SOME QUOTIENTS ON A BCK-ALGEBRA GENERATED BY A FUZZY SET

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Abstract. First we show that the cosets of a fuzzy ideal μ in a BCK-algebra X form another BCK-algebra $\frac{X}{\mu}$ (called the fuzzy quotient BCK-algebra of X by μ). Also we show that $\frac{X}{\mu}$ is a fuzzy partition of X and we prove several some isomorphism theorems. Moreover we prove that if the associated fuzzy similarity relation of a fuzzy partition P of a commutative BCK-algebra is compatible, then P is a fuzzy quotient BCK-algebra. Finally we define the notion of a coset of a fuzzy ideal and an element of a BCK-algebra and prove related theorems.

1. Introduction

In 1966, the notion of a BCK-algebra was introduced by Y. Imai and K. Iseki [4]. Zadeh in 1965 [13] introduced the notion of fuzzy subset of a nonempty set A as a function from A to [0,1]. Ougen Xi extended these ideas to BCK-algebra [11]. In this paper the notions of fuzzy quotient BCK-algebra induced by fuzzy ideals, and the concept of a quotient algebra of a BCK-algebra, generated by a fuzzy ideal and an element are defined and then related theorems are proved.

2. Preliminaries

Definition 2.1. [4, 7] (a) A BCK-algebra is a nonempty set X with a binary operation "*" and a constant 0 satisfying the following axioms:
(i) $((x*y)*(x*z))*(z*y)=0$
(ii) $(x*(x*y))*y=0$
(iii) $x*x=0$
(iv) $x*y=0$ and $y*x=0$ imply that $x=y$
(v) $0*x=0$, forall $x,y,z\in X$ ·
(b) A nonempty set A of a BCK-algebra is said to be an ideal of X if the following conditions hold:
(i) $0\in A$
(ii) $x\in X$, $y*x\in A$ imply that $y\in A$, forall $y\in X$
(c) A BCK-algebra X is said to be commutative if $x*(x*y)=y*(y*x)$, for all $x,y\in X$. $x*(x*y)$ is denoted by $x\wedge y$

Received: November 2003; Accepted: April 2004
Key words and Phrases: Fuzzy similarity relations, Fuzzy partitions, Fuzzy quotient, Fuzzy ideal, Cosets, Quotient algebra.

Lemma 2.2. [8] *Let X be a BCK-algebra. Then,*

(i) $x * 0 = x$, $\forall x \in X$

(ii) $[(y_1 * x) * (y_2 * x)] * (y_1 * y_2) = 0$, $\forall x, y_1, y_2 \in X$

(iii) $(x * y) * z = (x * z) * y$, $\forall x, y, z \in X$

(iv) $(x * y) * x = 0$ $\forall x, y \in X$

(v) $(x \wedge y) * x = (x \wedge y) * y$, $\forall x, y \in X$

Definition 2.3. [9, 13] (i) For $r \in [0, 1]$ fuzzy point x_r is defined to be fuzzy subset of X such that

$$x_r(y) = \begin{cases} r & \text{if} \quad y = x \\ 0 & \text{if} \quad y \neq x \end{cases}$$

(ii) If μ, η are two fuzzy subsets of X. Then

$$\mu \subseteq \eta \Leftrightarrow \mu(x) \leq \eta(x) \ , \ \forall x \in X$$

Definition 2.4. [11] A fuzzy sunset μ of a BCK-algebra X is a fuzzy ideal if it satisfies

(i) $\mu(0) = 1$, $\forall x \in X$

(ii) $\mu(x) \geq \min\{\mu(x * y), \mu(y)\}$, $\forall x, y \in X$

Lemma 2.5. [3] *Let X be a BCK-algebra and μ a fuzzy ideal of X. Then*

(i) $\mu(x * y) \geq \min\{\mu(x * z), \mu(y)(z * y)\}$, $\forall x, y, z \in X$

(ii) *if $x * y = 0$ then $\mu(x) \geq \mu(y)$* , $\forall x, y \in X$ $\cdot$

Definition 2.6. Let μ be a fuzzy subset of X and $\alpha \in [0, 1]$. Then by a level subset μ_α of μ we mean the set $\{x \in X : \mu(x) \geq \alpha\}$.

