FUZZY (POSITIVE, WEAK) IMPLICATIVE HYPER BCK-IDEALS

M. BAKHSHI, M. M. ZAHEDI AND R. A. BORZOOEI

Abstract. In this note first we define the notions of fuzzy positive implicative hyper BCK-ideals of types 1,2,3 and 4. Then we prove some theorems which characterize the above notions according to the level subsets. Also we obtain the relationships among these notions, fuzzy (strong, weak, reflexive) hyper BCK-ideals and fuzzy positive implicative hyper BCK-ideals of types 5,6,7 and 8. Then, we define the notions of fuzzy (weak) implicative hyper BCK-ideals and we obtain some related results. Finally, by considering the product of two hyper BCK-algebras we give some theorems which show that how the projections of a fuzzy (positive implicative, implicative) hyper BCK-ideal is again a fuzzy (positive implicative, implicative) hyper BCK-ideal.

1. Introduction

The study of BCK-algebras was initiated by Y. Imai and K. Iséki [6] in 1966 as a generalization of the concept of set-theoretic difference and propositional calculi. Since then a great deal of literature has been produced on the theory of BCK-algebras. In particular, emphasis seems to have been put on the ideal theory of BCK-algebras. The hyperstructure theory (called also multialgebras)was introduced in 1934 by F. Marty [13] at the 8th congress of Scandinavian Mathematiciens. Around the 40's, several authors worked on hypergroups, especially in France and in the United States, but also in Italy, Russia and Japan. Hyperstructures have many applications to several sectors of both pure and applied sciences. In [9] Jun, Zahedi, Borzooei et al. applied the hyperstructures to BCK-algebras, and introduced the notion of hyper BCK-algebra which is a generalization of BCK-algebra, and investigated some related properties. They also introduced the notion of hyper BCK-ideal and weak hyper BCK-ideal. In [1], Borzooei and Bakhshi introduced the notions of positive implicative hyper BCK-ideals of types 1, 2 ,..., 8 and obtained some relations between these notions and (weak, strong) hyper BCK-ideals. In [11], Jun et al. introduced the notion of implicative hyper BCK-ideal and gave some relations between this notion and hyper BCK-ideal. Also, they used the fuzzy structure on hyper BCK-algebras in [5, 12, 15]. Now, we define the notions of fuzzy positive implicative hyper BCK-ideals of types 1,2,3 and 4, fuzzy (weak) implicative hyper BCK-ideals and obtain some related results.

Received: May 2004; Accepted: September 2004

Keywords and Phrases: Hyper BCK-algebra, Fuzzy (strong, weak, reflexive) hyper BCK-ideal, Fuzzy (positive, weak) implicative hyper BCK-ideals .

2. Preliminaries

Definition 2.1. [9] By a *hyper BCK-algebra* we mean a non-empty set H endowed with a hyperoperation "$\circ$" and a constant 0 satisfying the following axioms:

(HK1) $(x \circ z) \circ (y \circ z) \ll x \circ y$,
(HK2) $(x \circ y) \circ z = (x \circ z) \circ y$,
(HK3) $x \circ H \ll \{x\}$,
(HK4) $x \ll y$ and $y \ll x$ imply $x = y$,

for all $x, y, z \in H$, where $x \ll y$ is defined by $0 \in x \circ y$ and for every $A, B \subseteq H$, $A \ll B$ is defined by $\forall a \in A$, $\exists b \in B$ such that $a \ll b$. In such case, we call "$\ll$" the *hyperorder* in H.

Proposition 2.2. [9] *In any hyper BCK-algebra H, the following hold:*

(i) $x \circ 0 = \{x\}$,
(ii) $x \circ y \ll x$,
(iii) $0 \circ A = \{0\}$,
(iv) $A \ll A$,
(v) $A \subseteq B$ *implies* $A \ll B$,
(vi) $A \circ \{0\} = \{0\}$ *implies* $A = \{0\}$,

for all $x, y, z \in H$ and for all non-empty subsets A and B of H.

Let I be a non-empty subset of hyper BCK-algebra H and $0 \in I$. Then I is called a *hyper BCK-subalgebra* of H, if $x \circ y \subseteq I$, for all $x, y \in I$, a *weak hyper BCK-ideal* of H, if $x \circ y \subseteq I$ and $y \in I$ imply $x \in I$, for all $x, y \in H$, a *hyper BCK-ideal* of H, if $x \circ y \ll I$ and $y \in I$ imply $x \in I$, for all $x, y \in H$, a *strong hyper BCK-ideal* of H, if $(x \circ y) \cap I \neq \emptyset$ and $y \in I$ imply $x \in I$, for all $x, y \in H$. I is said to be *reflexive*, if $x \circ x \subseteq I$ for all $x \in H$, *S-reflexive* if $(x \circ y) \cap I \neq \emptyset$ implies that $x \circ y \ll I$, for all $x, y \in H$, *closed*, if $x \ll y$ and $y \in I$ imply that $x \in I$, for all $x, y \in H$. It is easy to see that every S-reflexive subset of H is reflexive.

Lemma 2.3. [9, 10] *Let H be a hyper BCK-algebra. Then,*

(i) *any reflexive hyper BCK-ideal of H is a strong hyper BCK-ideal of H,*
(ii) *any strong hyper BCK-ideal of H is a hyper BCK-ideal of H,*
(iii) *any hyper BCK-ideal of H is a weak hyper BCK-ideal of H,*
(iv) *any weak hyper BCK-ideal of H which satisfies the closed condition is a hyper BCK-ideal of H.*

Definition 2.4. [1] Let H be a hyper BCK-algebra. Then H is said to be a *positive implicative hyper BCK-algebra*, if for all $x, , z \in H$, $(x \circ y) \circ z = (x \circ z) \circ (y \circ z)$.

Definition 2.5. [11] Let I be a non-empty subset of H and $0 \in I$. Then I is said to be a *weak implicative hyper BCK-ideal* of H, if $(x \circ z) \circ (y \circ x) \subseteq I$ and $z \in H$, then $x \in I$, an *implicative hyper BCK-ideal* of H, if $(x \circ z) \circ (y \circ x) \ll I$ and $z \in I$, then $x \in I$.

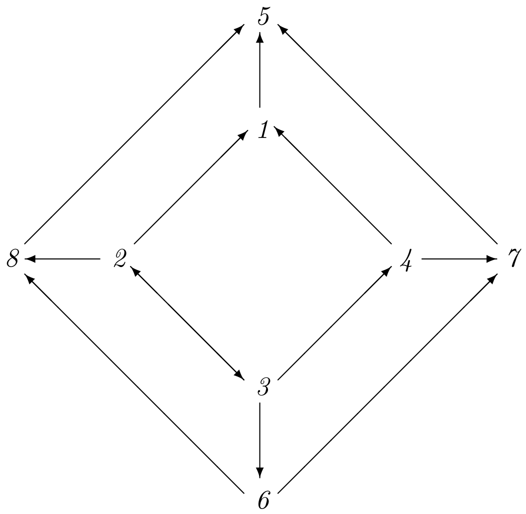
Definition 2.6. [1] Let I be a nonempty subset of H and $0 \in I$. Then I is said to be a *positive implicative hyper BCK-ideal* of

