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Ulam-Hyers-Rassias stability for fuzzy fractional integral equations

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Abstract

In this paper, we study the fuzzy Ulam-Hyers-Rassias stability for two kinds of fuzzy fractional integral equations by employing the fixed point technique.

Keywords: Fuzzy Ulam-Hyers-Rassias stability, fuzzy fractional integral equations, fixed point theory.

1 Introduction

Ulam [40] posed the question: "*When can we assert that an approximate solution of a functional equation can be approximated by a solution of the corresponding equation?*". This question was first answered by [19] and after that this result was improved by [8, 9] and [35]. Ulam-Hyers stability or Ulam-Hyers-Rassias stability for differential equations has been discussed by a lot of researchers (see [20, 21, 34, 4]) in recent years. Jung [20] proved the Hyers-Ulam-Rassias stability of a Volterra integral equation of the second kind by using the fixed point method. Also, Jung et al. [21] investigated the approximate solutions of the delay differential equation with an initial condition and showed that they can be "approximated" by solutions of the equation that are constant on the interval delay. Huang et al.[18] discussed some classes of linear functional differential equations by direct method. In recent years, fractional calculus and fractional differential equation models have been applied in various areas of engineering, mathematics, physics and bioengineering, and other applied sciences. For some fundamental results in the theory of fractional differential equations involving Riemann-Liouville fractional derivative and Caputo derivative, we refer the reader to monographs of Kilbas [22]. Recently, investigating the existence and stability of solution to initial and boundary value problems for functional differential and integral equations and inclusions involving the Riemann-Liouville, Caputo, and Hadamard fractional derivatives and integrals has attracted the attention of several researchers (see [1, 41, 42, 44] and the references therein). In particular, Sousa et al. [41] presented some results about the Hyers-Ulam stability and the Hyers-Ulam-Rassias stability of fractional Volterra integral differential equations by using fixed-point approach with the ψ-Hilfer fractional derivative. In addition, by means of Banach fixed point theorem, Sousa et al.[42] studied the existence and Ulam-Hyers-Rassias stability of the integro-differential equations with noninstantaneous impulsive effect under $\psi-$Hilfer fractional derivative. In addition, by using fixed point approach, Sousa et al. [44] presented the sufficient conditions for existence and uniqueness of solution and $\delta-$Ulam-Hyers-Rassias stability of an impulsive fractional differential equations with the $\psi-$Hilfer fractional derivative concept.

Fuzzy fractional integral and differential equations have been investigated by mathematicians. The reader can refer to the papers [5, 6, 7, 12, 13, 15, 17, 14, 16, 25, 10, 32, 45, 33] and the references therein. Recently, Shen et al. [37] established a new connection between the Ulam stability and fuzzy differential equations. He studied the Ulam stability problems of the first order linear fuzzy differential equations under some suitable conditions and generalized

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Hukuhara differentiability. Using theory of fixed point introduced by [11], Shen et al. [39] proved the Ulam stability of fuzzy differential equations. Ahmadian et al. [3] presented four new sorts fuzzy E_α-Ulam stabilities such as: fuzzy E_α-Ulam-Hyers stability, fuzzy generalized E_α-Ulam-Hyers stability, fuzzy E_α-Ulam-Hyers-Rassias stability and fuzzy generalized E_α-Ulam-Hyers-Rassias stability. Authors also proved the E_α-Ulam stability for the integro-differential equations based on the fuzzy fractional Caputo derivative. Long el al. [29] presented new concepts of the Ulam-Hyers stability for the Darboux problems for nonlinear fractional partial integro-differential equations in fuzzy setting under Caputo generalized fractional differentiability in the infinity domain by using fixed point approach in a weighted metric with exponential functions. In addition, Long el al. [30] investigated the existence, uniqueness results and the Ulam-Hyers stability for the nonlocal problem for fractional partial fuzzy integro-differential equations in the framework of partially ordered generalized metric space of fuzzy valued functions.

Motivated by Shen et al. [37, 39], Ahmadian et al. [3] and Hoa [12], we will propose fuzzy Ulam-Hyers-Rassias stability for the two kinds of fuzzy fractional integral equations by employing the fixed point technique.

The rest of this paper is organized as follows: In section 2, we recall some notations of the fuzzy number space and the fixed point theorem. In Sections 3 and 4, the Ulam-Hyers-Rassias stability is investigated for fuzzy fractional integral equations with $w-$increasing and with $w-$decreasing, respectively. Finally, some examples are presented to illustrate our results in Section 5.

2 Preliminaries

In this section, we introduce some definitions, theorems and lemmas which are used in this paper. For more details, we can see papers [11, 37, 29, 32] and the monograph of [23].

Definition 2.1. [11] *A function $d : \mathbb{X} \times \mathbb{X} \to [0, +\infty)$ is called a generalized metric on X if and only if d satisfies*

(1) $d(x, y) = 0$ if and only if $x = y$;

(2) $d(x, y) = d(y, x)$ for all $x, y, z \in \mathbb{X}$;

(3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in \mathbb{X}$.

