POINTWISE PSEUDO-METRIC ON THE \( L \)-REAL LINE

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ABSTRACT. In this paper, a pointwise pseudo-metric function on the \( L \)-real line is constructed. It is proved that the topology induced by this pointwise pseudo-metric is the usual topology.

1. Introduction

The \( L \)-fuzzy unit interval and the \( L \)-fuzzy real line are two important \( L \)-topological spaces. The \( L \)-fuzzy unit interval was defined by Hutton [2]. The \( L \)-fuzzy real line was respectively defined by Höhle [3] and Gantner et al. [4]. They are important not only in \( L \)-topology, but also in other fields.

To reflect the characteristics of pointwise \( L \)-topology, i.e., the relation between a fuzzy point and its Q-neighborhoods (or R-neighborhoods) [5], a theory of pointwise uniformities and a theory of pointwise metrics were introduced on completely distributive lattices and in \( L \)-fuzzy set theory (see [6, 7, 8, 9]). Many ideal results in general topology were generalized to \( L \)-topology. In [9], it was proved that the \( L \)-fuzzy real line is pointwise pseudo-metrizable, but no pointwise pseudo-metric function on the \( L \)-fuzzy real line was given. In this paper, our aim is to construct a pointwise pseudo-metric function in the \( L \)-real line and prove that the topology induced by this pointwise pseudo-metric function is the usual topology.

2. Preliminaries

Throughout this paper, \( L \) always denotes a completely distributive lattice with an order-reversing involution. \( M(\mathcal{L}^X) \) denotes the set of all non-zero \( \vee \)-irreducible elements in \( \mathcal{L}^X \). For \( A \in \mathcal{L}^X \), \( \beta(A) \) denotes the maximal minimal family of \( A \) (see [5]) and \( \beta^* (A) = \beta(A) \cap M(\mathcal{L}^X) \). It is easy to verify that for \( e \in M(\mathcal{L}) \), \( e \in \beta^* (A) \) if and only if \( a \ll A \), where \( \ll \) is the way below relation ([1]).

Definition 2.1 ([9]). A pointwise pseudo-quasi-metric on \( \mathcal{L}^X \) is a mapping \( d : M(\mathcal{L}^X) \times M(\mathcal{L}^X) \to [0, +\infty) \) satisfying the following (M1)–(M3):

\( (M1) \forall a \in M(\mathcal{L}^X), \ d(a, a) = 0. \)
\( (M2) \forall a, b, c \in M(\mathcal{L}^X), \ d(a, c) \leq d(a, b) + d(b, c). \)

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A pointwise pseudo-quasi-metric \( d \) is called a pointwise pseudo-metric if it satisfies the following conditions.

\[
\begin{align*}
(M3) & \quad \forall a, b \in M(L^X), \quad d(a, b) = \bigwedge_{c \leq b} d(a, c). \\
(M4) & \quad \forall a, b, c \in M(L^X), \quad a \leq b \text{ implies } d(a, c) \leq d(b, c). \\
(M5) & \quad \forall \lambda, \mu \in M(L^X), \quad \bigwedge_{a \leq \lambda} d(a, \mu) < r \text{ if and only if } \bigwedge_{b \geq \mu} d(b, \lambda) < r.
\end{align*}
\]

**Theorem 2.2** ([9]). Let \( d \) be a pointwise pseudo-metric on \( L^X \). \( \forall r \in (0, +\infty) \), define a mapping \( P_r : M(L^X) \to L^X \) by

\[
P_r(a) = \bigvee \{ b \in M(L^X) | d(a, b) \geq r \}.
\]

Then the family \( \{ P_r | r \in (0, +\infty) \} \) of R-nbd mappings of \( d \) satisfies the following conditions.

\[
\begin{align*}
(R1) & \quad \forall a \in M(L^X), \quad \bigwedge_{r > 0} P_r(a) = 0; \\
(R2) & \quad \forall a \in M(L^X), \forall r \in (0, +\infty), a \neq P_r(a); \\
(R3) & \quad \forall r, s \in (0, +\infty), P_r \circ P_r \geq P_{r+s}; \\
(R4) & \quad \forall a \in M, P_r(a) = \bigwedge_{s < r} P_s(a); \\
(R5) & \quad \forall r \in (0, +\infty), P_r \text{ is symmetric.}
\end{align*}
\]

**Theorem 2.3** ([9]). If \( \{ P_r | P_r : M(L^X) \to L^X, r \in (0, +\infty) \} \) is a family of mappings satisfying (R1)–(R5), and we define \( d : M(L^X) \times M(L^X) \to [0, +\infty) \) by

\[
d(a, b) = \bigwedge \{ r | b \leq P_r(a) \},
\]

then \( d \) is a pointwise pseudo-metric on \( L^X \) and the family of R-nbd mappings of \( d \) is exactly \( \{ P_r | r \in (0, +\infty) \} \).