(i) *type 1*, if $(x \circ y) \circ z \subseteq I$ and $y \circ z \subseteq I$ imply $x \circ z \subseteq I$,
(ii) *type 2*, if $(x \circ y) \circ z \ll I$ and $y \circ z \subseteq I$ imply $x \circ z \subseteq I$,
(iii) *type 3*, if $(x \circ y) \circ z \ll I$ and $y \circ z \ll I$ imply $x \circ z \subseteq I$,

(iv) *type 4*, if $(x \circ y) \circ z \subseteq I$ and $y \circ z \ll I$ imply $x \circ z \subseteq I$,
(v) *type 5*, if $(x \circ y) \circ z \subseteq I$ and $y \circ z \subseteq I$ imply $x \circ z \ll I$,
(vi) *type 6*, if $(x \circ y) \circ z \ll I$ and $y \circ z \ll I$ imply $x \circ z \ll I$,
(vii) *type 7*, if $(x \circ y) \circ z \subseteq I$ and $y \circ z \ll$ imply $x \circ z \ll I$,
(viii) *type 8*, if $(x \circ y) \circ z \ll I$ and $y \circ z \subseteq I$ imply $x \circ z \ll I$,

for all $x, y, z \in H$.

Theorem 2.7. [1] *Let H be a hyper BCK-algebra. Then, the following diagram shows the relationships between all of types of positive implicative hyper BCK-ideals:*



Theorem 2.8. [1] *Let I be a non-empty subset of a hyper BCK-algebra H. Then,*

(i) *if I satisfies the closed condition, then I is a positive implicative hyper BCK-ideal of type i if and only if it is a positive implicative hyper BCK-ideal of type j, for all $1 \leq i, j \leq 4$,*
(ii) *if I is a positive implicative hyper BCK-ideal of type 2 (type 1), then I is a (weak) hyper BCK-ideal,*
(iii) *if H is a positive implicative hyper BCK-algebra and I is a (weak) hyper BCK-ideal of H, then I is a positive implicative hyper BCK-ideal of type 2 (type 1).*

Definition 2.9. [12] Let H be a hyper BCK-algebra. Then we say that a fuzzy subset μ of H

(i) satisfies the sup property, if for any subset T of H there exists $x_0 \in T$ such that $\mu(x_0) = \sup\limits_{x \in T} \mu(x)$,
(ii) is *fuzzy closed*, if $x \ll y$ then $\mu(x) \geq \mu(y)$, for all $x, y \in H$.

Definition 2.10. [12] A fuzzy subset μ of hyper BCK-algebra H is called a,

(i) *fuzzy weak hyper BCK-ideal* of H, if for all $x, y \in H$,

$$\mu(0) \geq \mu(x) \geq \min(\inf_{a \in x \circ y} \mu(a), \mu(y))$$

(ii) *fuzzy hyper BCK-ideal* of H, if $x \ll y$ implies $\mu(x) \geq \mu(y)$ and for all $x, y \in H$,

$$\mu(x) \geq \min(\inf_{a \in x \circ y} \mu(a), \mu(y))$$

(iii) *fuzzy strong hyper BCK-ideal* of H, if for all $x, y \in H$,

$$\inf_{a \in x \circ x} \mu(a) \geq \mu(x) \geq \min(\sup_{b \in x \circ y} \mu(b), \mu(y))$$

(iv) *fuzzy reflexive hyper BCK-ideal* of H, if for all $x, y \in H$,

$$\inf_{a \in x \circ x} \mu(a) \geq \mu(y) \text{ and } \mu(x) \geq \min(\sup_{a \in x \circ y} \mu(a), \mu(y))$$

Theorem 2.11. [12] *Let H be a hyper BCK-algebra. Then the following statements hold:*

(i) *every fuzzy hyper BCK-ideal of H is a fuzzy weak hyper BCK-ideal of H,*
(ii) *every fuzzy strong hyper BCK-ideal of H is a fuzzy hyper BCK-ideal ,*
(iii) *every fuzzy reflexive hyper BCK-ideal of H is a fuzzy strong hyper BCK-ideal of H.*

Definition 2.12. Let μ be a fuzzy subset of hyper BCK-algebra H. Then, the level subset μ_t of H is defined by $\mu_t = \{x \in H : \mu(x) \geq t\}$, where $t \in [0, 1]$.

It is easy to see that if μ is fuzzy closed, then for all $t \in [0, 1]$, μ_t satisfies the closed condition.

Theorem 2.13. [12] *Let μ be a fuzzy subset of a hyper BCK-algebra H. Then,*

(i) *μ is a fuzzy (weak) hyper BCK-ideal of H if and only if for all $t \in [0, 1]$, $\mu_t \neq \emptyset$ is a (weak) hyper BCK-ideal of H,*
(ii) *if μ is a fuzzy strong hyper BCK-ideal of H, then $\mu_t \neq \emptyset$ is a strong hyper BCK-ideal of H, for all $t \in [0, 1]$,*
(iii) *if μ satisfies the sup property and for all $t \in [0, 1]$, $\mu_t \neq \emptyset$ is a strong hyper BCK-ideal of H, then μ is a fuzzy strong hyper BCK-ideal of H,*

Note. From now on in this paper we let H denote a hyper BCK-algebra.

3. Fuzzy positive implicative hyper BCK-ideals

Definition 3.1. Let μ be a fuzzy subset of H and $\mu(0) \geq \mu(x)$, for all $x \in H$. Then, μ is said to be a *fuzzy positive implicative hyper BCK-ideal* of

(i) *type 1*, if for all $t \in x \circ z$,

$$\mu(t) \geq \min(\inf_{a \in (x \circ y) \circ z} \mu(a), \inf_{b \in y \circ z} \mu(b)),$$

(ii) *type 2*, if for all $t \in x \circ z$,

$$\mu(t) \geq \min(\sup_{a \in (x \circ y) \circ z} \mu(a), \inf_{b \in y \circ z} \mu(b)),$$

(iii) *type 3*, if for all $t \in x \circ z$,

$$\mu(t) \geq \min(\sup_{a \in (x \circ y) \circ z} \mu(a), \sup_{b \in y \circ z} \mu(b)),$$

(iv) *type 4*, if for all $t \in x \circ z$,

$$\mu(t) \geq \min(\inf_{a \in (x \circ y) \circ z} \mu(a), \sup_{b \in y \circ z} \mu(b))$$

Definition 3.2. [14] Let μ be a fuzzy subset of H. Then μ is said to be a *fuzzy positive implicative hyper BCK-ideal* of

(i) *type 5*, if there exists $t \in x \circ z$ such that

$$\mu(t) \geq \min(\inf_{a \in (x \circ y) \circ z} \mu(a), \inf_{b \in y \circ z} \mu(b)),$$

(ii) *type 6*, if there exists $t \in x \circ z$ such that

$$\mu(t) \geq \min(\sup_{a \in (x \circ y) \circ z} \mu(a), \sup_{b \in y \circ z} \mu(b)),$$

(iii) *type 7*, if there exists $t \in x \circ z$ such that

$$\mu(t) \geq \min(\inf_{a \in (x \circ y) \circ z} \mu(a), \sup_{b \in y \circ z} \mu(b)),$$

(iv) *type 8*, if there exists $t \in x \circ z$ such that

$$\mu(t) \geq \min(\sup_{a \in (x \circ y) \circ z} \mu(a), \inf_{b \in y \circ z} \mu(b)).$$

Theorem 3.3. *Let μ be a fuzzy subset of H. Then,*

(i) *if μ is a fuzzy positive implicative hyper BCK-ideal of type 3, then μ is a fuzzy positive implicative hyper BCK-ideal of type 2, 4 and 6,*
(ii) *if μ is a fuzzy positive implicative hyper BCK-ideal of type 2 or 4, then μ is a fuzzy positive implicative hyper BCK-ideal of type 1,*
(iii) *if μ is a fuzzy positive implicative hyper BCK-ideal of type 2 (type 4), then μ is a fuzzy positive implicative hyper BCK-ideal of type 8 (type 7),*
(iv) *if μ is a fuzzy positive implicative hyper BCK-ideal of type 1, then μ is a fuzzy positive implicative hyper BCK-ideal of type 5.*

Proof. The proof is straightforward. □

Example 3.4. (i) Let $H = \{0, 1, 2, 3\}$. Then the following table shows a hyper BCK-algebra structure on H.