Theorem 2.2. [11] Let $d : \mathbb{X} \times \mathbb{X} \to [0, +\infty)$ be a generalized metric on $\mathbb{X}$ and $(\mathbb{X}, d)$ is a generalized complete metric space. Assume that $T : \mathbb{X} \to \mathbb{X}$ is a strictly contractive operator with the Lipschitz constant $L < 1$. If there exists a nonnegative integer n such that $d\big(T^{n+1}x, T^n x\big) < \infty$ for some $x \in \mathbb{X}$, then the following are true:

(i) the sequece $\{T^n x\}$ converges to a fixed point x^* of T;

(ii) x^* is the unique fixed point of T in $\mathbb{X}^* = \big\{y \in \mathbb{X} \,|\, d\big(T^n x, y\big) < \infty\big\}$;

(iii) if $y \in \mathbb{X}^*$, then we have $d(y, x^*) \leq \dfrac{1}{1-L} d\big(Ty, y\big)$.

Lemma 2.3. [39] *Let $\varphi : J \to [0, +\infty)$ be a continuous function. We define the set*

$$\mathbb{X} := \big\{x : J \to \mathbb{R}_{\mathscr{F}} \,|\, x \text{ is continuous function on } J\big\},$$

where $\mathbb{R}_{\mathscr{F}}$ is the space of fuzzy sets, equipped with the metric

$$d(x, y) = \inf\big\{\eta \in [0, +\infty) \cup \{+\infty\} \,|\, D\big(x(t), y(t)\big) \leq \eta\varphi(t),\ \forall t \in J\big\}.$$

Then, $(\mathbb{X}, d)$ is a complete generalized metric space.

Let $K_c(\mathbb{R}^d)$ denote the family of all nonempty, compact and convex subsets of $\mathbb{R}^d$. The addition and scalar multiplication in $K_c(\mathbb{R}^d)$ are defined as usual, *i.e.*, for $A, B \in K_c(\mathbb{R}^d)$ and $\lambda \in \mathbb{R}$,

$$A + B = \{a + b \mid a \in A, b \in B\}, \qquad \lambda A = \{\lambda a \mid a \in A\}.$$

The Hausdorff distance or Pompeiu-Hausdorff distance d_{H} in $K_c(\mathbb{R}^d)$ is defined as follows:

$$d_{\mathrm{H}}(A, B) = \max\Big\{\sup_{a\in A}\inf_{b\in B}\|a-b\|, \sup_{b\in B}\inf_{a\in A}\|a-b\|\Big\},$$

where $A, B \in K_c(\mathbb{R}^d)$, and $\|\cdot\|$ denotes usual Euclidean norm in $\mathbb{R}^d$.

Define $E^d = \{u : \mathbb{R}^d \to [0, 1]$ such that $u(z)$ satisfies (i)-(iv)$\}$ ([23]):

(i) u is normal, that is, there exists $z_0 \in \mathbb{R}^d$ such that $u(z_0) = 1$;

(ii) u is fuzzy convex, *i.e.*, $u(\lambda z_1 + (1-\lambda)z_2) \geq \min\{u(z_1), u(z_2)\}$ for any $z_1, z_2 \in \mathbb{R}^d$ and $\lambda \in [0,1]$;

(iii) u is upper semicontinuous;

(iv) $[u]^0 = \text{cl}\{z \in \mathbb{R}^d : u(z) > 0\}$ is compact, where "cl" denotes the closure in $(\mathbb{R}^d, \|\cdot\|)$.

The addition and scalar multiplication in fuzzy set space E^d, for $u, v \in E^d$ and $\alpha \in [0,1]$ are as following:

$$[u+v]^\alpha = [u]^\alpha + [v]^\alpha \quad \text{and} \quad [\lambda u]^\alpha = \lambda [u]^\alpha.$$

Let us denote by

$$D[u,v] = \sup\big\{d_{\text{H}}\big([u]^\alpha, [v]^\alpha\big) : 0 \leq \alpha \leq 1\big\},$$

the distance between u and v in E^d, where $d_{\text{H}}([u]^\alpha, [v]^\alpha)$ is the Pompeiu-Hausdorff distance between two sets $[u]^\alpha$, $[v]^\alpha$ of $K_c(\mathbb{R}^d)$. In fact (E^d, D) is a complete metric space.

For all $u_1, u_2, u_3, u_4 \in E^d$ and $\lambda \in \mathbb{R}$, we have the properties as following:

$$\begin{aligned} &D[u_1+u_3, u_2+u_3] = D[u_1, u_2],\\ &D[\lambda u_1, \lambda u_2] = |\lambda| D[u_1, u_2],\\ &D[u_1+u_3, u_2+u_4] \leq D[u_1, u_2] + D[u_3, u_4]. \end{aligned}$$

Let $u, v \in E^d$. If there exists $w \in E^d$ such that $u = v + w$, then w is called the Hukuhara difference of u, v and it is denoted by $u \ominus v$. Let us remark that $u \ominus v \neq u + (-1)v$.