Definition 2.7. Let X and Y be two sets, and f a function of X into Y. Let μ and η be fuzzy subsets of X and Y, respectively. Then $f(\mu)$ the image of μ under f, is a fuzzy subset of Y:

$$f(\mu)(y) = \begin{cases} \sup\limits_{f(x)=y} \mu(x) & \text{if} \quad f^{-1}(y) \neq \emptyset \\ 0 & \text{if} \quad f^{-1}(y) = \emptyset \ , \end{cases}$$

for all $y \in Y$, $f^{-1}(\eta)$ the pre-image of η under f, is a fuzzy subset of X such that

$$f^{-1}(\eta)(x) = \eta(f(x)) \ , \quad \forall x \in X \ \cdot$$

Lemma 2.8. [11] (i) *Let μ be a fuzzy ideal of BCK-algebra X. For all $\alpha \in [0, 1]$, if $\mu_\alpha \neq \Phi$, then μ_α is an ideal of X.*
(ii) *Let $f : X \to X'$ be an epimorphism of BCK-algebra and μ' a fuzzy ideal of X'. Then $f^{-1}(\mu')$ is a fuzzy ideal of X.*

Definition 2.9. [10] Let X be a nonempty set and R a fuzzy subset of $X \times X$. Then R is called a fuzzy similarity relation on X if

(i) $R(x, x) = 1$, $\forall x \in X$

(ii) $R(x, y) = R(y, x)$

(iii) $R(x, z) \geq \min\{R(x, y), R(y, z)\}$ $\cdot$

Definition 2.10. [2, 10, 12] A fuzzy partition of a set X is a subset P of $[0,1]^X$ whose members satisfy the following conditions:
(i) Every $N \in P$ is normalized; i.e.$N(x) = 1$, for at least one $N \in X$;
(ii) For each $x \in X$, there is exactly one $N \in P$ with $N(x) = 1$;
(iii) If $M, N \in P$ and, $x, y \in X$ are such that $M(x) = N(y) = 1$, Then

$$M(y) = N(x) = \sup\{\min\{M(z), N(z)\} : z \in X\} \cdot$$

Given a fuzzy partition P of X and element $x \in X$, we denote the unique element of P with value 1 at x by $[x]_p$. It is called the fuzzy similarity class of x.

Lemma 2.11. [10, 12] *A canonical one-to-one correspondence between fuzzy partition and fuzzy similarity relations is defined by sending a fuzzy partition P of X to its fuzzy similarity relation $R_P \in [0,1]^{X\times X}$, where for all $x, y \in X$, we have $R_P(x, y) = [x]_P(y)$.*

The inverse correspondence is defined by sending a fuzzy similarity relation R on X to its fuzzy partition $P_R \subseteq [0,1]^X$ given by $P_R = \{R\langle x\rangle : x \in X\}$, where $R\langle x\rangle$ is the fuzzy subset of X defined for all $y \in X$ by $R\langle x\rangle(y) = R(x, y)$.

Lemma 2.12. [10] *Let R be a fuzzy similarity relation on X, and $a, b \in X$. Then*

$$R\langle a\rangle = R\langle b\rangle \Leftrightarrow R(a, b) = 1 \cdot$$

Definition 2.13. Let X and X' be general sets, $f : X \to X'$ a function, and μ a fuzzy subset of X, If $f(x) = f(y)$ implies that $\mu(x) = \mu(y)$, then μ is called f-invariant.

Theorem 2.14. [5] *Let A be an ideal of X. The relation $\sim_A$ on X is defined by*

$$x \sim_A y \Leftrightarrow x * y \in A \ , \ y * x \in A \cdot$$

$i)$ The relation $\sim_A$ is an equivalence relation.
$ii)$ Let C_x be the equivalence class of x and $\frac{X}{A} = \{C_x : x \in X\}$.
Then $(\frac{X}{A}, o, C_o)$, is a BCK-algebra where $C_x o c_y = C_{x*y}, \forall x, y \in X$.