$\circ$	0	1	2	3
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{0\}$	$\{0\}$
2	$\{2\}$	$\{2\}$	$\{0\}$	$\{0\}$
3	$\{3\}$	$\{3\}$	$\{2,3\}$	$\{0,2,3\}$

Define the fuzzy subset μ of H by:

$$\mu(0) = \mu(1) = 1 \quad , \quad \mu(2) = \frac{1}{2} \quad , \quad \mu(3) = \frac{1}{3}$$

Then μ is a fuzzy positive implicative hyper BCK-ideal of type 1, while it is not of type 2. Because $3 \in 3 \circ 0$, but

$$\mu(3) = \frac{1}{3} \not\geq \frac{1}{2} = \min(\sup_{a \in (3 \circ 2) \circ 0} \mu(a), \inf_{b \in 2 \circ 0} \mu(b)).$$

(ii) Let $H = \{0, 1, 2\}$. Consider the following table:

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{0,1\}$
2	$\{2\}$	$\{1,2\}$	$\{0,1,2\}$

Then H is a hyper BCK-algebra. Define the fuzzy subset ν of H by:

$$\nu(0) = \nu(2) = 1 \ \ , \ \ \nu(1) = 0,$$

Then ν is a fuzzy positive implicative hyper BCK-ideal of type 4, while it is not of type 3. Because $1 \in 2 \circ 2$, but $\nu(1) = 0 \not\geq 1 = \min(\sup_{a \in (2 \circ 0) \circ 2} \nu(a), \sup_{b \in 0 \circ 2} \nu(b))$.

Theorem 3.5. [1, 12] *Let A, B and I are non-empty subsets of H. Then,*

(i) *if I is a hyper BCK-ideal of H and $A \ll I$, then $A \subseteq I$,*

(ii) *if I is a reflexive hyper BCK-ideal of H, then*

$$(x \circ y) \cap I \neq \emptyset \text{ implies that } x \circ y \ll I, \text{ for all } x, y \in H,$$

(iii) *if I is a hyper BCK-ideal of H, $A \circ B \ll I$ and $B \subseteq I$, then $A \subseteq I$.*

Theorem 3.6. *Let μ be a fuzzy subset of H. Then the following statements hold:*

(i) *if μ is a fuzzy positive implicative hyper BCK-ideal of type 1, then μ is a fuzzy weak hyper BCK-ideal of H,*

(ii) *if μ is a fuzzy positive implicative hyper BCK-ideal of type 2, then μ is a fuzzy strong hyper BCK-ideal of H,*

Proof. (i) Follows from [[8], Theorem 3.8] and Theorem 2.11(i).
(ii) Let μ be a fuzzy positive implicative hyper BCK-ideal of type 2. Similar to the proof of (i), we get that $\mu(x) \geq \min(\sup_{a \in x \circ y} \mu(a), \mu(y))$, for all $x, y \in H$. We show that for all $x, y \in H$ if $x \ll y$, then $\mu(x) \geq \mu(y)$. For this, let $x \ll y$ i.e. $0 \in x \circ y$. Since $\mu(0) \geq \mu(a)$, for all $a \in H$, then $\sup_{a \in x \circ y} \mu(a) = \mu(0)$ and so

$$\mu(x) \geq \min(\sup_{a \in x \circ y} \mu(a), \mu(y)) = \min(\mu(0), \mu(y)) = \mu(y)$$

Now, since by Proposition 2.2(v), $x \circ x \ll x$, then for all $a \in x \circ x$, $a \ll x$ and so, $\mu(a) \geq \mu(x)$. Hence $\inf_{a \in x \circ x} \mu(a) \geq \mu(x)$. Therefore, μ is a fuzzy strong hyper BCK-ideal of H. □

The following example shows that the converse of Theorem 3.6 is not true in general.

Example 3.7. Let $H = \{0, 1, 2, 3\}$. Consider the following table:

$\circ$	0	1	2	3
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{0\}$	$\{0\}$
2	$\{2\}$	$\{2\}$	$\{0\}$	$\{0\}$
3	$\{3\}$	$\{3\}$	$\{2\}$	$\{0, 2\}$

Then H is a hyper BCK-algebra. Define fuzzy subset μ of H by

$$\mu(0) = \mu(1) = 1 \quad , \quad \mu(2) = \mu(3) = \frac{1}{2}.$$

Then μ is a fuzzy weak hyper BCK-ideal of H, but it is not a fuzzy positive implicative hyper BCK-ideal type 1. Because $2 \in 3 \circ 2 = \{2\}$, but

$$\min(\inf_{a \in (3\circ 2)\circ 2} \mu(a), \inf_{b \in 2\circ 2} \mu(b)) = \min(\mu(0), \mu(0)) = 1 \not\leq \frac{1}{2} = \mu(2).$$

(ii) Let $H = \{0, 1, 2\}$. Then the following table shows the hyper BCK-algebra structure on H:

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{2\}$	$\{0, 2\}$

Let fuzzy subset μ on H is defined as follows:

$$\mu(0) = 1 \quad , \quad \mu(1) = 0 \quad , \quad \mu(2) = \frac{1}{2}.$$

Then, μ is a fuzzy strong hyper BCK-ideal of H, but it is not a fuzzy positive implicative hyper BCK-ideal of type 2. Since $2 \in 2 \circ 2$, but

$$\min(\sup_{a \in (2\circ 0)\circ 2} \mu(a), \inf_{b \in 0\circ 2} \mu(b)) = \min(\mu(0), \mu(0)) = 1 \not\leq \frac{1}{2} = \mu(2).$$

Theorem 3.8. *Let μ be a fuzzy subset of H. Then the following statements hold:*

(i) *μ is a fuzzy positive implicative hyper BCK-ideal of type 1 if and only if for all $t \in [0, 1]$, $\mu_t \neq \emptyset$ is a positive implicative hyper BCK-ideal of type 1,*

(ii) *If μ is a fuzzy positive implicative hyper BCK-ideal of type 2(3), then for all $t \in [0, 1]$, $\mu_t \neq \emptyset$ is a positive implicative hyper BCK-ideal of type 2(3),*

(iii) *If for all $t \in [0, 1]$, $\mu_t \neq \emptyset$ is a reflexive positive implicative hyper BCK-ideal of type 2(3) and μ satisfies the sup property, then μ is a fuzzy positive implicative hyper BCK-ideal of type 2(3),*