Let us denote

$$E^d \ni \hat{0}(z) = \begin{cases} 1 & \text{if} \quad z = 0 \\ 0 & \text{if} \quad z \neq 0, \end{cases}$$

where 0 is the zero element of $\mathbb{R}^d$.

Remark 2.4. *Let $u_1, u_2, u_3, u_4 \in E^d$. We have*

(P1) If $u_1 \ominus u_2$ and $u_1 \ominus u_3$ exist, then $D[u_1 \ominus u_2, \hat{0}] = D[u_1, u_2]$ and $D[u_1 \ominus u_2, u_1 \ominus u_3] = D[u_2, u_3]$.

(P2) If $u_1 \ominus u_2$ and $u_3 \ominus u_4$ exist, then $D_0[u_1 \ominus u_2, u_3 \ominus u_4] = D[u_1 + u_4, u_2 + u_3]$.

(P3) If $u_1 \ominus u_2$ and $u_1 \ominus (u_2 + u_3)$ exist, then there exist $(u_1 \ominus u_2) \ominus u_3$ and $(u_1 \ominus u_2) \ominus u_3 = u_1 \ominus (u_2 + u_3)$.

(P4) If $u_1 \ominus u_2$, $u_1 \ominus u_3$ and $u_3 \ominus u_2$ exist, then there exist $(u_1 \ominus u_2) \ominus (u_1 \ominus u_3)$ and $(u_1 \ominus u_2) \ominus (u_1 \ominus u_3) = u_3 \ominus u_2$.

Definition 2.5. The generalized Hukuhara difference of two fuzzy sets $u, v \in E^d$ (gH-difference for short) is defined as follows:

$$u \ominus_{gH} v = w \Leftrightarrow \begin{cases} \quad (i)\ u = v + w, \\ \text{or } (ii)\ v = u + (-1)w. \end{cases}$$

Let $[0,a]$ be a compact interval in $\mathbb{R}_+$. Denote by $diam[u(t)]^\alpha$ the diameter of fuzzy set u, for $t \in [0,a]$. A function $u : [0,a] \to E^d$ is called w-increasing (w-decreasing) on $[0,a]$ if for every $\alpha \in [0,1]$ the function $t \mapsto diam[u(t)]^\alpha$ is nondecreasing (nonincreasing) on $[0,a]$. If u is w-increasing or w-decreasing on $[0,a]$, then we say that u is w-monotone on $[0,a]$. Denote $C([0,a], E^d)$ by the space of continuous fuzzy functions.

Definition 2.6. [32] *The fuzzy fractional integral of the order $\beta > 0$ of the measurable and integrable bounded fuzzy mapping $F : [0,a] \to E^d$ at $t \in (0,a)$ is the fuzzy set $(I_0^\beta F)(t) \in E^d$ defined by*

$$\big(I_0^\beta F\big)(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} F(s) ds,$$

where $\Gamma(\beta)$ is the well-known gamma function.

In the sequel, we shall investigate the following fuzzy fractional integral equation (FFIE):

$$u(t) \ominus_{gH} u_0 = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} F\big(t, s, u(s), u(\tau(s))\big) ds, \tag{1}$$

where $\beta \in (0,1)$, the mapping $F : [0,a] \times [0,a] \times E^d \times E^d \to E^d$ is continuous on $[0,a]$ and $\tau : [0,a] \to [0,a]$ is a continuous delay function which satisfies $\tau(t) \le t$ for all $t \in [0,a]$ and $u_0 \in E^d$.

Remark 2.7. We observe that:
If $u \in C([0,a], E^d)$ is such that $t \mapsto diam[u(t)]^\alpha$ is nondecreasing on $[0,a]$, then (1) can be written as

$$u(t) = u_0 + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} F\big(t, s, u(s), u(\tau(s))\big) ds, \quad t \in [0,a]. \tag{2}$$

If $u \in C([0,a], E^d)$ is such that $t \mapsto diam[u(t)]^\alpha$ is nonincreasing on $[0,a]$, then (1) can be written as

$$u(t) = u_0 \ominus (-1) \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} F\big(t, s, u(s), u(\tau(s))\big) ds, \quad t \in [0,a]. \tag{3}$$

In the sequel, our aim score is to present the results for the existence and the stability of the problem (1). The methods to solve these problems are quite similar. However, since the conditions for the existence of solutions of fuzzy fractional integral equations (2) and (3) are dissimilar, we shall present the two kinds (2) and (3) in two separate sections.

3 Fuzzy Ulam-Hyers-Rassias stability for FFIEs (2)

Firstly, we present the definitions of fuzzy Ulam-Hyers stability and fuzzy Ulam-Hyers-Rassias stability.

