FUZZY PSEUDOTOPOLOGICAL HYPERGROUPOIDS

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Abstract. On a hypergroupoid one can define a topology such that the hyperoperation is pseudocontinuous or continuous. In this paper we extend this concepts to the fuzzy case. We give a connection between the classical and the fuzzy (pseudo)continuous hyperoperations.

1. Introduction

The theory of algebraic hyperstructures was initiated by Marty [28] in 1934 during the 8th Congress of Scandinavian Mathematicians, when he defined the hypergroups, giving some of their applications to non-commutative groups, algebraic functions, rational fractions. No sooner had Zadeh [31] introduced the fuzzy sets, than the reconsideration of the concept of classical mathematics began. Since then many mathematicians have studied the connections between fuzzy sets and hyperstructures; their works can be classified into three groups: the papers that consider *crisp hyperoperations* defined through fuzzy sets ([4, 5, 6, 7, 10, 11, 12, 30]), the papers which deal with *fuzzy hyperoperations*, that is hyperoperations which map a pair of elements of a set X to a fuzzy subset of X ([8, 24]), and finally papers that concern the *fuzzy hyperalgebras*, a direct extension of the concept of fuzzy algebras (fuzzy (sub)groups, fuzzy lattices, fuzzy rings etc.) ([13, 14, 16, 17, 18, 35]).

Using the structure of a fuzzy topological space and that of a fuzzy group (introduced by Rosenfeld [29]), Foster [20] defined the concept of *fuzzy topological group*. Later, Ma and Yu [27] changed Foster's definition in order to make sure that an ordinary topological group is a special case of a fuzzy topological group.

Inspired by the definition of the *topological groupoid*, Ameri [1] and later on Hoskova [22] have introduced the notions of *(strongly) pseudocontinuous* and *continuous hyperoperations* on a topological space. In this work we extend these notions on a *fuzzy topological space*.

The paper is organized as follows. First we review some basic definitions and results on fuzzy topological spaces, grouped in Section 2, and on topological hypergroupoids in Section 3. The main results are presented in Section 4: we define fuzzy (pseudo)continuous hyperoperations, we give relations between fuzzy continuous and continuous hyperoperations, between fuzzy continuous and fuzzy pseudocontinuous hyperoperations, respectively. Moreover, we see when a product hypergroupoid is a fuzzy pseudotopological hypergroupoid.

Key words and phrases: Hypergroupoid, (Fuzzy) pseudocontinuous hyperoperation, (Fuzzy) continuous hyperoperation, Fuzzy topological space.

2. **Fuzzy Topological Spaces**

In this section we recall some basic results on the fuzzy topological spaces that we use in the following.

Let X be a nonempty set. A fuzzy set A in X is characterized by a membership function $\mu_A : X \longrightarrow [0,1]$. We denote by $FS(X)$ the set of all fuzzy sets on X.

In this paper we use the definition of a fuzzy topological space given by Chang [2].

Definition 2.1. [2] A *fuzzy topology* on a set X is a collection $\mathcal{T}$ of fuzzy sets in X satisfying

(i) $\underline{0} \in \mathcal{T}$ and $\underline{1} \in \mathcal{T}$ (where $\underline{0}, \underline{1} : X \longrightarrow [0,1]$, $\underline{0}(x) = 0, \underline{1}(x) = 1$, for any $x \in X$).

(ii) If $A_1, A_2 \in \mathcal{T}$, then $A_1 \cap A_2 \in \mathcal{T}$.

(iii) If $A_i \in \mathcal{T}$ for any $i \in I$, then $\bigcup_{i \in I} A_i \in \mathcal{T}$,

where $\mu_{A_1 \cap A_2}(x) = \mu_{A_1}(x) \wedge \mu_{A_2}(x)$ and $\mu_{\bigcup_{i\in I} A_i}(x) = \bigvee_{i \in I} \mu_{A_i}(x)$.

The pair $(X, \mathcal{T})$ is called a *fuzzy topological space.*

In the definition of a fuzzy topology of Lowen [26], the condition (i) is substituted by

(i') for all $c \in [0,1]$, $k_c \in \mathcal{T}$, where $\mu_{k_c}(x) = c$, for any $x \in X$.

Example 2.2. We give some examples of fuzzy topologies on a set X.

(i) The family $\mathcal{T} = \{\underline{0}, \underline{1}\}$ is called the *indiscrete* fuzzy topology on X.