Definition 2.15. [8] A BCK-algebra X is called bounded if there is an element 1 of X such that $x * 1 = 0$ for all $x \in X$.

Lemma 2.16. [8] *Let X be a bounded and commutative BCK-algebra then*
(i) $(x \wedge y) \wedge z = x \wedge (y \wedge z)$ *for all* $x, y, z \in X$
(ii) $x \wedge 1 = 1 \wedge x = x$

Definition 2.17. [1] A fuzzy ideal μ of a BCK-algebra X is said to be prime if:

$$\mu(x \wedge y) = \mu(x) \text{ or } \mu(x \wedge y) = \mu(y) \ , \text{ for all } x, y \in X \cdot$$

3. **Fuzzy cosets**

From now on, X is a BCK-algebra and μ is a fuzzy ideal of X.

Definition 3.1. Let $x \in X$. Then the fuzzy subset μ_x which is defined by

$$\mu_x(y) = \min\{\mu(x * y), \mu(y * x)\}$$

is called a fuzzy coset of μ. The set of all fuzzy cosets of μ is denoted by $\dfrac{X}{\mu}$.

Lemma 3.2. *Let $\overline{\mu}$ be a fuzzy relation on X which is defined by*

$$\overline{\mu}(x, y) = \mu_x(y) \quad , \quad \forall x, y \in X \cdot$$

Then $\overline{\mu}$ is a fuzzy similarity relation on X.

Proof. Clearly the conditions (i) and (ii) of Definition 2.9 hold. Now by Lemma 2.5 (i), for all $x, y, z \in X$,

$$\mu(x * z) \geq \min\{\mu(x * y), \mu(y * z)\} \ , \ \mu(z * x) \geq \min\{\mu(z * y), \mu(y * x)\}$$

Therefore the condition (iii) of Definition 2.9 holds. □

Remark 3.3. Clearly $\overline{\mu}\langle x\rangle = \mu_x, \forall x \in X$.

Lemma 3.4. *Let $x, y_1, y_2 \in X$ and $\mu_{y_1} = \mu_{y_2}$. Then*

$$\mu_{x*y_1} = \mu_{x*y_2} \quad , \quad \mu_{y_1*x} = \mu_{y_2*x}$$

Proof. Since $\mu_{y_1} = \mu_{y_2}$, then by Lemma 2.12, we get that $\mu(y_1*y_2) = \mu(y_2*y_1) = 1$. On the other hand, from Definition 2.1 (a) (i) and Lemma 2.5 (ii) we obtain that:

$$\mu((x * y_1) * (x * y_2)) \geq \mu(y_2 * y_1) \cdot$$

Thus $\mu((x * y_1) * (x * y_2)) = 1$. Similarly $\mu((x * y_2) * (x * y_1)) = 1$.
Consequently $\overline{\mu}(x*y_1, x*y_2) = 1$ and hence by Remark 3.3 and Lemma 2.12 we have $\mu_{x*y_1} = \mu_{x*y_2}$. Similarly, by Lemma 2.2 (ii) we can show that $\mu_{y_1*x} = \mu_{y_2*x}$. □

Lemma 3.5. *Let $x, y, x', y' \in X, \mu_x = \mu_{x'}$ and $\mu_y = \mu_{y'}$. Then $\mu_{x*y} = \mu_{x'*y'}$.*

Proof. By Lemma 3.4 $\mu_{x*y} = \mu_{x'*y}$ and $\mu_{x'*y} = \mu_{x'*y'}$. Therefore $\mu_{x*y} = \mu_{x'*y'}$. □

Theorem 3.6. $(\dfrac{X}{\mu}, O, \mu_0)$ *is a BCK-algebra where*

$$O : \frac{X}{\mu} \times \frac{X}{\mu} \to \frac{X}{\mu}$$
$$(\mu_x, \mu_y) \mapsto \mu_{x*y} \cdot$$

Proof. The proof follows from Lemma 3.5. □

Theorem 3.7. $\dfrac{X}{\mu}$ *is a fuzzy partition of X.*

Proof. The proof follows from Lemmas 3.2 and 2.11. □

Theorem 3.8. *There exists an ideal K of $\dfrac{X}{\mu}$ such that*

$$\frac{(\frac{X}{\mu})}{K} \simeq \frac{X}{\mu_\alpha}$$

for all $\alpha \in [0, 1]$.