Definition 3.1. *We say that the equation (2) is the fuzzy Ulam-Hyers stability if there exists a constant $C_F > 0$ such that for each $\epsilon > 0$ and for each solution $v \in C([0,a], E^d)$ of the following inequality*

$$D\Big[v(t), v_0 + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} F\big(t, s, v(s), v(\tau(s))\big) ds\Big] \le \epsilon, \quad \forall t \in [0,a],$$

where $v_0 = u_0$, then there exists a unique solution $\widehat{u} : [0,a] \to E^d$ of the problem (2) which satisfies

$$\widehat{u}(t) = u_0 + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} F\big(t, s, \widehat{u}(s), \widehat{u}(\tau(s))\big) ds$$

and $D\big[\widehat{u}(t), v(t)\big] \le C_F \epsilon$, for any $t \in [0,a]$.

Definition 3.2. *Let $0 < \beta_0 < \beta < 1$ and $L^{\frac{1}{\beta_0}}([0,a], \mathbb{R}_+)$ be the space of real-valued measurable functions and $\frac{1}{\beta_0}$-integrable. We say that the equation (2) is the fuzzy Ulam-Hyers-Rassias stability with respect to $\varphi \in L^{\frac{1}{\beta_0}}([0,a], \mathbb{R}_+)$ if there exists a constant $C_{F,\varphi} > 0$ such that for each $\epsilon > 0$ and for each solution $v : [0,a] \to E^d$ of the following inequality*

$$D\Big[v(t), u_0 + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} F\big(t, s, v(s), v(\tau(s))\big) ds\Big] \le \varphi(t), \quad \forall t \in [0,a],$$

then there exists a unique solution $\widehat{u} : [0,a] \to E^d$ of the problem (2) which satisfies

$$\widehat{u}(t) = u_0 + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} F\big(t, s, \widehat{u}(s), \widehat{u}(\tau(s))\big) ds,$$

and $D\big[\widehat{u}(t), v(t)\big] \le C_{F,\varphi} \varphi(t)$, for any $t \in [0,a]$.

Remark 3.3. *We observe that Definition 3.2 $\Longrightarrow$ Definition 3.1.*

In the following theorem by the fixed point theory in a complete generalized space (see in [39]), we shall prove that FFIEs (2 is fuzzy Ulam-Hyers-Rassias stability on bounded interval.

Theorem 3.4. *Let $0 < \beta_0 < \beta < 1$ and we set $M = \dfrac{a^{\beta-\beta_0}}{\Gamma(\beta)}\left(\dfrac{1-\beta}{1-\beta_0}\right)^{1-\beta_0}$. Let L, K be non-negative constants with $0 < MLK < 1$. Assume that $\tau : [0,a] \to [0,a]$ is a continuous delay function which satisfies $\tau(t) \le t$ for all $t \in [0,a]$ and $F : [0,a] \times E^d \times E^d \to E^d$ is a continuous function which satisfies the following Lipschitz condition*

$$D\big[F(t,s,u(s),u(\tau(s)),F(t,s,v(s),v(\tau(s))\big] \le LD[u,v], \tag{4}$$

for any $t,s \in [0,a]$, $u,v \in E^d$. If a continuously $w-$increasing function $v : [0,a] \to E^d$ satisfies

$$D\Big[v(t), v_0 + \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}F\big(t,s,v(s),v(\tau(s))\big)ds\Big] \le \varphi(t), \tag{5}$$

for any $t \in [0,a]$, where $v_0 = u_0$ and $\varphi \in L^{\frac{1}{\beta_0}}([0,a],\mathbb{R}_+)$ which satisfies

$$\Bigg(\int_0^t \big(\varphi(s)\big)^{\frac{1}{\beta_0}}ds\Bigg)^{\beta_0} \le K\varphi(t), \tag{6}$$

for any $t \in [0,a]$, then there exists a unique solution $\widehat{u} : [0,a] \to E^d$ of the problem (2) which satisfies

$$\widehat{u}(t) = u_0 + \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}F\big(t,s,\widehat{u}(s),\widehat{u}(\tau(s))\big)ds, \tag{7}$$

and

$$D\big[\widehat{u}(t), v(t)\big] \le \frac{1}{1-MLK}\varphi(t), \quad \forall t \in [0,a]. \tag{8}$$