**Theorem 2.4** ([9]). If \( d \) is a pointwise pseudo-quasi-metric on \( L^X \), then

1. \( \{ P_r(a) | a \in M(L^X), r \in (0, +\infty) \} \) is a base for a co-topology on \( L^X \). This co-topology is denoted by \( \eta_d \); \n
2. \( \{ P_r(a) | r > 0 \} \) is a locally R-neighborhood base at \( a \) in the co-topology \( \eta_d \).

**Definition 2.5** ([3, 4]). The \( L \)-fuzzy real line \( \mathbb{R}_L \) is defined as the set of all equivalence classes of antitone maps \( \lambda : \mathbb{R} \to L \) satisfying

\[
\bigvee_{t \in \mathbb{R}} \lambda(t) = 1 \text{ and } \bigwedge_{t \in \mathbb{R}} \lambda(t) = 0,
\]

where the equivalence identifies two maps \( \lambda \) and \( \mu \) if and only if \( \forall t \in I, \lambda(t+) = \mu(t+) \). The canonical \( L \)-topology on \( \mathbb{R}_L \) is generated from the subbase \( \{ \mathcal{L}_t, \mathcal{R}_t | t \in \mathbb{R} \} \), where

\[
\mathcal{L}_t : I(L) \to L \text{ by } \mathcal{L}_t(\lambda) = \lambda(t-); \\
\mathcal{R}_t : I(L) \to L \text{ by } \mathcal{R}_t(\lambda) = \lambda(t+).
\]
3. Pointwise Pseudo-metric on the $L$-real Line

**Lemma 3.1.** Let $\mathbb{R}(L)$ be the $L$-real line. Define a mapping $\varepsilon : M(L^{R(L)}) \to \mathbb{R}$ and a mapping $\sigma : M(L^{R(L)}) \to \mathbb{R}$ such that for all $e \in M(L^{R(L)})$,

$$
\varepsilon(e) = \sup \{ t \mid e \leq \mathcal{L}_t \}, \quad \sigma(e) = \inf \{ t \mid e \leq \mathcal{R}_t \}.
$$

Then we have the following results:

1. $\varepsilon(e) = \max \{ t \mid e \leq \mathcal{L}'_t \}$, \quad $\sigma(e) = \min \{ t \mid e \leq \mathcal{R}'_t \}$.
2. If $a, b \in M(L^{R(L)})$ and $a \leq b$, then $\varepsilon(a) \geq \varepsilon(b)$ and $\sigma(a) \leq \sigma(b)$.
3. If $b \in M(L^{R(L)})$, then $\varepsilon(b) = \bigwedge_{c \leq b} \varepsilon(c)$ and $\sigma(b) = \bigvee_{c \leq b} \sigma(c)$.
4. $\forall \lambda, \mu \in M(L^{R(L)})$, there exists $a \not\leq \lambda'$ such that $\varepsilon(\mu) < \varepsilon(a) + r$ if and only if there exists $b \not\leq \mu'$ such that $\sigma(\lambda) > \sigma(b) - r$.

**Proof.** (1) and (2) are obvious. By (2) we can obtain that $\varepsilon(b) \leq \bigwedge_{c \leq b} \varepsilon(c)$ and $\sigma(b) \geq \bigvee_{c \leq b} \sigma(c)$. Thus in order to prove (3) we need only to prove that

$$
\varepsilon(b) \geq \bigwedge_{c \leq b} \varepsilon(c) \quad \text{and} \quad \sigma(b) \leq \bigvee_{c \leq b} \sigma(c).
$$

Suppose that $\varepsilon(b) < \bigwedge_{c \leq b} \varepsilon(c)$. Then there exists $s \in \mathbb{R}$ such that

$$
\varepsilon(b) = \max \{ t \mid b \leq \mathcal{L}'_t \} < s < \bigwedge_{c \leq b} \varepsilon(c).
$$

This implies that $b \not\leq \mathcal{L}'_s$. Further there exists $c \ll b$ such that $c \not\leq \mathcal{L}'_s$. Thus we have that $\varepsilon(c) < s$. By $s < \bigwedge_{c \leq b} \varepsilon(c)$ we obtain a contradiction. Therefore $\varepsilon(b) \geq \bigwedge_{c \leq b} \varepsilon(c)$. Similarly we can prove that $\sigma(b) \leq \bigvee_{c \leq b} \sigma(c)$. Hence (3) follows.