Proof. Let $\alpha \in [0,1]$. By Lemma 2.8 (i), μ_α is an ideal of X. Define $\varphi : \frac{X}{\mu} \to \frac{X}{\mu_\alpha}$ by $\varphi(\mu_x) = C_x$ for all $x \in X$. If $\mu_x = \mu_y$, then by Lemma 2.12 $\overline{\mu}(x,y) = 1$ and hence $\mu(x*y) = \mu(y*x) = 1 \geq \alpha$, which implies that $x*y \in \mu_\alpha$ and $y*x \in \mu_\alpha$. hence $C_x = C_y$. Thus φ is well-defined. Clearly φ is an epimorphism. Now let $K = Ker\varphi$. The theorem is proved. □

Definition 3.9. By μ^*, we mean the set $\{x \in X : \mu(x) = 1\}$. Clearly μ^* is an ideal of X.

Theorem 3.10. $\frac{X}{\mu} \simeq \frac{X}{\mu^*}$.

Proof. It is enough to show that the epimorphism φ, defined in the proof of theorem 3.8, is one-to-one. To do this, Let $C_x, C_y \in \frac{X}{\mu^*}$ be such that $C_x = C_y$ for $x, y \in X$. Then $x*y \in \mu^*$ and $y*x \in \mu^*$. In other words, $\mu(x*y) = \mu(y*x) = 1$ and hence by Remark 3.3 and Lemmas 2.12 and 3.2, $\mu_x = \mu_y$. □

Theorem 3.11. *Let f be a BCK-homomorphism from X onto X' and μ an f-invariant fuzzy ideal of X such that $\mu^* \subseteq Kerf$. Then $\frac{X}{\mu} \simeq X'$.*

Proof. Define $g : \frac{X}{\mu} \to X'$ by $g(\mu_x) = f(x)$. By Lemmas 2.12 and 3.2, we have for all $x, x' \in X$

$$\mu_x = \mu_{x'} \Rightarrow x*x', x'*x \in \mu^* \Rightarrow x*x', x'*x \in Kerf \Rightarrow f(x) = f(x')$$

Therefore g is well-defined. Clearly g is an epimorphism.
Now let $\mu_x \in Kerg$. Then $f(x) = f(0) = 0$. Since μ is f-invariant, hence $\mu(x) = \mu(0)$. From Definition 2.1 (a) (v) and Lemma 2.2 (i) we obtain that $\mu(x*0) = \mu(0*x) = \mu(0) = 1$.
Hence, $\overline{\mu}(x,0) = 1$, which implies that $\mu_x = \mu_0$, by Lemma 2.12. Thus $Kerg = \{\mu_0\}$, and hence g is one-to-one. □

Theorem 3.12. *Let f be a BCK-homomorphism from X onto X' and $\mu^* = Kerf$. Then*

$$\frac{X}{\mu} \simeq X' \cdot$$

Proof. Since $\frac{X}{Kerf} \simeq X'$, we conclude that $\frac{X}{\mu^*} \simeq X'$. Also by theorem 3.10 $\frac{X}{\mu} \simeq \frac{X}{\mu^*}$. Thus $\frac{X}{\mu} \cong X'$. □

Lemma 3.13. *Let $\prod_\mu : X \to \frac{X}{\mu}$ be a function defined by $\prod_\mu(x) = \mu_x$. Then*
(i) $0\prod_\mu$ *is an epimorphism*
(ii) *if $\mu = \chi_{\{0\}}$, then $\prod_\mu$ is an isomorphism. in other words,*

$$X \simeq \frac{X}{\mu}$$

Proof. (i) The proof is easy.
(ii) If $\mu_x = \mu_y$, for $x, y \in X$, then $\mu(x * y) = \mu(y * x) = 1$. Thus $x * y = y * x = 0$. Hence $x = y$, therefore $\prod_\mu$ is one-to-one. $\square$