(ii) The family of all fuzzy sets in X is called the *discrete* fuzzy topology on X.

(iii) If τ is a topology on X, then the collection $\mathcal{T} = \{A_O \mid O \in \tau\}$ of fuzzy sets in X, where μ_{A_O} is the characteristic function of the open set O, is a fuzzy topology on X.

(iv) The collection of all constant fuzzy sets in X is a fuzzy topology on X, where a constant fuzzy set A in X has the membership function μ_A defined as follows: $\mu_A : X \longrightarrow [0,1]$, $\mu_A(x) = k$, for any $x \in X$, with k a fix constant in $[0,1]$.

Definition 2.3. [2] Given two topological spaces $(X, \mathcal{T})$ and $(Y, \mathcal{U})$, a function $f : X \longrightarrow Y$ is *fuzzy continuous* if, for any fuzzy set $A \in \mathcal{U}$, the inverse image $f^{-1}[A]$ belongs to $\mathcal{T}$, where $\mu_{f^{-1}[A]}(x) = \mu_A(f(x))$, for any $x \in X$.

Proposition 2.4. [2] *A composition of fuzzy continuous functions is a fuzzy continuous function.*

Definition 2.5. [25] A *base* for a fuzzy topological space $(X, \mathcal{T})$ is a subcollection $\mathcal{B}$ of $\mathcal{T}$ such that each member A of $\mathcal{T}$ can be written as the union of members of $\mathcal{B}$.

A natural question is: 'How to judge whether some fuzzy subsets just form a base of some fuzzy topological space?' We have the following rule:

Proposition 2.6. [25] *A family $\mathcal{B}$ of fuzzy sets in X is a base for a fuzzy topology $\mathcal{T}$ on X if and only if it satisfies the following conditions:*

(i) *For any* $A_1, A_2 \in \mathcal{B}$*, we have* $A_1 \cap A_2 \in \mathcal{B}$.

(ii) $\bigcup_{A \in \mathcal{B}} A = \underline{1}$.

If $(X_1, \mathcal{T}_1)$ and $(X_2, \mathcal{T}_2)$ are fuzzy topological spaces, we can speak about the product fuzzy topological space $(X_1 \times X_2, \mathcal{T}_1 \times \mathcal{T}_2)$, where the product fuzzy topology is given by a base as in the following result, which can be generalized to a family of fuzzy topological spaces.

Proposition 2.7. [25] *Let* $(X_1, \mathcal{T}_1)$ *and* $(X_2, \mathcal{T}_2)$ *be fuzzy topological spaces. The product fuzzy topology* $\mathcal{T}$ *on the product space* $X = X_1 \times X_2$ *has the set of product fuzzy sets of the form* $A_1 \times A_2$*, with* $A_i \in \mathcal{T}_i$*,* $i = 1, 2$*, and* $\mu_{A_1 \times A_2}(x_1, x_2) = \mu_{A_1}(x_1) \wedge \mu_{A_2}(x_2)$ *as a base.*

Proposition 2.8. [20] *Let* $\{(X_i, \mathcal{T}_i)\}_{i \in I}, \{(Y_i, \mathcal{U}_i)\}_{i \in I}$ *be two families of fuzzy topological spaces and* $(X, \mathcal{T}), (Y, \mathcal{U})$ *the respective product fuzzy topological spaces. For each* $i \in I$*, let* $f_i : (X_i, \mathcal{T}_i) \longrightarrow (Y_i, \mathcal{U}_i)$*. Then the product mapping* $f = \times f_i : (x_i) \longrightarrow (f_i(x_i))$ *of* $(X, \mathcal{T})$ *into* $(Y, \mathcal{U})$ *is fuzzy continuous if* f_i *is fuzzy continuous, for each* $i \in I$.

3. **Topological Hypergroupoids**

A *hypergroupoid* is a nonempty set H together with a *hyperoperation*, i.e. a map $\circ : H \times H \longrightarrow \mathcal{P}^*(H)$, where by $\mathcal{P}^*(H)$ we mean the set of all nonempty subsets of H. The image of the pair $(x, y) \in H \times H$ is denoted by $x \circ y$. Moreover, for any $x \in H$ and $A, B \subseteq H$, by $A \circ B$, $A \circ x$ and $x \circ B$ we mean $A \circ B = \bigcup_{\substack{a \in A \\ b \in B}} a \circ b$, $A \circ x = A \circ \{x\}$ and $x \circ B = \{x\} \circ B$, respectively.

