269])
x_2	(0.4128, 0.5872], [0.1021, 0.2596])	([0.3502, 0.5003], [0.2498, 0.3997])
x_3	(0.2867, 0.6267], [0.2400, 0.3400])	([0.3622, 0.5176], [0.3059, 0.4059])
x_4	([0.5623, 0.8000], [0.0672, 0.2000])	([0.2057, 0.3962], [0.3094, 0.4868])
x_5	([0.4260, 0.5589], [0.3288, 0.4411])	([0.5679, 0.7652], [0.0695, 0.1695])
	C_3	C_4
x_1	([0.5223, 0.7000], [0.1223, 0.2223])	([0.4917, 0.6414], [0.1293, 0.2795])
x_2	([0.6738, 0.7738], [0.1262, 0.2262])	([0.4813, 0.6346], [0.2121, 0.3121])
x_3	([0.1000, 0.3886], [0.3380, 0.5000])	([0.3087, 0.4749], [0.3500, 0.5088])
x_4	([0.3905, 0.5381], [0.2095, 0.3952])	([0.5288, 0.6712], [0.1627, 0.3288])
x_5	([0.3348, 0.5541], [0.2652, 0.3892])	([0.1496, 0.3730], [0.3369, 0.5369])

Step 2. Utilize the IVIFPAM operator (8)

$$\tilde{r}_i = IVIFPAM(\tilde{r}_{i1}, \tilde{r}_{i2}, \tilde{r}_{i3}, \tilde{r}_{i4}) = \sum_{j=1}^{4} w_{ij}\tilde{r}_{ij} = \left(\left[\sum_{j=1}^{4} w_{ij}a_{ij}, \sum_{j=1}^{4} w_{ij}b_{ij}\right], \left[\sum_{j=1}^{4} w_{ij}c_{ij}, \sum_{j=1}^{4} w_{ij}d_{ij}\right]\right) \tag{32}$$

where $w_{ij} = T_{ij}/T_i, (j = 1, 2, 3, 4)$; $T_{i1} = 1$, $T_{i2} = s(\tilde{r}_{i1})$, $T_{i3} = s(\tilde{r}_{i1}) \times s(\tilde{r}_{i2})$, $T_{i4} = s(\tilde{r}_{i1}) \times s(\tilde{r}_{i2}) \times s(\tilde{r}_{i3})$ and $T_i = \sum_{j=1}^{4} T_{ij}$, $i = 1, 2, \cdots, 5$.

to aggregate all the preference values $\tilde{r}_{ij}(j = 1, 2, 3, 4)$ in the ith line of $\tilde{R}$, and get the overall preference values $\tilde{r}_i$ $(i = 1, 2, \cdots, 5)$:

$\tilde{r}_1 = ([0.3932, 0.5780], [0.2447, 0.3498])$, $\tilde{r}_2 = ([0.4444, 0.5975], [0.1616, 0.3009])$, $\tilde{r}_3 = ([0.2807, 0.5485], [0.2813, 0.3947])$, $\tilde{r}_4 = ([0.4150, 0.6151], [0.1775, 0.3360])$, $\tilde{r}_5 = ([0.4140, 0.5894], [0.2528, 0.3737])$.

Step 3. Calculate the scores of $\tilde{r}_i(i = 1, 2, 3, 4, 5)$ respectively: $S(\tilde{r}_1) = 0.5942$, $S(\tilde{r}_2) = 0.6449$, $S(\tilde{r}_3) = 0.5383$, $S(\tilde{r}_4) = 0.6291$, $S(\tilde{r}_5) = 0.5942$, $H(\tilde{r}_1) = 0.7829$ and $H(\tilde{r}_5) = 0.8150$. Since $S(\tilde{r}_2) > S(\tilde{r}_4) > S(\tilde{r}_5) = S(\tilde{r}_1) > S(\tilde{r}_3)$ and $H(\tilde{r}_5) > H(\tilde{r}_1)$, we have: $x_2 \succ x_4 \succ x_5 \succ x_1 \succ x_3$. So the best alternative is $x2$.