Theorem 3.14. *Let f be a BCK-homomorphism from X into X', μ a fuzzy ideal of X and μ' a fuzzy ideal of X' such $f(\mu) \subseteq \mu'$. Then there is a homomorphism of BCK-algebras $f^* : \dfrac{X}{\mu} \to \dfrac{X'}{\mu'}$ such that $f^* \prod_\mu = \prod_{\mu'} f$. In another words, the following diagram is commutative.*

$$\begin{array}{ccc} X & \xrightarrow{f} & X' \\ \downarrow \prod_\mu & & \downarrow \prod_{\mu'} \\ \frac{X}{\mu} & \overrightarrow{f^*} & \frac{X'}{\mu'} \end{array}$$

Proof. Define $f^* : \dfrac{X}{\mu} \to \dfrac{X'}{\mu'}$ by $f^*(\mu_x) = \mu'_{f(x)}$. At first we show that f^* is well-defined. To do this let $\mu_{x_1} = \mu_{x_2}$. Then by Lemma 2.12 $\overline{\mu}(x_1, x_2) = 1$, and hence $\mu(x_1 * x_2) = \mu(x_2 * x_1) = 1$. Now we have

$$\begin{aligned} \mu'(f(x_1) * f(x_2)) &= \mu'(f(x_1 * x_2)) \\ &= f^{-1}(\mu')(x_1 * x_2) \\ &\geq \mu(x_1 * x_2) \ , \text{ since } f(\mu) \subseteq \mu' \\ &= 1 \cdot \end{aligned}$$

Similarly $\mu'(f(x_2) * f(x_1)) = 1$, thus $\mu'_{f(x_1)} = \mu'_{f(x_2)}$ by Lemma 2.12. It is easily seen that f^* is a homomorphism and $f^* \prod_\mu = \prod_{\mu'} f$. $\square$

Theorem 3.15. *(Isomorphism theorem) Let $f : X \to X'$ be an epimorphism of BCK-algebras, and μ' a fuzzy ideal of X'. Then*

$$\frac{X}{f^{-1}(\mu')} \simeq \frac{X'}{\mu'} \ .$$

Proof. By Lemma 2.8 (ii), $\mu = f^{-1}(\mu')$ is a fuzzy ideal of X. Since f is onto, then

$$f(\mu) = f(f^{-1}(\mu')) = \mu' \ \cdot$$

By Theorem 3.14, the mapping f^* is a homomorphism. Clearly f^* is onto. To show that f^* is one-to-one, suppose that $\mu_a \in Kerf^*$, for $a \in X$ then we have $\mu'_0 = f^*(\mu_a) = \mu'_{f(a)}$ it follows that $\mu'(f(a) * 0) = 1$. In other words $\mu'(f(a)) = 1$. Hence $\mu(a * 0) = \mu(a) = (f^{-1}(\mu'))(a) = \mu'(f(a)) = 1$. On the other hand $1 = \mu(0) = \mu(0 * a)$. Consequently $\mu_a = \mu_0$. This completes the proof. $\square$

Corollary 3.16. *(Homomorphism Theorem). Let $f : X \to X'$ be an epimorphism of BCK-algebras. Then $\dfrac{X}{f^{-1}(\chi_{\{0\}})} \simeq X'$.*

Proof. The proof follows from Theorem 3.15 and Lemma 3.13 (ii). $\square$

Definition 3.17. A fuzzy similarity relation R on X is said to be compatible if for each $x, y, z \in X$ we have:

$$R(x * z, y * z) \geq R(x, y) \text{ and } R(z * x, z * y) \geq R(x, y)$$

Theorem 3.18. *Let R be a compatible fuzzy similarity on X. Then $R\langle 0\rangle$ is a fuzzy ideal of X.*