For more details on Hypergroups Theory and some of their applications the reader is refereed to [3], [9].

In various branches of mathematics we encounter important examples of topologico-algebraical structures like topological groupoids, groups, rings, fields etc. Our aim is to generalize the concept of topological groupoid to that of topological hypergroupoid.

Assume that $(G, \cdot)$ is a groupoid and τ is a topology on G. If the binary operation $\cdot : G \times G \to G$ is continuous, the triple $(G, \cdot, \tau)$ is called a *topological groupoid*.

The continuity of "$\cdot$" means that for every $O \in \tau$ the set $\{(x, y) \in G \times G : x \cdot y \in O\}$ is open in $G \times G$ (endowed with ordinary product topology $\tau \times \tau$).

Inspired by the definition of the topological groupoid the following notions has been introduced in [22].

Definition 3.1. [22] Let $(H, \cdot)$ be a hypergroupoid and (H, τ) be a topological space. The hyperoperation "$\cdot$" is called *pseudocontinuous* or for short *p-continuous* if for every $O \in \tau$ the set $O_* = \{(x, y) \in H \times H : x \cdot y \subseteq O\}$ is open in $H \times H$.

It is easy to verify that the hyperoperation "$\cdot$" is *p-continuous* if and only if, for any $O \in \tau$ and any pair $(x, y) \in H \times H$ such that $x \cdot y \subseteq O$ there exist $U, V \in \tau$, $x \in U$, $y \in V$ such that $u \cdot v \subseteq O$ for any $u \in U$ and $v \in V$.

As the hyperoperation "$\cdot$" is a mapping from $H \times H$ to $\mathcal{P}^*(H)$ giving topologies on H and $\mathcal{P}^*(H)$ we can speak about the continuity of "$\cdot$". Unfortunately, if a topology is given on H, there is no standard straightforward way of generating a topology on $\mathcal{P}^*(H)$. This leads to the following definition.

Definition 3.2. [22] Let $(H, \cdot)$ be a hypergroupoid, (H, τ) be a topological space and τ_* be a topology on $\mathcal{P}^*(H)$. The hyperoperation "$\cdot$" is called *τ_*-continuous* if the mapping $\cdot : H \times H \to \mathcal{P}^*(H)$ is continuous with respect to the topologies $\tau \times \tau$ and τ_*.

Definition 3.3. [22] Let $(H, \cdot)$ be a hypergroupoid, (H, τ) be a topological space and τ_* be a topology on $\mathcal{P}^*(H)$. The triple $(H, \cdot, \tau)$ is called a *pseudotopological hypergroupoid* if the hyperoperation "$\cdot$" is pseudocontinuous.
The quadruple $(H, \cdot, \tau, \tau_*)$ is called *τ_*-topological hypergroupoid* if the hyperoperation "$\cdot$" is τ_*-continuous.

Now, we will show that given a topology τ on H it is possible to find a topology τ_* in such a way that τ_*-continuity means just p-continuity.

First let us recall that a *base* $\mathcal{B}$ of a topological space (X, τ), i.e., a family of open sets such that any $O \in \tau$ is a union of sets from $\mathcal{B}$, has the following two properties [19]:

(B1) For any U_1, $U_2 \in \mathcal{B}$ and every point $x \in U_1 \cap U_2$ there exists a $U_3 \in \mathcal{B}$ such that $x \in U \subset U_1 \cap U_2$.

(B2) For every $x \in X$ there exists a $U_3 \in \mathcal{B}$ such that $x \in U$.

Conversely, any system of sets fulfilling (B1) and (B2) determines a unique topology τ on X such that this system is a base of τ.

In the following, a new topology on $\mathcal{P}^*(H)$ is introduced.

Lemma 3.4. [22] *Let (H, τ) be a topological space. Then the family $\mathcal{U}$ consisting of all sets $S_V = \{U \in \mathcal{P}^*(H) : U \subseteq V\}, V \in \tau$ is a base of a topology $\tau_{\mathcal{U}}$ on $\mathcal{P}^*(H)$.*

Definition 3.5. [22] The topology $\tau_{\mathcal{U}}$ on $\mathcal{P}^*(H)$ from the previous lemma is called *upper topology* on $\mathcal{P}^*(H)$ induced by the topology τ on H.

The pseudotopological hypergroupoids are characterized in the following result.