To prove (4) suppose that $\varepsilon(\mu) < \varepsilon(a) + r$. Then there is $t > 0$ such that $\varepsilon(\mu) < \varepsilon(a) + r - t$. This implies that

$$
\mu \not\leq \mathcal{L}_{\varepsilon(a)+r-t} \quad \text{or} \quad \mathcal{L}_{\varepsilon(a)+r-t} \not\leq \mu'.
$$

So there exists a point $b \leq \mathcal{L}_{\varepsilon(a)+r-t}$ such that $b \not\leq \mu'$. We obtain

$$
\sigma(b) \leq \varepsilon(a) + r - t \quad \text{or} \quad \sigma(b) - r < \varepsilon(a)
$$

since $\mathcal{L}_{\varepsilon(a)+r-t} \leq \mathcal{R}'_{\varepsilon(a)+r-t}$. By $a \leq \mathcal{L}_{\varepsilon(a)}$ we have that

$$
\lambda \not\leq a' \geq \mathcal{L}_{\varepsilon(a)} \geq \mathcal{R}'_{\varepsilon(a)+r-t}.
$$

Therefore $\sigma(\lambda) > \sigma(b) - r$.

\[\square\]

**Theorem 3.2.** Let $\mathbb{R}(L)$ be the $L$-real line. For all $a, b \in M(L^{R(L)})$, define

$$
d_1(a, b) = \max \{ \varepsilon(b) - \varepsilon(a), 0 \}, \quad d_2(a, b) = \max \{ \sigma(a) - \sigma(b), 0 \},
$$

Then $d_1, d_2$ are pointwise pseudo-quasi-metrics, $\{ \mathcal{L}_t \mid t \in \mathbb{R} \}$ is the topology induced by $d_1$ and $\{ \mathcal{R}_t \mid t \in \mathbb{R} \}$ is the topology induced by $d_2$. 

\[\square\]
Proof. We only prove that $d_1$ is a pointwise pseudo-quasi-metric. The proof for $d_2$ is similar. Obviously, by (2) in Lemma 3.1 we know that $a \leq b \Rightarrow d_1(a, b) = 0$. Thus (M1) is true. (M2) can be obtained as follows.

\[
d_1(a, c) = \max \{ \varepsilon(c) - \varepsilon(a), 0 \}
\]

\[
= \max \{ \varepsilon(c) - \varepsilon(b) + \varepsilon(b) - \varepsilon(a), 0 \}
\]

\[
\leq \max \{ \varepsilon(b) - \varepsilon(a), 0 \} + \max \{ \varepsilon(c) - \varepsilon(b), 0 \}
\]

\[
= d_1(a, b) + d_1(b, c)
\]

(M3) can be obtained as follows:

\[
d_1(a, b) = \max \{ \varepsilon(b) - \varepsilon(a), 0 \}
\]

\[
= \max \{ \bigwedge_{c \leq b} (\varepsilon(c) - \varepsilon(a)), 0 \}
\]

\[
= \bigwedge_{c \leq b} \max \{ \varepsilon(c) - \varepsilon(a), 0 \} = \bigwedge_{c \leq b} d_1(a, c).
\]

In order to prove that \{ $L_{t}$ \mid $t \in \mathbb{R}$ \} is the topology induced by $d_1$ and \{ $R_{t}$ \mid $t \in \mathbb{R}$ \} is the topology induced by $d_2$, we only need to prove that the family \{ $P_{d_1}^{r}$ \mid $r > 0$ \} of R-nbd mappings of $d_1$ and the family \{ $P_{d_2}^{r}$ \mid $r > 0$ \} of R-nbd mappings of $d_2$ satisfy the following condition:

\[
P_{d_1}^{r}(a) = L'_{\varepsilon(a)+r} \quad \text{and} \quad P_{d_2}^{r}(a) = R'_{\sigma(a)-r}.
\]

In fact, $\forall a, b \in M(L^{R}(L))$, we have:

\[
b \leq P_{d_1}^{r}(a) \quad \Leftrightarrow \quad d_1(a, b) \geq r
\]

\[
\Leftrightarrow \quad \varepsilon(b) - \varepsilon(a) \geq r
\]

\[
\Leftrightarrow \quad \varepsilon(b) \geq \varepsilon(a) + r
\]

\[
\Leftrightarrow \quad b \leq L'_{\varepsilon(a)+r}
\]

and

\[
b \leq P_{d_2}^{r}(a) \quad \Leftrightarrow \quad d_2(a, b) \geq r
\]

\[
\Leftrightarrow \quad \sigma(a) - \sigma(b) \geq r
\]

\[
\Leftrightarrow \quad \sigma(b) \leq \sigma(a) - r
\]

\[
\Leftrightarrow \quad b \leq R'_{\sigma(a)-r}
\]

The result follows.

Remark 3.3. When $L = 2$, $d_1$ and $d_2$ are conjugate pseudo-quasi-metrics in the usual sense.