Proof. Clearly $R\langle 0\rangle(0) = 1$. Now let $x, y \in X$ we have

$$\begin{aligned} R\langle 0\rangle(x) &= R(0, x) \geq \min\{R(0, x * y), R(x * y, x)\} \text{ , by Definition 2.9} \cdot (iii) \\ &= \min\{R(0, x * y), R(x * y, x * 0)\} \quad \text{by Lemma 2.2}(i) \\ &\geq \min\{R(0, x * y), R(y, 0)\} \quad \text{by Definition 3.17} \\ &= \min\{R\langle 0\rangle(x * y), R\langle 0\rangle(y)\} \cdot \end{aligned}$$

□

Theorem 3.19. *Let X be a commulative BCK-algebra, P a fuzzy partition of X such that its fuzzy similarity R_P (see Lemma 2.11) is compatible. Then $\dfrac{X}{R_P\langle 0\rangle} = P$.*

Proof. For simplicity of notation, we will denote $R_P\langle 0\rangle$ by η. At first we show that $P \subseteq \dfrac{X}{\eta}$. To do this, let $M \in P$. Then by Definition 2.10 (i) there exists $x \in X$ such that $M(x) = 1$. On the other hand, for all $y \in X$, $[y]_P(y) = 1$. Thus by Definition 2.1- (iii) and 3.17 we have:

$$M(y) = [y]_P(x) = R_P(x, y) = R_P(y, x) \leq R_P(y * y, x * y) = R_P(0, x * y) \text{ ,}$$

and also

$$M(y) = R_p(x, y) \leq R_p(x * x, y * x) = R_P(0, y * x) \cdot$$

Therefore

(1) $$M(y) \leq \eta_x(y) \quad , \quad \forall y \in X \cdot$$

On the other hand we obtain that

$$\eta_x(y) \leq R_P(0, x * y) \leq R_P(x * 0, x * (x * y)) = R_P(x, x * (x * y))$$

and

$$\eta_x(y) \leq R_P(0, y * x) \leq R_P(y * 0, y * (y * x)) = R_P(y, y * (y * x)) \cdot$$

Since X is commutative, it follows that

$$\eta_x(y) \leq \min\{R_P(x, x \wedge y), R_P(x \wedge y, y)\} \leq R_P(x, y)$$

Hence

(2) $$\eta_x(y) \leq R_P(x, y) = [y]_P(x) = M(y) \text{ , } \forall y \in X \cdot$$

From (1) and (2) we obtain that

$$M = \eta_x \quad , \quad \exists x \in X \cdot$$

Thus

(3) $$P \subseteq \frac{X}{\eta} \cdot$$

Now let $\eta_x \in \dfrac{X}{\eta}$. Then by Definition 2.10 (ii), there exists $N \in P$ such that $N(x) = 1$. As we have proved, $N = \eta_x$, which implies that

$$\frac{X}{\eta} \subseteq P \cdot \tag{4}$$

Now the proof follows from (3) and (4). □

4. Cosets of a BCK-algebra generated by a fuzzy ideal and an element

Definition 4.1. Let $a \in X$. We define the relation "$\sim_a$" on X as follows.

$$x \sim_a y \Leftrightarrow \mu(x * y) \geq \mu(a), \mu(y * x) \geq \mu(a) \quad \text{for all } x, y \in X$$

Theorem 4.2. *$x \sim_a y$ is an equivalence relation on X.*

Proof. By Definition 2.1 and 2.4 (i), "$\sim_a$" is reflexive and clearly "$\sim_a$" is symmetric. Now we prove that "$\sim_a$" is transitive. To do this let $x, y, z \in X$, $x \sim_a y$ and $y \sim_a z$. Then we have

$$\begin{aligned} \mu((x * z) * (x * y)) &\geq \min\{\mu(((x * z) * (x * y)) * (y * z)), \mu(y * z)\} \\ &\qquad \text{by Definition } 2.4(ii) \\ &= \min\{\mu(0), \mu(y * z)\}\ , \quad \text{by Definition } 2.1(i) \\ &= \mu(y * z)\ , \quad \text{by Definition } 2.1(i) \\ &\geq \mu(a)\ , \quad \text{since } y \sim_a z\ \cdot . \end{aligned}$$

Hence:

$$\begin{aligned} \mu(x * z) &\geq \min\{\mu((x * z) * (x * y)), \mu(x * y)\} \\ &= \min\{\mu(0), \mu(a)\}\ , \quad \text{since } x \sim_a y \end{aligned}$$

Therefor

$$\mu(x * z) \geq \mu(a) \cdot$$

Similarly by Lemma 2.2 (ii), we can show that

$$\mu(z * x) \geq \mu(a) \cdot$$

Hence $x \sim_a z$. □

Definition 4.3. Let $a \in X$. For $x \in X$, the equivalence class of x with respect to "$\sim_a$" is denoted by $C_x(a, \mu)$ and it is called the coset of x in X and generated by a and μ.