Theorem 3.6. [22] *Let $(H, \cdot)$ be a hypergroupoid and (H, τ) be a topological space. Then the triple $(H, \cdot, \tau)$ is a pseudotopological hypergroupoid if and only if the quadruple $(H, \cdot, \tau, \tau_{\mathcal{U}})$ is $\tau_{\mathcal{U}}$-topological hypergroupoid.*

4. Fuzzy Pseudotopological Hypergroupoids

Definition 4.1. Let $(H, \circ)$ be a hypergroupoid, $\mathcal{T}$ and $\mathcal{U}$ be fuzzy topologies on H and $\mathcal{P}^*(H)$, respectively. The hyperoperation "$\circ$" is called *$\mathcal{U}$-fuzzy continuous* if the map $\circ : H \times H \longrightarrow \mathcal{P}^*(H)$ is fuzzy continuous with respect to the fuzzy topologies $\mathcal{T} \times \mathcal{T}$ and $\mathcal{U}$.

Example 4.2. Let $(H, \circ)$ be a hypergroupoid and $\mathcal{T}$ be a fuzzy topology on H. We denote by h the map $\circ : H \times H \longrightarrow \mathcal{P}^*(H)$, i.e. $h(x, y) = x \circ y$.

(i) Let $\mathcal{U} = \{\underline{0}, \underline{1}\}$ be the indiscrete fuzzy topology on $\mathcal{P}^*(H)$. Then the hyperoperation "$\circ$" is $\mathcal{U}$-fuzzy continuous.
Indeed, if $A \in \mathcal{U}$, then $A = \underline{0}$ or $A = \underline{1}$. If $A = \underline{0}$, then $\mu_{h^{-1}[A]}(x, y) = \mu_A(x \circ y) = 0$, for any $(x, y) \in H^2$, thus $h^{-1}[A] = \underline{0} \times \underline{0} \in \mathcal{T} \times \mathcal{T}$.
Similarly, if $A = \underline{1}$, then $h^{-1}[A] = \underline{1} \times \underline{1} \in \mathcal{T} \times \mathcal{T}$.

(ii) Let $\mathcal{U}$ be an arbitrary fuzzy topology on $\mathcal{P}^*(H)$. If $\mathcal{T} = FS(H)$ is the discrete fuzzy topology on H, then $\mathcal{T} \times \mathcal{T}$ is the discrete fuzzy topology on $H \times H$. Thus, for any $A \in \mathcal{U}$, $h^{-1}[A] \in \mathcal{T} \times \mathcal{T}$, and so the hyperoperation "$\circ$" is $\mathcal{U}$-fuzzy continuous.

(iii) Let $(H, \circ)$ be a finite hypergroupoid and α be a fixed real number in the interval $(1/2, 1)$. Consider on H the fuzzy topology $\mathcal{T} = \{A \mid \mu_A(z) \in [0, \alpha] \cup \{1\}, \forall z \in H\}$ and on $\mathcal{P}^*(H)$ the fuzzy topology $\mathcal{U} = \{U_k \mid \mu_{U_k}(X) = k = \textit{constant}, \forall X \subset H, k \in [0, \alpha]\} \cup \{\underline{1}\}$. Moreover, denote by h the hyperoperation "$\circ$" as in Example 4.2. It is obvious that, for any $k \in [0, \alpha]$, there exists a constant fuzzy set $A_k \in \mathcal{T}$ such that $\mu_{A_k}(z) = k$, for any $z \in H$. Thus, for any $U_k \in \mathcal{U}$ and any $(x, y) \in H^2$, we obtain:

$$\mu_{h^{-1}[U_k]}(x, y) = \mu_{U_k}(x \circ y) = k = \mu_{A_k}(x) \wedge \mu_{A_k}(y) = \mu_{A_k \times A_k}(x, y),$$

and so $h^{-1}[U_k] = A_k \times A_k \in \mathcal{T} \times \mathcal{T}$, that is, the hyperoperation "$\circ$" is $\mathcal{U}$-fuzzy continuous.

For any topology τ on a set X, we denote by $\mathcal{T}_c$ the fuzzy topology formed with the characteristic functions of the open sets of τ (see Example 2.2). In the following result we give a relation between the continuity and fuzzy continuity of a hyperoperation.

