

# Continuity of super- and sub-additive transformations of continuous functions

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## Abstract

We prove a continuity inheritance property for super- and sub-additive transformations of non-negative continuous multivariate functions defined on the domain of all non-negative points and vanishing at the origin. As a corollary of this result we obtain that super- and sub-additive transformations of continuous aggregation functions are again continuous aggregation functions.

*Keywords:* Super-additive transformation, sub-additive transformation, continuity inheritance, aggregation function.

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## 1 Introduction and preliminaries

Super-additive and sub-additive transformations of aggregation functions were introduced in [3] on the basis of a motivation originating in economy. Since various aspects of such functions and their transformations have been dealt with, respectively, in [1, 2, 6] and [3, 5, 7], in order to avoid repetitiousness we skip further introductory exposition and refer the interested reader to the quoted works and references therein for more details. Instead, in the interest of brevity, in this short contribution we will aim right to the point and introduce the basic concepts.

We begin by noting that the concept of super- and sub-additive transformation can be applied to a broader class of function than originally considered in [3]. To demonstrate this, let  $D$  be a ‘rectangular’ subset of  $[0, \infty[^d$  of the form  $D = \prod_{i=1}^d [0, a_i]$  for  $a_i > 0$ ,  $i = 1, 2, \dots, d$ . Let  $A: D \rightarrow [0, \infty[$  be an arbitrary function. The *super-additive transformation* of  $A$  is the mapping  $A^*: D \rightarrow [0, \infty[$  defined by

$$A^*(\mathbf{x}) = \sup \left\{ \sum_{j=1}^k A(\mathbf{x}^{(j)}) : \sum_{j=1}^k \mathbf{x}^{(j)} = \mathbf{x}, \mathbf{x}^{(j)} \in D, k \in \mathbb{N} \right\}.$$

Note that, as defined, this transformation might furnish an infinite value. If there exists at least one point  $\mathbf{x} \in D$  such that  $A^*(\mathbf{x})$  is unbounded we will say that  $A^*$  *escapes* at  $\mathbf{x}$ . It follows from [4] that if  $A^*$  escapes at some point  $\mathbf{x} \in D$ , then  $A^*$  escapes at every point of  $D$  distinct from the origin  $\mathbf{0}$ . Note also that if  $A(\mathbf{0}) > 0$ , then  $A^*$  automatically escapes at every point of  $D$ .

The *sub-additive transformation* of  $A$ , on the other hand, is defined by

$$A_*(\mathbf{x}) = \inf \left\{ \sum_{j=1}^k A(\mathbf{x}^{(j)}) : \sum_{j=1}^k \mathbf{x}^{(j)} = \mathbf{x}, \mathbf{x}^{(j)} \in D, k \in \mathbb{N} \right\}.$$

The sub-additive transformation  $A_*$  is well-defined for any  $A$  as above because of the simple fact that  $A_*$  is bounded above by  $A$ . Analogous definitions of  $A^*$  and  $A_*$  applies if  $D = [0, \infty[^d$ .

As already alluded to, the concepts of super- and sub-additive transformations were originally introduced in [3] for the more restricted *aggregation functions*, that is, functions  $A$  defined on  $D$  that are increasing in every coordinate and vanish at the origin.

In connection with any operator defined on a space of functions one of the first and natural questions is whether or not the operator preserves continuity. This is also the case of the two operators  $A \mapsto A^*$  and  $A \mapsto A_*$  considered in this paper. The question was actually already addressed in [4] in the special case of super- and sub-additive transformations of one-dimensional aggregation functions. The aim of this note is to show that continuity inheritance of  $A^*$  and  $A_*$  extends to *arbitrary*  $d$ -dimensional continuous functions  $A : D \rightarrow [0, \infty[$  as above (and not just to aggregation functions).

## 2 The result

We will use standard notational conventions. For example, for any set  $D$  as above and any  $\mathbf{x}, \mathbf{y} \in D$ , by  $\mathbf{x} \wedge \mathbf{y}$  we denote element-wise infimum of the two points; clearly,  $\mathbf{x} \wedge \mathbf{y} \in D$  and  $\mathbf{x} \wedge \mathbf{y} \leq \mathbf{x}, \mathbf{y}$ , where  $\leq$  denotes a partial order on  $D$  and is defined by  $\mathbf{x} \leq \mathbf{y}$  if and only if all coordinates of  $\mathbf{x}$  are less or equal to corresponding coordinates of  $\mathbf{y}$ . We will also use the  $L_\infty$  norm  $\|\cdot\|$  on  $D$  throughout.