Remark 4.4. The set of all cosets generated by a and μ is denoted by $C_X(a, \mu)$.

Corollary 4.5. *$C_X(a, \mu)$ is a partition for X.*

Proposition 4.6. *Let $a, x \in X$. Then $a \in C_x(a, \mu)$ if and only if*

$$C_x(a, \mu) = C_0(a, \mu) \cdot$$

Proof. Let $C_x(a,\mu) = C_0(a,\mu)$. By Definition 2.1 (iv) and Lemma 2.2. (i) it follows that $a \in C_0(a,\mu)$. Hence $a \in C_x(a,\mu)$. Conversely, let $a \in C_x(a,\mu)$. Then we have

$$\mu(x*0) = \mu(x) \geq \min\{\mu(x*a), \mu(a)\} \geq \mu(a) \cdot$$

On the other hand

$$\mu(0*x) = \mu(0) \geq \mu(a) \cdot$$

Hence $0 \sim_a x$, therefore $C_0(a,\mu) = C_x(a,\mu)$. □

Proposition 4.7. *For all $x, y, a \in X$, $x*y, y*x \in C_0(a,\mu)$ if and only if*

$$C_x(a,\mu) = C_y(a,\mu) \cdot$$

Proof. The proof follows from Definition 2.1 (iii) and Lemma 2.2. (i). □

Proposition 4.8. *$y \in C_x(0,\mu)$ implies that $\mu(x) = \mu(y)$.*

Proof. The proof follows from Lemma 2.5 (ii), and Definition 2.4 (ii). □

Proposition 4.9. *For $a \in X$, $C_0(a,\mu)$ is a subalgebra and an ideal of X.*

Proof. Let $x, y \in C_0(a,\mu)$. Then by Lemma 2.2 (iv) and 2.5 (ii) we have $\mu(x*y) \geq \mu(x) \geq \mu(a)$ and $\mu(y*x) \geq \mu(y) \geq \mu(a)$. Since

$$\mu((x*y)*0) = \mu(x*y), \mu((y*x)*0) = \mu(y*x) ,$$

hence $x*y \in C_0(a,\mu)$. Therefore $C_0(a,\mu)$ is a subalgebra of X. From Definition 2.5 and some calculation we get that, $C_0(a,\mu)$ is an ideal of X. □

Lemma 4.10. *For all $x, a, b \in X$,*
(i) $C_x(a \wedge b, \mu) \subseteq C_x(a,\mu) \cap C_x(b,\mu)$,
(ii) if μ is a fuzzy prime ideal of X, then $C_x(a \wedge b, \mu) = C_x(a,\mu) \cap C_x(b,\mu)$.

Proof. (i) From Lemma 2.2. (v) and 2.5 (ii) we can prove (i).
(ii) follows from Definition 2.17. □

Theorem 4.11. *Let X be a bounded, commutative BCK-algebra, μ a fuzzy prime ideal of X, $x \in X$, and $C_x(X,\mu) = \{C_x(a,\mu) : a \in X\}$. Define the operation "." On $C_x(X,\mu)$ as follows: $C_x(a,\mu) \cdot C_x(b,\mu) = C_x(a \wedge b, \mu)$. Then $(C_x(X,\mu), \cdot)$ is a monoid.*

Proof. The proof follows from Lemmas 4.10 (ii) and 2.16 (ii). □

Lemma 4.12. *Let $a, y_1, y_2 \in X$ and $y_1 \sim_a y_2$. Then*

$$x*y_1 \sim_a x*y_2 \quad \textit{and} \quad y_1*x \sim_a y_2*x \ , \ \textit{for all } x \in X \cdot$$