Note that Yu et al. ([19]) utilized the IVIFPWA (5) and IVIFPWG (6) operator to solve the IVIF MCDM problem about talent introduction and derived the same ranking of the alternatives.

The proposed IVIFPAM (8) mentioned above is used to deal with the IVIF MCDM problem in which only one criterion exists at each priority level. Taking into account the situation that more than one criteria exist at some priority level, we take the following two steps to calculate overall value for any alternative.

Corollary 5.1. *Suppose a collection of criteria partitioned into q distinct categories $H_1, H_2, \cdots, H_q$, such that $H_i = \{C_{i1}, C_{i2}, \cdots, C_{in_i}\}$, where C_{ij} are the criteria in category H_i, and there are a prioritization between these categories $H_1 \succ H_2 \succ \cdots \succ H_q$. The criteria in the class H_i have a higher priority than those in H_k if $i < k$. The total set of criteria is $C = \cup_{i=1}^{q} H_i$. The total number of criteria is $n = \sum_{i=1}^{q} n_i$. For any alternative $x \in X$, an IVIFV $\tilde{\alpha}_{ij}(x) \in \Omega$ indicates its satisfaction to criteria C_{ij}. we take the following two steps to calculate overall $\alpha(x)$ for any alternative x.*

Step 1. *Utilize the IVIFWAM (7) to aggregate the IVIFVs $\tilde{\alpha}_{ij}(x) = ([a_{ij}, b_{ij}], [c_{ij}, d_{ij}])$ $(j = 1, 2, \cdots, n_i)$ with the same priority H_i, i.e., for $i = 1, 2, \cdots, q.$,*

$$\tilde{\beta}_i(x) = IVIFWAM(\tilde{\alpha}_{i1}(x), \tilde{\alpha}_{i2}(x), \cdots, \tilde{\alpha}_{in_i}(x)) = \sum_{j=1}^{n_i} \frac{1}{n_i}\tilde{\alpha}_j = \left(\left[\sum_{j=1}^{n_i} \frac{a_{ij}}{n_i}, \sum_{j=1}^{n_i} \frac{b_{ij}}{n_i}\right], \left[\sum_{j=1}^{n_i} \frac{c_{ij}}{n_i}, \sum_{j=1}^{n_i} \frac{d_{ij}}{n_i}\right]\right)$$

Step 2. *Utilize the IVIFPAM (8) to aggregate the IVIFVs $\tilde{\beta}_i = ([a'_i, b'_i], [c'_i, d'_i])$ $(i = 1, 2, \cdots, q)$ with the different priority expressed by the linear ordering $H_1 \succ H_2 \succ \cdots \succ H_q$, i.e.,*

$$\tilde{\alpha}(x) = IVIFPAM\left(\tilde{\beta}_1(x), \tilde{\beta}_2(x), \cdots, \tilde{\beta}_q(x)\right) = \left(\left[\sum_{i=1}^{q} w_i a'_i, \sum_{i=1}^{q} w_i b'_i\right], \left[\sum_{i=1}^{q} w_i c'_i, \sum_{i=1}^{q} w_i d'_i\right]\right)$$

where $w_i = T_i/T$, $T_1 = 1$, and for $i > 1$, $T_i = \prod_{k=1}^{i-1} S(\tilde{\beta}_k)$, $T = \sum_{i=1}^{q} T_j$ and $S(\tilde{\beta}_k) = (2 + a'_k + b'_k - c'_k - d'_k)/4$.