Let us begin by proving an equivalent but somewhat more restrictive condition for uniform continuity of a real function defined on  $D$ , which may be known to specialists but we could not locate it in the literature.

**Proposition 2.1.** *Suppose that for every  $\eta > 0$  there exists a  $\gamma = \gamma(\eta) > 0$  such that for every  $\mathbf{z}, \mathbf{z}' \in D$  satisfying  $\mathbf{z}' \leq \mathbf{z}$ , the inequality  $\|\mathbf{z} - \mathbf{z}'\| < \gamma$  implies that  $|f(\mathbf{z}) - f(\mathbf{z}')| < \eta$ . Then,  $f$  is uniformly continuous on  $D$ .*

*Proof.* We need to show that for every  $\varepsilon > 0$  there exists a  $\delta > 0$  such that, for every  $\mathbf{x}, \mathbf{x}' \in D$ , from  $\|\mathbf{x} - \mathbf{x}'\| < \delta$  it follows that  $|f(\mathbf{x}) - f(\mathbf{x}')| < \varepsilon$ . Thus, let  $\varepsilon > 0$  be given. We let  $\eta = \varepsilon/2$  and choose  $\delta = \gamma(\eta)$ . Let now  $\mathbf{x}, \mathbf{x}' \in D$  be arbitrary points such that  $\|\mathbf{x} - \mathbf{x}'\| < \delta$ . Letting  $\mathbf{y} = \mathbf{x} \wedge \mathbf{x}'$ , note that  $\mathbf{y} \in D$  and, obviously,  $\|\mathbf{y} - \mathbf{x}\| < \delta$  and  $\|\mathbf{y} - \mathbf{x}'\| < \delta$ . Since  $\delta = \gamma(\eta)$ , by our assumption we have  $|f(\mathbf{y}) - f(\mathbf{x})| < \eta$ , and  $|f(\mathbf{y}) - f(\mathbf{x}')| < \eta$ , so that  $|f(\mathbf{x}) - f(\mathbf{x}')| < 2\eta = \varepsilon$ . This completes the proof.  $\square$

Before stating and proving our main result let us establish a few useful inequalities involving differences of the form  $\mathbf{x} - \mathbf{x}'$  for  $\mathbf{x}, \mathbf{x}' \in D$  as these will play an important role in what follows.

**Lemma 2.2.** *Let  $\mathbf{x}, \mathbf{x}', \mathbf{y} \in D$  be such that  $\mathbf{x}' \leq \mathbf{x}$  and  $\mathbf{y} \leq \mathbf{x}$ . Then it follows that  $\mathbf{0} \leq \mathbf{y} - \mathbf{x}' \wedge \mathbf{y} \leq \mathbf{x} - \mathbf{x}'$ .*

*Proof.* The assertion is obvious if  $\mathbf{x}' \leq \mathbf{y}$ , so in what follows we assume that this is not the case. Then, without loss of generality, we may assume that exactly the first  $m \geq 1$  coordinates of  $\mathbf{x}'$  are larger than those of  $\mathbf{y}$ , so that

$$\mathbf{x}' \wedge \mathbf{y} = (y_1, \dots, y_m, x'_{m+1}, \dots, x'_n)$$

where the last stretch of coordinates of  $\mathbf{x}'$  is empty if  $m = n$ . Now, from  $\mathbf{x}' \wedge \mathbf{y} \leq \mathbf{y}$  and  $\mathbf{x}', \mathbf{y} \leq \mathbf{x}$  it follows that

$$\mathbf{0} \leq \mathbf{y} - \mathbf{x}' \wedge \mathbf{y} = (0, \dots, 0, y_{m+1} - x'_{m+1}, \dots, y_n - x'_n) \leq (x_1 - x'_1, \dots, x_m - x'_m, x_{m+1} - x'_{m+1}, \dots, x_n - x'_n) = \mathbf{x} - \mathbf{x}'$$