(iv) *If μ is a fuzzy closed and fuzzy positive implicative hyper BCK-ideal of type 4, then for all $t \in [0, 1]$, $\mu_t \neq \emptyset$ is a positive implicative hyper BCK-ideal of type 4,*

(v) *Let μ be a fuzzy closed and satisfies the sup property. If for all $t \in [0, 1]$, $\mu_t \neq \emptyset$ is a reflexive positive implicative hyper BCK-ideal of type 4, then μ is a fuzzy positive implicative hyper BCK-ideal of type 4,* *Proof.* (i) Let μ be a fuzzy positive implicative hyper BCK-ideal of type 1, $(x \circ y) \circ z \subseteq \mu_t$ and $y \circ z \subseteq \mu_t$, for $x, y, z \in H$. Then $\mu(a) \geq t$, $\mu(b) \geq t$, for all $a \in (x \circ y) \circ z$ and $b \in y \circ z$. Hence, $\inf\limits_{a \in (x \circ y) \circ z} \mu(a) \geq t$ and $\inf\limits_{b \in y \circ z} \mu(b) \geq t$. Thus, by hypothesis for all $u \in x \circ z$,

$$\mu(u) \geq \min(\inf_{a \in (x \circ y) \circ z} \mu(a), \inf_{b \in y \circ z} \mu(b))) \geq t$$

and so $u \in \mu_t$. Therefore, $x \circ z \subseteq \mu_t$ and this implies that μ_t is a positive implicative hyper BCK-ideal of type 1.

Conversely, let for all $t \in [0, 1]$, $\mu_t \neq \emptyset$ be a positive implicative hyper BCK-ideal of type 1 and $\lambda = \min(\inf\limits_{a \in (x \circ y) \circ z} \mu(a), \inf\limits_{b \in y \circ z} \mu(b))$. Then $\inf\limits_{a \in (x \circ y) \circ z} \mu(a) \geq \lambda$ and $\inf\limits_{b \in y \circ z} \mu(b) \geq \lambda$. So, $\mu(a) \geq \lambda$ and $\mu(b) \geq \lambda$, for all $a \in (x \circ y) \circ z$ and $b \in y \circ z$. Hence $(x \circ y) \circ z \subseteq \mu_\lambda$ and $y \circ z \subseteq \mu_\lambda$ and so by hypothesis $x \circ z \subseteq \mu_\lambda$. Thus, for all $u \in x \circ z$,

$$\mu(u) \geq \lambda = \min(\inf_{a \in (x \circ y) \circ z} \mu(a), \inf_{b \in y \circ z} \mu(b)).$$

Therefore, μ is a fuzzy positive implicative hyper BCK-ideal of type 1.
(ii) Let μ be a fuzzy positive implicative hyper BCK-ideal of type 2, $t \in [0, 1]$, $(x \circ y) \circ z \ll \mu_t$ and $y \circ z \subseteq \mu_t$, for $x, y, z \in H$. Then for all $a \in (x \circ y) \circ z$ there exists $u \in \mu_t$ such that $a \ll u$. Since by Theorem 3.6(ii) and Theorem 2.11(ii), μ is a fuzzy hyper BCK-ideal of H, then $\mu(a) \geq \mu(u) \geq t$ and so $\sup\limits_{a \in (x \circ y) \circ z} \mu(a) \geq t$. Moreover, since $y \circ z \subseteq \mu_t$ then $\mu(b) \geq t$, for all $b \in y \circ z$. Hence, $\inf\limits_{b \in y \circ z} \mu(b) \geq t$ and so for all $v \in x \circ z$, $\mu(v) \geq \min(\sup\limits_{a \in (x \circ y) \circ z} \mu(a), \inf\limits_{b \in y \circ z} \mu(b)) \geq t$. Thus, $v \in \mu_t$ and so $x \circ z \subseteq \mu_t$. Therefore, μ_t is a positive implicative hyper BCK-ideal of type 2.
The proof of type 3 is similar to type 2.
(iii) Let for all $t \in [0, 1]$, $\mu_t \neq \emptyset$ be a positive implicative hyper BCK-ideal of type 2 and for $x, y, z \in H$, $\lambda = \min(\sup\limits_{a \in (x \circ y) \circ z} \mu(a), \inf\limits_{b \in y \circ z} \mu(b))$. Since μ satisfies the sup property, then there exists $a_0 \in (x \circ y) \circ z$ such that $\mu(a_0) = \sup\limits_{a \in (x \circ y) \circ z} \mu(a) \geq \lambda$ and so $a_0 \in \mu_\lambda$. Hence, $((x \circ y) \circ z) \cap \mu_\lambda \neq \emptyset$. Since by Theorem 2.8(ii), μ_λ is a hyper BCK-ideal of H and so by hypothesis and Theorem 3.5(ii), $(x \circ y) \circ z \ll \mu_\lambda$. Moreover, since for all $u \in y \circ z$, $\mu(u) \geq \inf\limits_{b \in y \circ z} \mu(b) \geq \lambda$, we get that $u \in \mu_\lambda$ and so $y \circ z \subseteq \mu_\lambda$. Since μ_λ is a positive implicative hyper BCK-ideal of type 2, then $x \circ z \subseteq \mu_\lambda$ and this implies that for all $v \in x \circ z$,

$$\mu(v) \geq \lambda = \min(\sup_{a \in (x \circ y) \circ z} \mu(a), \inf_{b \in y \circ z} \mu(b)).$$

Therefore, μ is a fuzzy positive implicative hyper BCK-ideal of type 2.
The proof of type 3 is similar as above.
(iv) By considering the fuzzy closed condition, the proof is similar to the proof of (ii) by some modifications.
(v) Let for all $t \in [0, 1]$, $\mu_t \neq \emptyset$ be a reflexive positive implicative hyper BCK-ideal

of type 4 and

$$\lambda = \min(\inf_{a\in(x\circ y)\circ z} \mu(a), \sup_{b\in y\circ z} \mu(b)).$$

Then $\inf\limits_{a\in(x\circ y)\circ z} \mu(a) \geq \lambda$ and $\sup\limits_{b\in y\circ z} \mu(b) \geq \lambda$. Hence, for all $a \in (x \circ y) \circ z$, $\mu(a) \geq \lambda$ and so $(x \circ y) \circ z \subseteq \mu_\lambda$. Moreover, since μ satisfies the sup property, then there exists $b_0 \in y \circ z$ such that $\mu(b_0) \geq \lambda$ and so $b_0 \in \mu_\lambda$. Hence $(y \circ z) \cap \mu_\lambda \neq \emptyset$. Since μ_λ is a positive implicative hyper BCK-ideal of type 4 and hence of type 1, then by Theorem 2.8(ii), μ_λ is a weak hyper BCK-ideal of H. Also, since μ is fuzzy closed, then μ_λ is closed and so by Lemma 2.3(iv), μ_λ is a hyper BCK-ideal of H. Now, μ_λ is a reflexive hyper BCK-ideal of H and $(y \circ z) \cap \mu_\lambda \neq \emptyset$ implies that $y \circ z \ll \mu_\lambda$, by Theorem 3.5(ii). Since μ_λ is a positive implicative hyper BCK-ideal of type 4, then $x \circ z \subseteq \mu_\lambda$. Hence, for all $t \in x \circ z$,

$$\mu(t) \geq \lambda = \min(\inf_{a\in(x\circ y)\circ z} \mu(a), \sup_{b\in y\circ z} \mu(b)).$$

Therefore, μ is a fuzzy positive implicative hyper BCK-ideal of type 4.