Theorem 3.5. *Let L, K be non-negative constants with $0 < LK < 1$. Assume that $\tau : [0,a] \to [0,a]$ is a continuous delay function which satisfies $\tau(t) \le t$ for all $t \in [0,a]$ and $F : [0,a] \times E^d \times E^d \to E^d$ is a continuous function which satisfies the following Lipschitz condition*

$$D\big[F(t,s,u(s),u(\tau(s)),F(t,s,v(s),v(\tau(s))\big] \le LD[u,v], \tag{9}$$

for any $t,s \in [0,a]$, $u,v \in E^d$. If a continuously $w-$increasing function $v : [0,a] \to E^d$ satisfies

$$D\Big[v(t), v_0 + \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}F\big(t,s,v(s),v(\tau(s))\big)ds\Big] \le \varphi(t),$$

for any $t \in [0,a]$, where $v_0 = u_0$ and $\varphi : [0,a] \to \mathbb{R}_+$ is a continuous function which satisfies

$$\left|\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\varphi(s)ds\right| \le K\varphi(t), \tag{10}$$

for any $t \in [0,a]$, then there exists a unique solution $\widehat{u} : [0,a] \to E^d$ of the problem (2) which satisfies

$$\widehat{u}(t) = u_0 + \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}F\big(t,s,\widehat{u}(s),\widehat{u}(\tau(s))\big)ds,$$

and $D\big[\widehat{u}(t), v(t)\big] \le \dfrac{1}{1-LK}\varphi(t), \quad \forall t \in [0,a]$.

Since the way to prove the assertions of Theorem 3.5 and Theorem 3.4 is quite similar, we shall prove Theorem 3.4.

Proof of Theorem 3.4. Let us consider the space of all continuous fuzzy functions $u : J \to E^d$ by

$$\mathbb{X} = \big\{u : J \to E^d \,|\, u \text{ is continuous on } [0, a]\big\},$$

equipped by the metric

$$d(u, v) = \inf \big\{C \in [0, +\infty) \cup \{+\infty\} \,|\, D\big[u(t), v(t)\big] \le C\varphi(t),\ \forall t \in J\big\}.$$

By Lemma 2.3, we observe that $(\mathbb{X}, d)$ is also a complete generalized metric space. We define an operator $\mathbb{Q} : \mathbb{X} \to \mathbb{X}$ by

$$\big(\mathbb{Q}u\big)(t) = u_0 + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} F\big(t, s, u(s), u(\tau(s))\big) ds, \quad \forall t \in [0, a]. \tag{11}$$

Because F is a continuous fuzzy function, the right hand side of (11) is also continuous on $[0, a]$. This yields that $\mathbb{Q}u$ is a continuous on $[0, a]$. So, the operator Q is well-defined.

To apply Theorem 2.2 in the proof of this theorem, we need the operator $\mathbb{Q}$ to be strict contractive on $\mathbb{X}$. For any $u, v \in \mathbb{X}$ and let $C_{u,v} \in [0, +\infty) \cup \{+\infty\}$ such that

$$d(u, v) \le C_{u,v}, \quad \forall t \in [0, a].$$

Then by the definition of d, we have

$$D\big[u(t), v(t)\big] \le C_{u,v}\varphi(t), \quad \forall t \in [0, a]. \tag{12}$$

From the definition of the operator $\mathbb{Q}$ and assumptions (4)-(6), we have the following estimation

$$\begin{aligned} D\big[\big(\mathbb{Q}u\big)(t), \big(\mathbb{Q}v\big)(t)\big] &= D\Big[\frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} F\big(t, s, u(s), u(\tau(s))\big) ds, \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} F\big(t, s, v(s), v(\tau(s))\big) ds\Big] \\ &\le \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} D\big[F\big(t, s, u(s), u(\tau(s))\big), F\big(t, s, v(s), v(\tau(s))\big)\big] ds \\ &\le \frac{L}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} D\big[u(s), v(s)\big] ds. \end{aligned} \tag{13}$$

Combining (12) and (13), we obtain for $t \in [0, a]$

$$\begin{aligned} D\big[\big(\mathbb{Q}u\big)(t), \big(\mathbb{Q}v\big)(t)\big] &\le \frac{LC_{uv}}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \varphi(s) ds \\ &\le \frac{LC_{uv}}{\Gamma(\beta)} \Big(\int_0^t (t-s)^{\frac{\beta-1}{1-\beta_0}} ds \Big)^{1-\beta_0} \Big(\int_0^t \big(\varphi(s)\big)^{\frac{1}{\beta_0}} ds \Big)^{\beta_0} \\ &\le \frac{1}{\Gamma(\beta)} \Big(\frac{1-\beta}{1-\beta_0} \Big)^{1-\beta_0} t^{\beta-\beta_0} LKC_{uv}\varphi(t) \le MLKC_{uv}\varphi(t), \end{aligned} \tag{14}$$

where $M := \dfrac{a^{\beta-\beta_0}}{\Gamma(\beta)} \Big(\dfrac{1-\beta}{1-\beta_0} \Big)^{1-\beta_0}$ with $0 < \beta_0 < \beta < 1$. That is, $d\big(\mathbb{Q}u, \mathbb{Q}v\big) \le MLKC_{uv}\varphi(t)$ for any $t \in [0, a]$. Hence, we can conclude that $d\big(\mathbb{Q}u, \mathbb{Q}v\big) \le MLKd(u, v)$ for any $u, v \in E^d$, where $0 < MLK < 1$. So the operator $\mathbb{Q}$ is a strictly contractive mapping on $\mathbb{X}$.