Theorem 3.4. Let $\mathbb{R}(L)$ be the L-real line. For all $a, b \in M(L^{R}(L))$, define

\[
d(a, b) = \max \{ \varepsilon(b) - \varepsilon(a), \sigma(a) - \sigma(b), 0 \} = \max \{ d_1(a, b), d_2(a, b) \}.
\]

Then $d$ is a pointwise pseudo-metric and $d$ exactly induces the topology on $\mathbb{R}(L)$. 

□
Thus (M1) is true. (M2) can be obtained as follows:

\[ d(a, c) = \max\{\varepsilon(c) - \varepsilon(a), \sigma(a) - \sigma(c), 0\} \]

\[ = \max\{\varepsilon(c) - \varepsilon(b) + \varepsilon(b) - \varepsilon(a), \sigma(a) - \sigma(b) + \sigma(b) - \sigma(c), 0\} \]

\[ \leq \max\{\varepsilon(b) - \varepsilon(a), \sigma(a) - \sigma(b), 0\} + \max\{\varepsilon(c) - \varepsilon(b), \sigma(b) - \sigma(c), 0\} \]

\[ = d(a, b) + d(b, c) \]

(M3) can be obtained as follows:

\[ d(a, b) = \max\{\varepsilon(b) - \varepsilon(a), \sigma(a) - \sigma(b), 0\} \]

\[ = \max\{\bigwedge_{c \in b} (\varepsilon(c) - \varepsilon(a), \sigma(a) - \varepsilon(c), 0)\} \quad \text{by Lemma 3.1} \]

\[ = \bigwedge_{c \in b} \max\{\varepsilon(c) - \varepsilon(a), \sigma(a) - \sigma(c), 0\} = \bigwedge_{c \in b} d(a, c) \]

(M4) can be obtained from (2) in Lemma 3.1.

To prove (M5), we note that \( \forall \lambda, \mu \in M(L^R(L)) \), if

\[ \bigwedge_{a \in \lambda'} d(a, \mu) = \bigwedge_{a \in \lambda'} \max\{\varepsilon(\mu) - \varepsilon(a), \sigma(\mu) - \sigma(a), 0\} < r, \]

then there exists \( a \not\in \lambda' \) such that

\[ \max\{\varepsilon(\mu) - \varepsilon(a), \sigma(\mu) - \sigma(a), 0\} < r, \]

i.e.,

\[ \varepsilon(\mu) - \varepsilon(a) < r, \ \sigma(\mu) - \sigma(a) < r. \]

Hence we have that

\[ \varepsilon(\mu) < \varepsilon(a) + r, \ \sigma(\mu) > \sigma(a) - r. \]

By (4) in Lemma 3.1 we know that there exist \( b \not\in \mu' \) and \( c \not\in \mu' \) such that

\[ \sigma(\lambda) > \sigma(b) - r, \ \varepsilon(\lambda) < \varepsilon(c) + r. \]

Thus, since \( \mu' \) is a prime element, \( b \land c \not\in \mu' \). Take a point \( d \leq b \land c \) such that \( d \not\in \mu' \). Then

\[ \sigma(\lambda) > \sigma(b) - r \geq \sigma(d) - r, \ \varepsilon(\lambda) < \varepsilon(c) + r \leq \varepsilon(d) + r. \]

This implies that

\[ \bigwedge_{d \in \mu'} d(d, \lambda) = \bigwedge_{d \in \mu'} \max\{\varepsilon(\lambda) - \varepsilon(d), \sigma(\lambda) - \sigma(d), 0\} < r. \]

In order to prove that \( \{L_t, R_t \mid t \in R\} \) is a subbase of the topology induced by \( d \), we only need to prove that the family \( \{P_r^d \mid r > 0\} \) of R-nbd mappings of \( d \) satisfies the following condition:

\[ P_r^d(a) = L_{\varepsilon(a) + r} \lor R_{\sigma(a) - r}. \]
In fact, \( \forall a, b \in M(L^{R(L)}) \) we have:

\[
\begin{align*}
  b \leq P_{\epsilon}(a) & \iff d(a, b) \geq r \\
                              & \iff \varepsilon(b) - \varepsilon(a) \geq r \quad \text{or} \quad \sigma(a) - \sigma(b) \geq r \\
                              & \iff \varepsilon(b) \geq \varepsilon(a) + r \quad \text{or} \quad \sigma(b) \leq \sigma(a) - r \\
                              & \iff b \leq L_{\varepsilon(a) + r} \lor R_{\sigma(a) - r}
\end{align*}
\]

The result follows. \( \square \)

**Remark 3.5.** When \( L = 2 \), the pointwise pseudo-metric \( d \) in Theorem 3.4 can be regarded as the usual pseudo-metric defined by \( d(a, b) = |a - b| \).

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**References**


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