□

Example 3.9. The following tables show two hyper BCK-algebra structures $(H, \circ_1)$ and $(H, \circ_2)$ on $H = \{0, 1, 2, 3\}$.

$\circ_1$	0	1	2	3
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{0\}$	$\{0\}$
2	$\{2\}$	$\{3\}$	$\{0,3\}$	$\{1,3\}$
3	$\{3\}$	$\{1,3\}$	$\{0,1,3\}$	$\{0,1,3\}$

$\circ_2$	0	1	2	3
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{0\}$	$\{0\}$
2	$\{2\}$	$\{1,2\}$	$\{0,1,2\}$	$\{1,2,3\}$
3	$\{3\}$	$\{3\}$	$\{0,3\}$	$\{0,3\}$

Let the fuzzy subset μ on $(H, \circ_1)$ is defined by $\mu(0) = \mu(1) = 1$, $\mu(2) = \frac{1}{2}$, $\mu(3) = \frac{1}{3}$. Then μ is not a fuzzy positive implicative hyper BCK-ideal of type 4 because, $3 \in 2 \circ 3$ and $\mu(3) = \frac{1}{3} \ngeq \min(\inf\limits_{a\in(2\circ1)\circ3} \mu(a), \inf\limits_{b\in1\circ3} \mu(b))$. But, for all $t \in [0, 1]$, $\mu_t = H$ or $\{0, 1\}$ is a positive implicative hyper BCK-ideal of type 4. Moreover, $\mu_{\frac{1}{2}} = \{0, 1\}$ is not reflexive. Therefore, in (v) the reflexive condition is necessary. Also, the fuzzy subset ν on H which is defined by $\nu(0) = \nu(1) = 1$, $\nu(2) = \nu(3) = 0$ is not a fuzzy positive implicative hyper BCK-ideal of type 2, because $3 \in 3 \circ 3$ and $\mu(3) = 0 \ngeq 1 = \min(\sup\limits_{a\in(3\circ1)\circ3} \nu(a), \inf\limits_{b\in1\circ3} \nu(b))$. But, $\nu_0 = H$ and for all $t \in (0, 1]$, $\mu_t = \{0, 1\}$ is a positive implicative hyper BCK-ideal of type 3 (and so of type 2). Moreover, for all $t \in [0, 1]$, μ_t is not reflexive. Therefore, in (iii) the reflexive condition is necessary.

Corollary 3.10. *Let μ be a fuzzy subset of H which satisfies the sup property and for all $t \in [0, 1]$, $\mu_t \neq \emptyset$ be reflexive. Then, μ is a fuzzy positive implicative hyper BCK-ideal of type 2 if and only if μ is a fuzzy positive implicative hyper BCK-ideal of type 3.*

Proof. Let μ be a fuzzy positive implicative hyper BCK-ideal of type 2. Then by Theorem 3.8(ii), $\mu_t \neq \emptyset$ is a positive implicative of type 2 and so of type 3. Hence, by Theorem 3.8(iii), μ is a fuzzy positive implicative hyper BCK-ideal of type 3. The converse follows from Theorem 3.3(i). □

Theorem 3.11. *Let H be a positive implicative hyper BCK-algebra. Then the following statements are equivalent.*

(i) *μ is a fuzzy weak hyper BCK-ideal of H,*
(ii) *μ is a fuzzy positive implicative hyper BCK-ideal of type 1,*

Proof. (i) $\Rightarrow$ (ii) If μ is a fuzzy weak hyper BCK-ideal of H, then by Theorem 2.13(i), $\mu_t \neq \emptyset$ is a weak hyper BCK-ideal of H and so by Theorem 2.8(iii), μ_t is a positive implicative hyper BCK-ideal of type 1.
(ii) $\Rightarrow$(i) The proof follows from Theorem 3.6(i). □

Theorem 3.12. *Let H be a positive implicative hyper BCK-algebra. Then the following statements are equivalent.*

(i) *μ is a fuzzy hyper BCK-ideal of H,*
(ii) *μ is a fuzzy positive implicative hyper BCK-ideal of type 2 (type 3),*

Proof. The proof is similar to the proof of Theorem 3.11. □

4. **Fuzzy (weak) implicative hyper BCK-ideals**

Definition 4.1. Let μ be a fuzzy subset of H and for all $x \in H$, $\mu(0) \geq \mu(x)$. Then μ is called a

- [?] *fuzzy weak implicative hyper BCK-ideal* of H, if

$$\mu(x) \geq \min(\inf_{a \in (x \circ z) \circ (y \circ x)} \mu(a), \mu(z))$$

- *fuzzy implicative hyper BCK-ideal* of H, if

$$\mu(x) \geq \min(\sup_{a \in (x \circ z) \circ (y \circ x)} \mu(a), \mu(z))$$

for all $x, y, z \in H$,

Theorem 4.2. *Every fuzzy implicative hyper BCK-ideal of H is a fuzzy weak implicative hyper BCK-ideal.*

Proof. The proof is straightforward. □

Example 4.3. Let $H = \{0, a, b\}$. The following table shows a hyper BCK-algebra structure on H:

$\circ$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0, a\}$	$\{0, a\}$
b	$\{b\}$	$\{a\}$	$\{0, a\}$

Define the fuzzy subset μ on H by $\mu(0) = 1$, $\mu(a) = 0$, $\mu(b) = \frac{1}{2}$. Then μ is a fuzzy weak implicative hyper BCK-ideal of H but it is not a fuzzy implicative hyper BCK-ideal. Because,

$$\min(\sup_{t\in(a\circ 0)\circ(a\circ a)} \mu(t), \mu(0)) = \min(\sup_{t\in\{0,a\}} \mu(t), \mu(0)) = \min(\mu(0), \mu(0)) = \mu(0) \not\leq \mu(a)$$

Theorem 4.4. *(i) Every fuzzy implicative hyper BCK-ideal of H is a fuzzy strong hyper BCK-ideal .*
(ii) Every fuzzy weak implicative hyper BCK-ideal of H is a fuzzy weak hyper BCK-ideal .