which proves our claim.  $\square$

**Lemma 2.3.** *Let  $\mathbf{x}, \mathbf{x}' \in D$  be such that  $\mathbf{x}' \leq \mathbf{x}$ . Further, let  $\{\mathbf{x}^{(j)}\}_{j=1}^k$  be a sequence of points from  $D$  such that  $\sum_{j=1}^k \mathbf{x}^{(j)} = \mathbf{x}$ . Then, there exists a sequence  $\{\mathbf{y}^{(j)}\}_{j=1}^k$  of points of  $D$  such that*

$$\sum_{j=1}^k \mathbf{y}^{(j)} = \mathbf{x}' \quad \text{and} \quad \mathbf{0} \leq \mathbf{x}^{(j)} - \mathbf{y}^{(j)} \leq \mathbf{x} - \mathbf{x}'$$

for every  $j = 1, 2, \dots, k$ .

*Proof.* Consider, e.g.,  $y_i^{(j)} = x_i^{(j)} \frac{x'_i}{x_i}$ , for  $i = 1, 2, \dots, d$  and  $j = 1, 2, \dots$  with a convention that  $0/0 = 1$ .  $\square$

With this preparation we are ready to state and prove our main result.

**Theorem 2.4.** *Let  $A : D \rightarrow [0, \infty[$  be a continuous function. If  $A^*$  does not escape, then  $A^*$  is continuous.*

*Proof.* It is sufficient to prove that  $A^*$  is uniformly continuous on  $D$ . Note that  $D$  is of the form  $\prod_{i=1}^d [0, a_i]$  for  $a_i > 0$  for  $i = 1, 2, \dots, d$ , as considered before. Let an arbitrary  $\varepsilon > 0$  be given, and let  $\mathbf{x} \neq \mathbf{0}$  be an arbitrary point in  $D$ . By definition of the super-additive transformation  $A^*$ , there exists a  $k$ -tuple  $\{\mathbf{x}^{(j)}\}_{j=1}^k$  of points  $\mathbf{x}^{(j)} \in D$  such that

$$\sum_{j=1}^k \mathbf{x}^{(j)} = \mathbf{x} \quad \text{and} \quad A^*(\mathbf{x}) - \sum_{j=1}^k A(\mathbf{x}^{(j)}) < \frac{\varepsilon}{2}. \quad (1)$$

For  $k = k(\varepsilon)$  from (1) and for the quantity  $\varepsilon' = \varepsilon/(2k)$ , by the assumed (uniform) continuity of our function  $A$  on  $D$  and by virtue of Proposition 2.1 there exists  $\delta = \delta(\varepsilon')$  such that, for any  $\mathbf{z}, \mathbf{z}' \in D$  such that  $\mathbf{z}' \leq \mathbf{z}$  and  $\|\mathbf{z} - \mathbf{z}'\| < \delta$ , one has  $|A(\mathbf{z}) - A(\mathbf{z}')| < \varepsilon'$ .

Let now  $\mathbf{x}' \in D$  be an arbitrary point distinct from  $\mathbf{x}$  such that  $\mathbf{x}' \leq \mathbf{x}$  and  $\|\mathbf{x} - \mathbf{x}'\| < \delta$ . To demonstrate uniform continuity of  $A^*$  it is sufficient to show, by Proposition 2.1, that  $A^*(\mathbf{x}) - A^*(\mathbf{x}') < \varepsilon$ , which we will do in what follows.

By Lemma 2.3, to our given  $\mathbf{x}, \mathbf{x}' \in D$  with  $\mathbf{x}' \leq \mathbf{x}$  and to the sequence  $\{\mathbf{x}^{(j)}\}_{j=1}^k$  of points in  $D$  obtained from (1) there exists a sequence  $\{\mathbf{y}^{(j)}\}_{j=1}^k$  of points of  $D$  summing to  $\mathbf{x}'$  and such that  $\mathbf{0} \leq \mathbf{x}^{(j)} - \mathbf{y}^{(j)} \leq \mathbf{x} - \mathbf{x}'$  for every  $j = 1, 2, \dots, k$ . Since, by definition of  $A^*$ , one has  $\sum_{j=1}^k A(\mathbf{y}^{(j)}) \leq A^*(\mathbf{x}')$ , the difference  $A^*(\mathbf{x}) - A^*(\mathbf{x}')$  may be bounded above (with the help of a pair of canceling sums) by