For an arbitrary $w \in \mathbb{X}$ and from the definition of $\mathbb{X}$ and $\mathbb{Q}$, it follows that there exists a constant $0 < C_w < \infty$ such that

$$D\big[\big(\mathbb{Q}w\big)(t), w(t)\big] = D\Big[u_0 + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} F\big(t, s, w(s), w(\tau(s))\big) ds, w(t)\Big] \le C_w\varphi(t),$$

for any $t \in [0,a]$, since F and w are bounded on $[0,a]$ and the minimum of $\varphi(t) > 0$ on $t \in [0,a]$. Then, we infer that $d\big(\mathbb{Q}w, w\big) \leq C_w < \infty$.

Therefore, according to (i) and (ii) of Theorem 2.3, there exists a continuously function $\widehat{u} : [0,a] \to E^d$ such that $\mathbb{Q}^n w \to \widehat{u}$ in the space $(\mathbb{X}, d)$ as $n \to \infty$ and $\mathbb{Q}\widehat{u} = \widehat{u}$, that is, $\widehat{u}$ satisfies the problem (7) for any $t \in [0,a]$. Now, we shall confirm that $\big\{u \in \mathbb{X} \,|\, d(w,u) < \infty\big\} = \mathbb{X}$. For an arbitrary $u \in E^d$, since u and w are bounded on $[0,a]$ and $\min_{t\in[0,a]} \varphi(t) > 0$, there exists a constant $0 < C_u < \infty$ such that $D\big[w(t), u(t)\big] \leq C_u\varphi(t)$, for any $t \in [0,a]$. Therefore, we have $d(w,u) < \infty$ for any $u \in E^d$, that is, $\big\{u \in \mathbb{X} \,|\, d(w,u) < \infty\big\} = \mathbb{X}$. By Theorem 2.3-(ii), we conclude that $\widehat{u}$ is the unique fixed point of $\mathbb{Q}$ on $\mathbb{X}$.

On the other hand, from the inequality (5) it follows that

$$d(u, \mathbb{Q}u) \leq 1. \tag{15}$$

Finally, By Theorem 2.3-(iii) and from the estimation (15), it implies that

$$d\big(\widehat{u}(t), u(t)\big) \leq \frac{1}{1-MLK} d\big(u, \mathbb{Q}u\big) \leq \frac{1}{1-MLK},$$

which means the estimation (8) holds true for any $t \in [0,a]$. This completes the proof.

4 Fuzzy Ulam-Hyers-Rassias stability for FFIEs (3)

Theorem 4.1. *Suppose that τ, φ and F satisfy all the conditions of Theorem (3.4) and let L, K be non-negative constants with $0 < LK < 1$. Assume that for each $t \in [0,a]$ and for each continuous fuzzy function $z : [0,a] \to E^d$, if the Hukuhara difference $z(0) \ominus (-1)\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} F\big(t, s, z(s), z(\tau(s))\big)ds$ exists and a continuously $w-$nonincreasing function $v : [0,a] \to E^d$ satisfies*

$$D\Big[v(t), v_0 \ominus (-1)\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} F\big(t, s, v(s), v(\tau(s))\big)ds\Big] \leq \varphi(t), \tag{16}$$

for any $t \in [0,a]$, where $v_0 = u_0$, then there exists a unique solution $\widehat{u} : [0,a] \to E^d$ of the problem (3) which satisfies

$$\widehat{u}(t) = u_0 \ominus (-1)\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} F\big(t, s, \widehat{u}(s), \widehat{u}(\tau(s))\big)ds,$$

and $D\big[\widehat{u}(t), v(t)\big] \leq \dfrac{1}{1-MLK}\varphi(t)$, for any $t \in [0,a]$, where $M := \dfrac{a^{\beta-\beta_0}}{\Gamma(\beta)}\left(\dfrac{1-\beta}{1-\beta_0}\right)^{1-\beta_0}$ with $0 < \beta_0 < \beta < 1$.

Theorem 4.2. *Suppose that τ, φ and F satisfy all the conditions of Theorem (3.5) and let L, K be non-negative constants with $0 < LK < 1$. Assume that for each $t \in [0,a]$ and each continuous fuzzy function $z : [0,a] \to E^d$, if the Hukuhara difference $z(0) \ominus (-1)\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} F\big(t, s, z(s), z(\tau(s))\big)ds$ exists and a continuously $w-$nonincreasing function $v : [0,a] \to E^d$ which satisfies,*

$$D\Big[v(t), v_0 \ominus (-1)\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} F\big(t, s, v(s), v(\tau(s))\big)ds\Big] \leq \varphi(t), \tag{17}$$

for any $t \in [0,a]$, where $v_0 = u_0$, then there exists a unique solution $\widehat{u} : [0,a] \to E^d$ of the problem (3) which satisfies

$$\widehat{u}(t) = u_0 \ominus (-1)\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} F\big(t, s, \widehat{u}(s), \widehat{u}(\tau(s))\big)ds, \tag{18}$$

and

$$D\big[\widehat{u}(t), v(t)\big] \leq \frac{1}{1-LK}\varphi(t), \quad t \in [0,a]. \tag{19}$$