Proof. Let μ be a fuzzy implicative hyper BCK-ideal of H. Then for all $x, y, z \in H$ we have

$$\mu(x) \geq \min(\sup_{a\in(x\circ z)\circ(y\circ x)} \mu(a), \mu(z))$$

By putting $z = y$ and $y = 0$ in the above inequality, we get that

$$\mu(x) \geq \min(\sup_{a\in(x\circ y)\circ(0\circ x)} \mu(a), \mu(y)) = \min(\sup_{a\in x\circ y} \mu(a), \mu(y)) \quad (1)$$

Now, let $x \ll y$, for $x, y \in H$. Then $0 \in x \circ y$ and so from (1) we get

$$\mu(x) \geq \min(\sup_{a\in x\circ y} \mu(a), \mu(y)) = \min(\mu(0), \mu(y)) = \mu(y) \quad (2)$$

Now, let $a \in x \circ x$. Since $x \circ x \ll x$, then $a \ll x$ and so by (2) we have $\mu(a) \geq \mu(x)$. Hence, $\inf_{a\in x\circ x} \mu(a) \geq \mu(x)$. Therefore, μ is a fuzzy strong hyper BCK-ideal of H.
(ii) Follows from [[7], Theorem 3.7] and Theorem 2.11(i) $\square$

Example 4.5. Let $H = \{0, a, b, c\}$. Consider the following table:

$\circ$	0	a	b	c
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0\}$	$\{0\}$	$\{0\}$
b	$\{b\}$	$\{b\}$	$\{0\}$	$\{0\}$
c	$\{c\}$	$\{c\}$	$\{b, c\}$	$\{0, b, c\}$

Then $(H, \circ)$ is a hyper BCK-algebra. Define the fuzzy subset μ of H by

$$\mu(0) = \mu(a) = 1 \quad , \quad \mu(b) = \mu(c) = \frac{1}{3}$$

Thus μ is a fuzzy strong hyper BCK-ideal (and so is a fuzzy weak hyper BCK-ideal) but it is not a fuzzy weak implicative (and so is not a fuzzy implicative) hyper BCK-ideal of H. Because,

$$\min(\inf_{t\in(b\circ 0)\circ(c\circ b)} \mu(t), \mu(0)) = \min(\mu(0), \mu(0)) = \mu(0) \not\leq \mu(b)$$

This implies that the converse of Theorem 4.4, is not correct in general.

Theorem 4.6. [11] *Let I be a non-empty subset of H. Then,*

(i) *if I is an implicative hyper BCK-ideal of H, then I is a hyper BCK-ideal of H,*

(ii) *if I is a hyper BCK-ideal of H, then I is an implicative hyper BCK-ideal of H if and only if $x \circ (y \circ x) \ll I$ implies $x \in I$, for all $x, y \in H$,*
(iii) *if I is an implicative and reflexive hyper BCK-ideal of H, then I is a positive implicative hyper BCK-ideal of type 3.*

Theorem 4.7. *Let μ be a fuzzy subset of H. Then*

(i) *if μ is a fuzzy implicative hyper BCK-ideal of H, then for all $t \in [0,1]$, $\mu_t \neq \emptyset$ is an implicative hyper BCK-ideal of H.*
(ii) *if μ satisfies the sup property and for all $t \in [0,1]$, $\mu_t \neq \emptyset$ is an S-reflexive implicative hyper BCK-ideal of H, then μ is a fuzzy implicative hyper BCK-ideal of H.*
(iii) *μ is a fuzzy weak implicative hyper BCK-ideal of H if and only if for all $t \in [0,1]$, $\mu_t \neq \emptyset$ is a weak implicative hyper BCK-ideal of H.*

Proof. (i) Let μ be a fuzzy implicative hyper BCK-ideal of H. Since by Theorem 4.4, μ is a fuzzy strong hyper BCK-ideal and so is a fuzzy hyper BCK-ideal of H, then by Theorem 2.13(i), for all $t \in [0,1]$, μ_t is a hyper BCK-ideal of H. Now, by Theorem 4.6(ii), it is enough to show that, if $x \circ (y \circ x) \ll \mu_t$, then $x \in \mu_t$, for all $t \in [0,1]$. Let $x \circ (y \circ x) \ll \mu_t$, for $x, y \in H$. Then for all $a \in x \circ (y \circ x)$ there exists $b \in \mu_t$ such that $a \ll b$. Since μ is a fuzzy hyper BCK-ideal of H, then $\mu(a) \geq \mu(b) \geq t$ and so $\sup\limits_{a \in x \circ (y \circ x)} \mu(a) \geq t$. Hence, by hypothesis,

$$\mu(x) \geq \min(\sup_{a \in (x \circ 0) \circ (y \circ x)} \mu(a), \mu(0)) = \sup_{a \in x \circ (y \circ x)} \mu(a) \geq t$$

i.e, $x \in \mu_t$.
(ii) Let for all $t \in [0,1]$, $\mu_t \neq \emptyset$ be an S-reflexive implicative hyper BCK-ideal of H and for $x, y, z \in H$,

$$t = \min(\sup_{a \in (x \circ z) \circ (y \circ x)} \mu(a), \mu(z))$$

Since μ has the sup property, then there exists $a_0 \in (x \circ z) \circ (y \circ x)$ such that $\mu(a_0) = \sup\limits_{a \in (x \circ z) \circ (y \circ x)} \mu(a) \geq t$ and so $a_0 \in \mu_t$. Hence, by (HK2) we have

$$((x \circ (y \circ x)) \circ z) \cap \mu_t = ((x \circ z) \circ (y \circ x)) \cap \mu_t \neq \emptyset$$

Thus there exists $a \in x \circ (y \circ x)$ such that $(a \circ z) \cap \mu_t \neq \emptyset$. Since by Theorem 4.6(i), μ_t is a hyper BCK-ideal of H and by hypothesis is S-reflexive, then it is a reflexive hyper BCK-ideal of H and so is a strong hyper BCK-ideal of H. Since $z \in \mu_t$, then $a \in \mu_t$ and so $(x \circ (y \circ x)) \cap \mu_t \neq \emptyset$ and since μ_t is S-reflexive, then $x \circ (y \circ x) \ll \mu_t$. Since μ_t is an implicative hyper BCK-ideal of H, then $x \in \mu_t$ and so

$$\mu(x) \geq t = \min(\sup_{a \in (x \circ z) \circ (y \circ x)} \mu(a), \mu(z))$$

which shows that μ is a fuzzy implicative hyper BCK-ideal of H.
(iii) Let μ be a fuzzy weak implicative hyper BCK-ideal of H, $t \in [0,1]$ and for

$x, y, z \in H$, $(x \circ z) \circ (y \circ x) \subseteq \mu_t$ and $z \in \mu_t$. Then for all $a \in (x \circ z) \circ (y \circ x)$, $\mu(a) \geq t$. Hence, $\inf\limits_{a \in (x \circ z) \circ (y \circ x)} \mu(a) \geq t$ and $\mu(z) \geq t$. Thus,

$$\mu(x) \geq \min(\inf_{a \in (x \circ z) \circ (y \circ x)} \mu(a), \mu(z)) \geq t$$

i.e, $x \in \mu_t$ which implies μ_t is a weak implicative hyper BCK-ideal of H.

Conversely, let for all $t \in [0, 1]$, μ_t be a weak implicative hyper BCK-ideal of H and for $x, y, z \in H$,

$$t = \min(\inf_{a \in (x \circ z) \circ (y \circ x)} \mu(a), \mu(z))$$

Thus $\mu(z) \geq t$ and for all $a \in (x \circ z) \circ (y \circ x)$, $\mu(a) \geq \inf\limits_{u \in (x \circ z) \circ (y \circ x)} \mu(u) \geq t$. This implies that $(x \circ z) \circ (y \circ x) \subseteq \mu_t$ and $z \in \mu_t$ and so by hypothesis $x \in \mu_t$. Therefore,

$$\mu(x) \geq t = \min(\inf_{a \in (x \circ z) \circ (y \circ x)} \mu(a), \mu(z))$$

which shows that μ is a fuzzy weak implicative hyper BCK-ideal of H. □

Example 4.8. Let μ be fuzzy subset defined in Example 4.5. Then for all $t \in [0, 1]$, $\mu_t (= \{0, a\}$ or $H)$ is an implicative hyper BCK-ideal of H. But μ is not a fuzzy implicative hyper BCK-ideal of H. Since $\mu_t = \{0, a\}$ is not S-reflexive, then Theorem 4.7(ii) is not correct in general.