$$A^*(\mathbf{x}) - A^*(\mathbf{x}') \leq \left( A^*(\mathbf{x}) - \sum_{j=1}^k A(\mathbf{x}^{(j)}) \right) + \left( \sum_{j=1}^k A(\mathbf{x}^{(j)}) - \sum_{j=1}^k A(\mathbf{y}^{(j)}) \right). \quad (2)$$

On the right-hand side of (2) the leftmost term in brackets is simply bounded above by  $\varepsilon/2$  according to (1). To deal with the rightmost term of (2) we may apply, for every  $j = 1, 2, \dots, k$  the inequality  $\mathbf{0} \leq \mathbf{x}^{(j)} - \mathbf{y}^{(j)} \leq \mathbf{x} - \mathbf{x}'$  mentioned above, and together with our assumption  $\|\mathbf{x} - \mathbf{x}'\| < \delta$  and the implied inequality  $|A(\mathbf{x}^{(j)}) - A(\mathbf{y}^{(j)})| < \varepsilon'$  for  $\|\mathbf{x}^{(j)} - \mathbf{y}^{(j)}\| < \delta$  by continuity of  $A$  we may modify (2) into

$$A^*(\mathbf{x}) - A^*(\mathbf{x}') \leq \frac{\varepsilon}{2} + \sum_{j=1}^k \left( A(\mathbf{x}^{(j)}) - A(\mathbf{y}^{(j)}) \right) \leq \frac{\varepsilon}{2} + k \frac{\varepsilon}{2k} = \varepsilon.$$

By all the assumptions made, this shows that  $A^*$  is continuous if  $A$  is. □

**Corollary 2.5.** *If  $A: [0, \infty[^d \rightarrow [0, \infty[$  is a continuous function such that  $A^*$  does not escape, then  $A^*$  is continuous.*

*Proof.* It is sufficient to prove that  $A^*$  is uniformly continuous for any region  $D$  of the form  $\prod_{i=1}^d [0, a_i]$ ,  $a_i > 0$  for  $i = 1, 2, \dots, d$ , and that is guaranteed by the previous theorem. □

**Corollary 2.6.** *If  $A: [0, \infty[^d \rightarrow [0, \infty[$  is a continuous aggregation function and  $A^*$  does not escape, then its super-additive transformation  $A^*$  is continuous aggregation function.*

**Example 2.7.** *Consider a one-dimensional aggregation function  $A_h: [0, \infty[ \rightarrow [0, \infty[$  given by  $A_h(x) = \sqrt{1+x} - 1 + h\mathbb{K}_{[1, \infty[}$  for  $h \geq 0$ . Even though this function is discontinuous for every  $h > 0$ , it is easy to see that  $A_h^*$  is a continuous aggregation function for  $h \leq 3/2 - \sqrt{2}$ ; in particular,  $A_h^*(x) = x/2$ . For  $h > 3/2 - \sqrt{2}$  the function  $A_h^*$  is obviously discontinuous at  $x = 1$ .*

A similar result holds for inheritance of continuity of  $A$  to its sub-additive transformation  $A_*$ . To prove it, it is sufficient to modify Lemma 1 by replacing the symbol  $\wedge$  by  $\vee$ .

**Theorem 2.8.** *If  $A: D \rightarrow [0, \infty[$  is a continuous function then  $A_*$  is also continuous.*

**Corollary 2.9.** *If  $A: [0, \infty[^d \rightarrow [0, \infty[$  is a continuous aggregation function then its sub-additive transformation  $A_*$  is continuous aggregation function.*

### 3 Summary

In this contribution we extended the continuity inheritance property by super- and sub-additive transformations of one-dimensional aggregation functions (established in [4]) to the multi-dimensional case. Our results not only contribute to broadening of theoretical foundations of aggregation functions and their transformations but may also be of future use in applications of these transformations.

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