Proof of Theorem 4.2. We consider the complete generalized space $(\mathbb{X}, d)$ defined as in the proof of Theorem 3.5. Define the operator $\mathbb{P} : \mathbb{X} \to \mathbb{X}$ as follows:

$$\big(\mathbb{P}u\big)(t) = u_0 \ominus (-1)\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} F\big(t, s, u(s), u(\tau(s))\big)ds, t \in [0, a]. \tag{20}$$

Since the function F is continuous on $[0, a]$ and the Hukuhara difference $u_0 \ominus (-1)\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} F\big(t, s, u(s), u(\tau(s))\big)ds$ exists, similarly to Theorem 3.5, it follows that $\mathbb{P}u$ is well-defined on $[0, a]$ or $\mathbb{P}u$ is continuous on $[0, a]$. Now, we observe that the operator $\mathbb{P}$ is strictly contractive on $\mathbb{X}$. Indeed, for any $u, v \in \mathbb{X}$ and let $C_{u,v} \in [0, +\infty) \cup \{+\infty\}$ be an arbitrary constant with $d(u, v) \leq C_{u,v}$ for $t \in [0, a]$, that is, let us assume that

$$D\big[u(t), v(t)\big] \leq C_{u,v}\varphi(t), \tag{21}$$

for $t \in [0, a]$. Furthermore, from (20), (21) and by the Lipschitz condition of F, we have the following estimation

$$\begin{aligned} D\big[\big(\mathbb{P}u\big)(t), \big(\mathbb{P}v\big)(t)\big] &= D\Big[u_0 \ominus (-1)\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} F\big(t, s, u(s), u(\tau(s))\big)ds, u_0 \ominus (-1)\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} F\big(t, s, v(s), v(\tau(s))\big)ds\Big] \\ &\leq \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} D\big[F\big(t, s, u(s), u(\tau(s))\big), F\big(t, s, v(s), v(\tau(s))\big)\big]ds \\ &\leq \frac{L}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} D\big[u(s), v(s)\big]ds, \quad \forall t \in [0, a]. \end{aligned} \tag{22}$$

By the assumption (10) and (22), (21), we obtain

$$D\big[\big(\mathbb{P}u\big)(t), \big(\mathbb{P}v\big)(t)\big] \leq \frac{L}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} D\big[u(s), v(s)\big]ds \leq \frac{LC_{u,v}}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\varphi(s)ds \leq LKC_{u,v}\varphi(t)$$

for $t \in [0, a]$. This means that $d\big(\mathbb{P}u, \mathbb{P}v\big) \leq LKC_{u,v}$. Therefore, we can conclude that $d\big(\mathbb{P}u, \mathbb{P}v\big) \leq LKd(u, v)$. Hence, the operator $\mathbb{P}$ is a strictly contractive mapping on $\mathbb{X}$ by the assumption $0 < LK < 1$. Similar to the proof of Theorem 3.4, we can show that each $w \in \mathbb{X}$ satisfies $d\big(\mathbb{P}w, w\big) < \infty$. Hence, by Theorem 2.3-(i), it implies that there exists a continuously function $\widehat{u} : [0, a] \to E^d$ such that $\mathbb{P}^n w \to \widehat{u}$ in $(\mathbb{X}, d)$ as $n \to \infty$, and such that $\mathbb{P}\widehat{u} = \widehat{u}$, that is, $\widehat{u}$ satisfies (18) for $t \in [0, a]$. Similar to the proof of Theorem 3.4, we observe that there exists a constant $C_w > 0$ such that $D\big[w(t), u(t)\big] \leq C_w$, for any $t \in [0, a]$. This means that $d(w, u) < \infty$ for each $u \in E^d$, or equivalently, $\big\{u \in \mathbb{X} \,|\, d(w, u) < \infty\big\} = \mathbb{X}$. Furthermore, by Theorem 2.3-(ii), we imply that $\widehat{u}$ is a unique continuous function which satisfies (18). Moreover, By Theorem 2.3-(iii) we also obtain

$$d\big(\widehat{u}(t), u(t)\big) \leq \frac{1}{1-LK} d\big(u, \mathbb{Q}u\big) \leq \frac{1}{1-LK},$$

which means the estimation (19) holds true for any $t \in [0, a]$. This completes the proof.