Proposition 4.3. *Let $(H, \circ)$ be a hypergroupoid, τ and τ^* be topologies on H and $\mathcal{P}^*(H)$, respectively. Let $\mathcal{T}_c$ and $\mathcal{U}_c$ be the fuzzy topologies on H and $\mathcal{P}^*(H)$, respectively, generated by τ and τ^*, respectively. The hyperoperation "$\circ$" is τ^*-continuous if and only if it is $\mathcal{U}_c$-fuzzy continuous.*

Proof. We denote by h the mapping that assigns the hyperproduct $x \circ y$ to the pair (x, y).

Sufficiency. Let $A^*_X \in \mathcal{U}_c$, with $X \in \tau^*$, i.e. $\mu_{A^*_X} : \mathcal{P}^*(H) \longrightarrow [0, 1]$, $\mu_{A^*_X}(Y) = \begin{cases} 1, & \text{if } Y \in X \\ 0, & \text{if } Y \notin X \end{cases}$. Since the hyperoperation "$\circ$" is τ^*-continuous, it follows that $h^{-1}(X) \in \tau \times \tau$; thus there exist $O_1, O_2 \in \tau$ such that $h^{-1}(X) = O_1 \times O_2 = \{(x, y) \in H \times H \mid x \circ y \in X\}$. Then $A_{O_1} \times A_{O_2} \in \mathcal{T}_c \times \mathcal{T}_c$, with $\mu_{A_{O_1} \times A_{O_2}}(x, y) = \mu_{A_1}(x_1) \wedge \mu_{A_2}(y) = \begin{cases} 1, & \text{if } (x, y) \in O_1 \times O_2 \\ 0, & \text{if } (x, y) \notin O_1 \times O_2 \end{cases} = \begin{cases} 1, & \text{if } x \circ y \in X \\ 0, & \text{if } x \circ y) \notin X \end{cases} = \mu_{A^*_X}(x \circ y) = \mu_{A^*_X}(h(x, y)) = \mu_{h^{-1}[A^*_X]}(x, y)$. Therefore $h^{-1}[A^*_X] = A_{O_1} \times A_{O_2} \in \mathcal{T}_c \times \mathcal{T}_c$ and then the hyperoperation "$\circ$" is $\mathcal{U}_c$-fuzzy continuous.

Necessity. Let $X \in \tau^*$; then $A^*_X \in \mathcal{U}_c$. Since the hyperoperation "$\circ$" is $\mathcal{U}_c$-fuzzy continuous, it follows that $h^{-1}[A^*_X] \in \mathcal{T}_c \times \mathcal{T}_c$; therefore, there exist $O_1, O_2 \in \tau$ such that $h^{-1}[A^*_X] = A_{O_1} \times A_{O_2}$, with $\mu_{A_{O_1} \times A_{O_2}}(x, y) = \mu_{A^*_X}(x \circ y) =$

$\begin{cases} 1, & \text{if } (x,y) \in O_1 \times O_2 \\ 0, & \text{if } (x,y) \notin O_1 \times O_2 \end{cases} = \begin{cases} 1, & \text{if } x \circ y \in X \\ 0, & \text{if } x \circ y) \notin X \end{cases}$. Therefore, $(x,y) \in O_1 \times O_2$ if and only if $x \circ y \in X$, that is, $h^{-1}(X) = O_1 \times O_2 \in \tau \times \tau$. So the hyperoperation "$\circ$" is τ^*-fuzzy continuous. $\square$

Proposition 4.4. *Let $(H, \mathcal{T})$ be a fuzzy topological space. Then the family $\mathcal{B} = \{\tilde{A} \in FS(\mathcal{P}^*(H)) \mid A \in \mathcal{T}\}$, where $\mu_{\tilde{A}}(X) = \bigwedge_{x \in X} \mu_A(x)$, is a base for a fuzzy topology $\mathcal{T}^*$ on $\mathcal{P}^*(H)$.*

Proof. $\mathcal{B}$ is a base for a fuzzy topology on $\mathcal{P}^*(H)$ because it satisfies the conditions of Proposition 2.6:

(i) For any $\tilde{A_1}, \tilde{A_2} \in \mathcal{B}$, with $A_1, A_2 \in \mathcal{T}$, it follows that $\tilde{A_1} \cap \tilde{A_2} \in \mathcal{B}$, because $\tilde{A_1} \cap \tilde{A_2} = \widetilde{A_1 \cap A_2}$ and $A_1 \cap A_2 \in \mathcal{T}$.
Indeed, for any $X \in \mathcal{P}^*(H)$, we have

$$\begin{aligned} \mu_{\widetilde{A_1 \cap A_2}}(X) = & \bigwedge_{x \in X} \mu_{A_1 \cap A_2}(x) = \bigwedge_{x \in X} (\mu_{A_1}(x) \wedge \mu_{A_2}(x)) = \\ & = \left(\bigwedge_{x \in X} \mu_{A_1}(x) \right) \wedge \left(\bigwedge_{x \in X} \mu_{A_2}(x) \right) = \mu_{\tilde{A_1}}(X) \wedge \mu_{\tilde{A_2}}(X) = \\ & = \mu_{\tilde{A_1} \cap \tilde{A_2}}(X). \end{aligned}$$

(ii) Since $\underline{1} \in \mathcal{T}$ it follows that $\mu_{\tilde{\underline{1}}}(X) = 1$, for any $X \in \mathcal{P}^*(H)$ and thus $\bigcup_{\tilde{A} \in \mathcal{B}} = \tilde{\underline{1}}$.

$\square$

Definition 4.5. Let $(H, \circ)$ be a hypergroupoid endowed with a fuzzy topology $\mathcal{T}$. The hyperoperation "$\circ$" is called *fuzzy pseudocontinuous* (or briefly *fuzzy p-continuous*) if, for any $A \in \mathcal{T}$, the fuzzy set A_* in $H \times H$ belongs to $\mathcal{T} \times \mathcal{T}$, where $\mu_{A_*}(x,y) = \bigwedge_{u \in x \circ y} \mu_A(u)$.
The triple $(H, \circ, \mathcal{T})$ is called a *fuzzy pseudotopological hypergroupoid* if the hyperoperation "$\circ$" is fuzzy p-continuous.

Now, we characterize a fuzzy pseudotopological hypergroupoid $(H, \circ, \mathcal{T})$ using the $\mathcal{T}^*$-fuzzy continuity of the hyperoperation "$\circ$", where the fuzzy topology $\mathcal{T}^*$ is that one given in Proposition 4.4.

Theorem 4.6. *Let $(H, \circ)$ be a hypergroupoid and $\mathcal{T}$ be a fuzzy topology on H. Then the triple $(H, \circ, \mathcal{T})$ is a fuzzy pseudotopological hypergroupoid if and only if the hyperoperation "$\circ$" is $\mathcal{T}^*$-fuzzy continuous.*

Proof. We denote again by H the mapping that assigns the hyperproduct $x \circ y$ to the pair (x,y). For any $\tilde{A} \in \mathcal{T}^*$, with $A \in \mathcal{T}$, we have

$$\mu_{h^{-1}[\tilde{A}]}(x,y) = \mu_{\tilde{A}}(x \circ y) = \bigwedge_{u \in x \circ y} \mu_A(u) = \mu_{A_*}(x,y),$$

thus $h^{-1}[\tilde{A}] = A_*$ and then h is fuzzy p-continuous if and only if it $\mathcal{T}^*$-fuzzy continuous. $\square$

Example 4.7. Let $(H, \circ)$ be the finite hypergroupoid defined by: $x \circ x = \{x\}$ and $x \circ y = \{x, y\}$, for any $x, y \in H$. Let $\mathcal{T}$ be a fuzzy topology on H and let $\mathcal{T}^*$ be the fuzzy topology on $\mathcal{P}^*(H)$ given in Proposition 4.4. Then the triple $(H, \circ, \mathcal{T})$ is a fuzzy pseudotopological hypergroupoid. It is enough to prove that the hyperoperation " $\circ$ " is $\mathcal{T}^*$-fuzzy continuous.
Indeed, denote by h the hyperoperation " $\circ$ " as in Example 4.2 and let $\tilde{A}$ be an arbitrary fuzzy set in $\mathcal{T}^*$. For any $(x, y) \in H^2$, we have

$$\mu_{h^{-1}[\tilde{A}]}(x, y) = \mu_{\tilde{A}}(h(x, y)) = \bigwedge_{u \in x \circ y} \mu_A(u) = \mu_A(x) \wedge \mu_A(y) = \mu_{A \times A}(x, y)$$

and therefore $h^{-1}[\tilde{A}] = A \times A \in \mathcal{T} \times \mathcal{T}$.

We now briefly discuss products of fuzzy pseudotopological hypergroupoids.