Time complexity depends mainly on the number of distinct categories, q and the number of the criteria in category H_i, n_i. The time complexity of aggregating the IVIFVs $\tilde{\alpha}_{ij}(x)$ $(j = 1, 2, \cdots, n_i)$ with the same priority H_i in Step 1 is $O(n_i)$ $(i = 1, 2, \ldots, q)$. In Step 2, the time complexity of aggregating the IVIFVs $\tilde{\beta}_i$ $(i = 1, 2, \cdots, q)$ with the different priority expressed by the linear ordering $H_1 \succ H_2 \succ \cdots \succ H_q$ is $O(q)$. So the total time complexity of the proposed approach is the largest one of both steps.

Example 5.2. *Consider the following prioritized collection of criteria:*
$H_1 = \{C_{11}, C_{12}\}, H_2 = \{C_{21}\}, H_3 = \{C_{31}, C_{32}, C_{33}\}, H_4 = \{C_{41}, C_{42}\}$.
Assume for alternative x we have
$\tilde{\alpha}_{11} = ([0.4, 0.6], [0.2, 0.3])$, $\tilde{\alpha}_{12} = ([0.5, 0.6], [0.3, 0.4])$,
$\tilde{\alpha}_{21} = ([0.6, 0.7], [0.1, 0.3])$,
$\tilde{\alpha}_{31} = ([0.5, 0.7], [0.1, 0.3])$, $\tilde{\alpha}_{32} = ([0.5, 0.6], [0.1, 0.2])$, $\tilde{\alpha}_{33} = ([0.4, 0.5], [0.4, 0.5])$,
$\tilde{\alpha}_{41} = ([0.4, 0.5], [0.4, 0.5])$, $\tilde{\alpha}_{42} = ([0.6, 0.7], [0.2, 0.3])$.
In the first step we have
$\tilde{\beta}_1 = \sum_{j=1}^{2} \frac{1}{2}\tilde{\alpha}_{1j} = ([0.45, 0.6], [0.25, 0.35])$, $\quad \tilde{\beta}_2 = \tilde{\alpha}_{21} = ([0.6, 0.7], [0.1, 0.3])$,
$\tilde{\beta}_3 = \sum_{j=1}^{3} \frac{1}{3}\tilde{\alpha}_{3j} = ([0.4667, 0.6], [0.2, 0.3333])$, $\quad \tilde{\beta}_4 = \sum_{j=1}^{2} \frac{1}{2}\tilde{\alpha}_{4j} = ([0.5, 0.6], [0.3, 0.4])$.
In the second step, we calculate
$S(\tilde{\beta}_1) = 0.6125, \quad S(\tilde{\beta}_2) = 0.7250, \quad S(\tilde{\beta}_3) = 0.6333.$
Using this we get
$T_1 = 1,\ T_2 = S(\tilde{\beta}_1) = 0.6125,\ T_3 = S(\tilde{\beta}_1)S(\tilde{\beta}_2) = 0.4441,\ T_4 = S(\tilde{\beta}_1)S(\tilde{\beta}_2)S(\tilde{\beta}_3) = 0.2812,\ T = \Sigma_{k=1}^{4} T_k = 2.9708.$
From this we obtain
$w_1 = T_1/T = 0.3366,\ w_2 = T_2/T = 0.2062,\ w_3 = T_3/T = 0.2440,\ w_4 = T_4/T = 0.2132.$
In this case then we have $\tilde{\alpha}(x) = \sum_{i=1}^{4} w_i \tilde{\beta}_i = ([0.4957, 0.6206], [0.2175, 0.3463])$.

6 Conclusions

In this paper,we have developed a prioritized arithmetic mean to deal with IVIF MCDM in the situation in which there exists a prioritization of criteria. The proposed operator can capture the prioritization phenomenon among the aggregated IVIFVs in Ω instead of ones in the lattice of nonempty IVIFVs $\tilde{L}$. Meanwhile, it also has overcome the drawback of the existing IVIF prioritized aggregation operators. Some of its desirable properties have been investigated in detail. Finally, we have proposed a prioritized aggregation operator-based two-step procedure to the IVIF MCDM in the situation that more than one criteria exist at some priority level.

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