Theorem 4.9. (i) *Let μ be a fuzzy subset of H has the sup property and for all $t \in [0, 1]$, μ_t be reflexive. If μ is a fuzzy implicative hyper BCK-ideal of H, then μ is a fuzzy positive implicative hyper BCK-ideal of type 3.*
(ii) *Let H be a positive implicative hyper BCK-algebra. If μ is a fuzzy weak implicative hyper BCK-ideal of H, then μ is a fuzzy positive implicative hyper BCK-ideal of type 1.*

Proof. Let μ be a fuzzy implicative hyper BCK-ideal of H. Then by Theorem 4.7(i), for all $t \in [0, 1]$, $\mu_t \neq \emptyset$ is an implicative hyper BCK-ideal of H. Since μ_t is reflexive then by Theorem 4.6(iii), μ_t is a positive implicative hyper BCK-ideal of type 3 and so by Theorem 3.8(iii), μ is a fuzzy positive implicative hyper BCK-ideal of type 3.
(ii) Let μ be a fuzzy weak implicative hyper BCK-ideal of H and for $x, y, z \in H$,

$$t = \min(\inf_{a \in (x \circ y) \circ z} \mu(a), \inf_{b \in y \circ z} \mu(b))$$

Thus for all $a \in (x \circ y) \circ z$ and $b \in y \circ z$, $\mu(a) \geq t$ and $\mu(b) \geq t$. Since H is a positive implicative hyper BCK-algebra, then $(x \circ z) \circ (y \circ z) = (x \circ y) \circ z \subseteq \mu_t$ and $y \circ z \subseteq \mu_t$. Since by Theorems 4.4(ii) and 2.13(i), μ_t is a weak hyper BCK-ideal of H then, $x \circ z \subseteq \mu_t$. Therefore, for all $s \in x \circ z$,

$$\mu(s) \geq t = \min(\inf_{a \in (x \circ y) \circ z} \mu(a), \inf_{b \in y \circ z} \mu(b))$$

which shows that μ is a fuzzy weak hyper BCK-ideal of H. □

5. Product of fuzzy hyper BCK-ideals

Definition 5.1. [4] Let $(H_1, \circ_1, 0_1)$ and $(H_2, \circ_2, 0_2)$ are hyper BCK-algebras and $H = H_1 \times H_2$. We define a hyperoperation "$\circ$" on H by

$$(a_1, b_1) \circ (a_2, b_2) = (a_1 \circ a_2, b_1 \circ b_2)$$

for all $(a_1, b_1), (a_2, b_2) \in H$, where for $A \subseteq H_1$ and $B \subseteq H_2$ by (A, B) we mean

$$(A, B) = \{(a, b) : a \in A, b \in B\}$$

and $0 = (0_1, 0_2)$ and a hyperorder "$\ll$" on H by

$$(a_1, b_1) \ll (a_2, b_2) \Leftrightarrow a_1 \ll a_2 \text{ and } b_1 \ll b_2.$$

Thus $(H, \circ, 0)$ is a hyper BCK-algebra.

Definition 5.2. Let μ and ν be fuzzy subsets of hyper BCK-algebras H_1 and H_2, respectively. Then $\mu \times \nu$, the product of μ and ν, of $H = H_1 \times H_2$ is defined as follows:

$$\mu \times \nu((x, y)) = \min(\mu(x), \nu(y)).$$

From now on we let H_1 and H_2 are hyper BCK-algebras and $H = H_1 \times H_2$.

Definition 5.3. Let μ be a fuzzy subset of H. Then fuzzy subsets μ_1 and μ_2 on H_1 and H_2, respectively, are defined as follows:

$$\mu_1(x) = \mu((x, 0)), \quad \mu_2(y) = \mu((0, y))$$

which are called the *projections* of μ.

Theorem 5.4. *Let μ be a fuzzy subset of H. If μ is a fuzzy (strong) hyper BCK-ideal of H, then μ_1 and μ_2 are fuzzy (strong) hyper BCK-ideal of H and $\mu = \mu_1 \times \mu_2$.*

Proof. Let μ be a fuzzy hyper BCK-ideal of H and $x \ll x'$, for $x, x' \in H_1$. Then $(x, 0) \ll (x', 0)$ and so $\mu_1(x) = \mu((x, 0)) \geq \mu((x', 0)) = \mu_1(x')$.
Now, let $t = \min(\inf\limits_{a \in x \circ y} \mu_1(a), \mu_1(y))$. Then $\mu_1(a) \geq \inf\limits_{u \in x \circ y} \mu_1(u) \geq t$ and $\mu_1(y) \geq t$ and so $\mu((a, 0)) \geq t$ and $\mu((y, 0)) \geq t$, for all $a \in x \circ y$. Thus $(x, 0) \circ (y, 0) \subseteq \mu_t$ and $(y, 0) \in \mu_t$. Since by Theorem 2.13(i), μ_t is a hyper BCK-ideal and so is a weak hyper BCK-ideal of H, then $(x, 0) \in \mu_t$ and so $\mu_1(x) = \mu((x, 0)) \geq t = \min(\inf\limits_{a \in x \circ y} \mu_1(a), \mu_1(y))$. This shows that μ_1 is a fuzzy hyper BCK-ideal of H_1. Similarly, we can show that μ_2 is a fuzzy hyper BCK-ideal of H_2.
Now, let μ be a fuzzy strong hyper BCK-ideal of H and $x, z \in H_1$, $y, w \in H_2$. Thus

$$\inf_{(a,b) \in (x,y) \circ (x,y)} \mu((a, b)) \geq \mu((x, y)) \geq \min(\sup_{(u,v) \in (x,y) \circ (z,w)} \mu((u, v)), \mu((z, w)))$$

If we let $y = w = 0$, then we get that

$$\begin{aligned}
\inf_{a\in x\circ x}\mu_1(a) &= \inf_{(a,b)\in(x,0)\circ(x,0)}\mu((a,b))\\
&\geq \mu((x,0)) = \mu_1(x)\\
&\geq \min(\sup_{(u,v)\in(x,0)\circ(z,0)}\mu((u,v)),\mu((z,0))\\
&= \min(\sup_{u\in x\circ z}\mu_1(u),\mu_1(z))
\end{aligned}$$

which shows that μ_1 is a fuzzy strong hyper BCK-ideal of H_1. Similarly, we can show that μ_2 is a fuzzy strong hyper BCK-ideal of H_2.
Now, since μ is a fuzzy strong hyper BCK-ideal of H, then

$$\mu((x,y)) \geq \min(\sup_{(a,b)\in(x,y)\circ(z,w)}\mu((a,b)),\mu((z,w)))$$

If we let $z = 0$ and $w = y$, then

$$\begin{aligned}
\mu((x,y)) &\geq \min(\sup_{(a,b)\in(x,y)\circ(0,y)}\mu((a,b)),\mu((0,y)))\\
&= \min(\sup_{(a,b)\in(x\circ 0,y\circ y)}\mu((a,b)),\mu((0,y)))\\
&= \min(\mu((x,0)),\mu((0,y))) \quad \text{since } 0\in y\circ y \text{ and } x\circ 0=\{x\}\\
&= (\mu_1\times\mu_2)((x,y))
\end{aligned}$$