5 Examples

Example 5.1. Let $\beta \in (0, 1)$ and $M = \frac{1}{\Gamma(\beta)}\left(\frac{1-\beta}{1-\beta_0}\right)^{1-\beta_0}$ with $0 < \beta_0 < \beta < 1$. Consider the fractional fuzzy integral equation as follows:

$$u(t) = u_0 + \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} F(t, s, u(s), u(\tau(s)))ds, \quad t \in [0, 1], \tag{23}$$

where $u_0 = (-1, 0, 1) \in E^1$, $F(t, s, u(s), u(\tau(s))) = Lu(s^3)$, $L > 0$ and $\alpha(t) = t^3$ for $t, s \in [0, 1]$.

It is easy to see that $\alpha(t) = t^3 \leq t$ for any $t \in [0, 1]$ and for each $u, v \in E^1$, F satisfies Lipschitz condition with Lipschitz constant $L \in \big(0, \min\{1, M^{-1}\}\big)$.

Let $\varphi(t) = e^{-\beta t}$ for $t \in [0, 1]$. We obtain

$$\left(\int_0^t \big(\varphi(s)\big)^{\frac{1}{\beta_0}} ds\right)^{\beta_0} = \left(\int_0^t e^{-\frac{\beta s}{\beta_0}} ds\right)^{\beta_0} = \frac{\beta_0}{\beta}\left(1 - e^{-\frac{\beta}{\beta_0}t}\right) \leq e^{-\beta t},$$

for any $t \in [0,1]$. Hence, the inequality (6) holds with $K = 1$.

Furthermore, if a continuous w-nondecreasing function $v : [0,1] \to E^1$ satisfies

$$D\Big[v(t), (-1,0,1) + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} L v(s^3) ds\Big] \le e^{-\beta t} \tag{24}$$

for any $t \in [0,1]$, then by Theorem 3.4 there exists a unique solution of (23) such that $D\big[u(t), v(t)\big] \le \dfrac{e^{-\beta t}}{1-L}, \quad t \in [0,1]$.

Example 5.2. Let $\beta \in (0,1)$, $\tau > 0$ and consider $F : [0,a]\times[0,a]\times E^1 \times E^1 \to E^1$ defined as $F(t,s,u(s),u(\tau(s))) = u(t) + Lu(s-\tau)$, $L > 0$ and $u_0 \in E^1$. Suppose that the Hukuhara difference $u_0 \ominus (-1)\frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1}[u(t) + Lu(s-\tau)]ds$ exists on $[0,a^*]$. Let us consider the fuzzy fractional integral equation as following

$$u(t) = u_0 \ominus (-1)\frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1}[u(t) + Lu(s-\tau)]ds, \quad t \in [0,a^*]. \tag{25}$$

It is also easy to see that $\alpha(t) = t - \tau \le t$ for any $t \in [0,a^*]$ and for each $u, v \in E^1$, F satisfies Lipschitz condition.

Let $\varphi(t) = e^{-\beta t}$, $t \in [0,a^*]$ and for $C > 0$ does not depend on $\beta \in (0,1)$ and $\epsilon > 0$, and using Lemma 2.3 in [29] we have the following estimate

$$\left|\frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1}\varphi(s)ds\right| = \left|\frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1}e^{-\beta s}ds\right| \le \widehat{M} e^{-\beta t},$$

where $\widehat{M} := \dfrac{1}{\Gamma(\beta+1)}\left[2\left(\dfrac{C}{\beta^{\frac{1}{1+\epsilon}}}\right)^{\frac{\beta}{2}} + \dfrac{1}{\beta}\left(\dfrac{C}{\beta^{\frac{1}{1+\epsilon}}}\right)^{\beta}\right]$. If we choose K and β such that $\widehat{M} t^{\beta-1} - Ke^{-\beta t} \le 0$ has at least one positive solution on $[0,a]$, then we have

$$\left|\frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1}e^{-\beta s}ds\right| \le \widehat{M} e^{-\beta t} \le K e^{-\beta t}, \quad \forall t \in [0,a].$$

Now, it remains to choose a constant $L > 0$ such that $0 < LK < 1$. If a continuously w-nonincreasing function $v : [0,a^*] \to E^1$ satisfies

$$D\Big[v(t), u_0 \ominus (-1)\frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1}[v(t) + Lv(s-\tau)]ds\Big] \le e^{-\beta t}, \quad t \in [0,a^*], \tag{26}$$

then by Theorem 4.2 there exists a unique solution of (25) such that $D\big[u(t), v(t)\big] \le \dfrac{e^{-\beta t}}{1-KL}, t \in [0,a^*]$.

6 Conclusion

In the recent years, the study of the stability of solutions of differential equations is intensifying and several researchers have presented new and interesting results involving fractional derivatives in fuzzy setting (see [39, 37, 30, 31]). In this paper, we presented an investigation of the Ulam-Hyers stability of the fuzzy solution of fuzzy fractional integral equations by using the fixed point technique. This result can be used to study fractional fuzzy differential equations with other types of derivative concepts in fuzzy setting, for example, Riemann-Liouville, Caputo-Hadamard and Hadamard generalized Hukuhara differentiability (see [17, 7]).

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