Proof. Since $y_1 \sim_a y_2$, we have $\mu(y_1*y_2) \geq \mu(a)$ and $\mu(y_2*y_1) \geq \mu(a)$. On the other hand by Definition 2.1 (i) we get that

$$((x*y_1)*(x*y_2))*(y_2*y_1) = 0 \cdot$$

Hence from Definition 2.4.

$$\begin{aligned}\mu((x*y_1)*(x*y_2)) &\geq \min\{\mu[((x*y_1)*(x*y_2))*(y_2*y_1)], \mu(y_2*y_1)\} \\ &\geq \min\{\mu(0), \mu(a)\} \ , \ \text{since } y_1 \sim_a y_2 = \mu(a) \\ &= \mu(a) \cdot\end{aligned}$$

Similarly we have $\mu((x * y_2) * (x * y_1)) \geq \mu(a)$. Consequently

$$x * y_1 \sim_a x * y_2$$

Similarly from Lemma 2.2 (ii) we get that $y_1 * x \sim_a y_2 * x$. □

Lemma 4.13. *Let $a \in X$, We define $\oplus : C_X(a,\mu) \times C_X(a,\mu) \to C_X(a,\mu)$ as follows*

$$C_x(a,\mu) \oplus C_y(a,\mu) = C_{x*y}(a,\mu) \quad , \quad \forall x, y \in X \cdot$$

Then $\oplus$ is an operation on $C_X(a,\mu)$.

Proof. Let $C_{x_1}(a,\mu) = C_{x_2}(a,\mu)$ and $C_{y_1}(a,\mu) = C_{y_2}(a,\mu)$. Then $x_1 \sim_a x_2$ and $y_1 \sim_a y_2$. Hence by Lemma 4.12 we have $x_1 * y_1 \sim_a x_2 * y_2$. ALso $y_1 \sim_a y_2$ implies that $x_1 * y_1 \sim_a x_1 * y_2$. Therefore from Theorem 4.2 we have $x_1 * y_1 \sim_a x_2 * y_2$. In other words $C_{x_1 * y_1}(a,\mu) = C_{x_2 * y_2}(a,\mu)$. □

Theorem 4.14. *Let $a \in X$. Then $(C_X(a,\mu), C_0(a,\mu), \oplus)$ is a BCK-algebra called the quotient algebra generated by μ and a.*

Proof. Clearly the axioms $(i), (ii), (iii), (v)$ of Definition 2.1 hold. Now let

$$C_x(a,\mu) \oplus C_y(a,\mu) = C_y(a,\mu) \oplus C_x(a,\mu) = C_0(a,\mu)$$

Then

$$C_{x*y}(a,\mu) = C_0(a,\mu) = C_{y*x}(a,\mu) \cdot$$

Hence by proposition 4.7 we have:

$$C_x(a,\mu) = C_y(a,\mu) \cdot$$

□

Theorem 4.15. *Let $f : X \to X'$ be an epimorphism of BCK-algebras and μ be f-invariant and $\mu_{\mu(a)} \subseteq Kerf$, for $a \in X$. Then $C_x(a,\mu) \simeq X'$.*

Proof. Define $\psi : C_X(a,\mu) \to X'$ by, $\psi(C_x(a,\mu)) = f(x)$, for all $x \in X$.
Let $C_x(a,\mu) = C_y(a,\mu)$, where $x, y \in X$. Then $x * y, y * x \in \mu_{\mu(a)} \subseteq Kerf$.
Hence $f(x) = f(y)$. In other words ψ is well defined. Clearly ψ is an epimorphism.
Now let $f(x) = f(y)$, for $x, y \in X$. Then $f(x * y) = f(y * x) = f(0)$.
Since μ is f-invariant, we have

$$\mu(x * y) = \mu(0) \geq \mu(a) \ , \ \mu(y * x) = \mu(0) \geq \mu(a) \cdot$$

Therefore $C_x(a,\mu) = C_y(a,\mu)$, which implies that ψ is one-to-one. □

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