Conversely, since $(x,0) \ll (x,y)$ and $(0,y) \ll (x,y)$, then $\mu((x,0)) \geq \mu((x,y))$ and $\mu((0,y)) \geq \mu((x,y))$. Hence,

$$(\mu_1\times\mu_2)((x,y)) = \min(\mu_1(x),\mu_2(y)) = \min(\mu((x,0)),\mu((0,y))) \geq \mu((x,y))$$

Therefore, $\mu = \mu_1 \times \mu_2$. □

Theorem 5.5. *Let μ be a fuzzy subset of H. Then,*

(i) *if μ is a fuzzy positive implicative hyper BCK-ideal of type i of H, then μ_1 and μ_2 are respectively fuzzy positive implicative hyper BCK-ideals of type i of H_1 and H_2, for $i = 1, 2, \ldots, 8$,*

(ii) *if μ is a fuzzy positive implicative hyper BCK-ideal of type 2, then $\mu_1\times\mu_2 = \mu$.*

Proof. (i) Let μ be a fuzzy positive implicative hyper BCK-ideal of type 1 and $x, y, z \in H_1$. Then for all $a \in x \circ z$,

$$\begin{aligned}
\mu_1(a) = \mu((a,0)) &\geq \min(\inf_{(u,v)\in((x,0)\circ(y,0))\circ(z,0)}\mu((u,v)),\inf_{(w,h)\in(y,0)\circ(z,0)}\mu((w,h)))\\
&= \min(\inf_{u\in(x\circ y)\circ z}\mu((u,0)),\inf_{w\in y\circ z}\mu((w,0)))\\
&= \min(\inf_{u\in(x\circ y)\circ z}\mu_1(u),\inf_{w\in y\circ z}\mu_1(w))
\end{aligned}$$

Similarly, we can show that μ_2 is a fuzzy positive implicative hyper BCK-ideal of type 1 of H_2. The proof of the other cases is similar.
(ii) Let μ be a fuzzy positive implicative hyper BCK-ideal of type 2. Since by

Theorem 3.6(ii), μ is a fuzzy strong hyper BCK-ideal of H, then by Theorem 5.4, $\mu = \mu_1 \times \mu_2$. □

Theorem 5.6. *Let μ be a fuzzy subset of H. If μ is a fuzzy (weak) implicative hyper BCK-ideal of H, then μ_1 and μ_2 are respectively fuzzy (weak) implicative hyper BCK-ideals of H_1 and H_2, and $\mu_1 \times \mu_2 = \mu$.*

Proof. Let μ be a fuzzy implicative hyper BCK-ideal of H and $x, y, z \in H_1$. Then for all $a \in x \circ z$,

$$\begin{aligned}\mu_1(a) = \mu((a,0)) &\geq \min(\sup_{(u,v)\in((x,0)\circ(z,0))\circ((y,0)\circ(x,0))} \mu((u,v)), \mu((z,0))) \\ &= \min(\sup_{u\in(x\circ z)\circ(y\circ x)} \mu(u), \mu(z))\end{aligned}$$

Similarly, we can show that μ_2 is a fuzzy implicative hyper BCK-ideal of H_2. Now, let μ be a fuzzy implicative hyper BCK-ideal of H. Since by Theorem 4.4(i), μ is a fuzzy strong hyper BCK-ideal , then by Theorem 5.4, $\mu = \mu_1 \times \mu_2$. □

Acknowledgements. Authors would like to express their sincere thanks to the referees for their valuable suggestions and comments.

References

[1] R. A. Borzooei, M. Bakhshi, *On positive implicative hyper BCK-ideals*, Scientiae Mathematicae Japonicae, Vol. **9**(2003), 303-314.

[2] R.A. Borzooei, M. Bakhshi, *Some Results on Hyper BCK-algebras*, Quasigroups and Related Systems, Vol.**11** (2004), 9-24.

[3] R. A. Borzooei, M. Bakhshi, *(Weak) Implicative Hyper BCK-ideals*, Quasigroups and Related Systems, Vol.**12**, to appear

[4] R. A. Borzooei, A. Hasankhani, M. M. Zahedi, Y. B. Jun, *On hyper K-algebras*, Math. Japon., Vol. **52**, No.**1**(2000), 13-121.

[5] R. A. Borzooei, M. M. Zahedi, *Fuzzy structures on hyper K-algebras*, International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems, Vol. **10**, No. **1**(2002) 77-93.

[6] Y. Imai, K. Iséki, *On axiom systems of propositional calculi XIV*, Proc. Japan Academy, **42** (1966) 19-22.

[7] Y. B. Jun, W. H. Shim, *Fuzzy implicative hyper BCK-ideals of hyper BCK-algebras*, Internat. J. Math. Sci., **29** (2002), No. 2, 63-70.

[8] Y. B. Jun, W. H. Shim, *Fuzzy structures of $PI(\ll, \subseteq, \subseteq)_{BCK}$-ideals in hyper BCK-algebras*, Internat. J. Math. Sci. (2003), No. **9**, 549-556.

[9] Y. B. Jun, M. M. Zahedi, X. L. Xin, R. A. Borzooei, *On hyper BCK-algebras*, Italian Journal of Pure and Applied Mathematics, No. **10** (2000), 127-136.

[10] Y. B. Jun, X. L. Xin, E. H. Roh, M. M. Zahedi, *Strong hyper BCK-ideals of hyper BCK-algebras*, Mathematicae Japonicae, Vol.**51**, **3** (2000), 493-498.

[11] Y. B. Jun, X. L. Xin, *Implicative hyper BCK-ideals of hyper BCK-algebras*, Mathematicae Japonicae, Vol. **52**, No. **3**, (2000), 435-443.

[12] Y. B. Jun, X. L. Xin, *Fuzzy hyper BCK-ideals of hyper BCK-algebras*, Scientiae Mathematicae Japonicae, **53**, No. **2**(2001), 353-360.

[13] F. Marty, *Sur une generalization de la notion de groups*, 8th congress Math. Scandinaves, Stockholm (1934) 45-49.

[14] M. M. Zahedi, M. Bakhshi, R. A. Borzooei, *Fuzzy positive implicative hyper BCK-ideals of types 5,6,7 and 8*, Journal of Basic Science, University of Mazandaran, Vol. **2**, No. **2**, (2003), 61-69.

[15] M. M. Zahedi, R. A. Borzooei, H. Rezaei, *Fuzzy positive implicative hyper K-ideals of type 1,2,3 and 4*, 9th IFSA World Congress and 20th NAFIPS 2001, July 25-28 (2001), Vancover (Canada), 1210-1215.

Mahmood Bakhshi*, Rajab Ali Borzooei, Department of Mathematics, Sistan and Baluchestan University, Zahedan, Iran
E-mail address: mbakhshi@hamoon.usb.ac.ir, borzooei@hamoon.usb.ac.ir

Mohammad Mehdi Zahedi, Department of Mathematics, Shahid Bahonar University of Kerman, Kerman, Iran
E-mail address: zahedi_mm@mail.uk.ac.ir

*Corresponding author