Proposition 4.8. *Let $(H_1, \mathcal{T}_1)$ and $(H_2, \mathcal{T}_2)$ be two fuzzy topological spaces. We denote $H = H_1 \times H_2$ and $\mathcal{T} = \mathcal{T}_1 \times \mathcal{T}_2$. Then the mapping $\alpha : (H, \mathcal{T}) \times (H, \mathcal{T}) \longrightarrow (H_1 \times H_1, \mathcal{T}_1 \times \mathcal{T}_1) \times (H_2 \times H_2, \mathcal{T}_2 \times \mathcal{T}_2)$, defined by $\alpha((x_1, x_2), (y_1, y_2)) = ((x_1, y_1), (x_2, y_2))$ is fuzzy continuous.*

Proof. We prove that, for any fuzzy set $F \in (\mathcal{T}_1 \times \mathcal{T}_1) \times (\mathcal{T}_2 \times \mathcal{T}_2)$, the inverse image $\alpha^{-1}[F] \in (\mathcal{T}_1 \times \mathcal{T}_2) \times (\mathcal{T}_1 \times \mathcal{T}_2)$.

Let $F = (A_1 \times B_1) \times (A_2 \times B_2) \in (\mathcal{T}_1 \times \mathcal{T}_1) \times (\mathcal{T}_2 \times \mathcal{T}_2)$. Then

$$\begin{aligned} \mu_{\alpha^{-1}[F]}((u_1, u_2), (v_1, v_2)) &= \mu_F(\alpha((u_1, u_2), (v_1, v_2))) = \mu_F((u_1, v_1), (u_2, v_2)) = \\ &= \mu_{A_1}(u_1) \wedge \mu_{B_1}(v_1) \wedge \mu_{A_2}(u_2) \wedge \mu_{B_2}(v_2) = \\ &= \mu_{A_1 \times A_2}(u_1, u_2) \wedge \mu_{B_1 \times B_2}(v_1, v_2) = \\ &= \mu_{(A_1 \times A_2) \times (B_1 \times B_2)}((u_1, u_2), (v_1, v_2)), \end{aligned}$$

therefore $\alpha^{-1}[F] = (A_1 \times A_2) \times (B_1 \times B_2) \in (\mathcal{T}_1 \times \mathcal{T}_2) \times (\mathcal{T}_1 \times \mathcal{T}_2)$. $\square$

Let $(H_1, \circ_1)$ and $(H_2, \circ_2)$ be two hypergroupoids. The product hypergroupoid $(H_1 \times H_2, \otimes)$ has the hyperoperation defined by $(x_1, x_2) \otimes (y_1, y_2) = (x_1 \circ_1 y_1, x_2 \circ_2 y_2)$, for any $(x_1, x_2), (y_1, y_2) \in H_1 \times H_2$.

Theorem 4.9. *If $(H_1, \circ_1, \mathcal{T}_1)$ and $(H_2, \circ_2, \mathcal{T}_2)$ are fuzzy pseudotopological hypergroupoids, then the product hypergroupoid $(H_1 \times H_2, \otimes, \mathcal{T}_1 \times \mathcal{T}_2)$ is a fuzzy pseudotopological hypergroupoid.*

Proof. We denote by h_1 the mapping which assigns the hyperproduct $x_1 \circ_1 y_1$ to the pair $(x_1, y_1) \in H_1 \times H_1$, by h_2 the mapping which assigns the hyperproduct $x_2 \circ_2 y_2$ to the pair $(x_2, y_2) \in H_2 \times H_2$ and by h the mapping which assigns the hyperproduct $(x_1, x_2) \otimes (y_1, y_2)$ to the pair $((x_1, x_2), (y_1, y_2)) \in (H_1 \times H_2) \times (H_1 \times H_2)$.

Since the hypergroupoids $(H_1, \circ_1, \mathcal{T}_1)$ and $(H_2, \circ_2, \mathcal{T}_2)$ are fuzzy pseudotopological, it results that the mappings h_1 and h_2 are fuzzy p-continuous and, moreover, by Theorem 4.6, they are $\mathcal{T}_1^*$, respectively $\mathcal{T}_2^*$-fuzzy continuous. Therefore, the product mapping
$h_1 \times h_2 : (H_1 \times H_1, \mathcal{T}_1 \times \mathcal{T}_1) \times (H_2 \times H_2, \mathcal{T}_2 \times \mathcal{T}_2) \longrightarrow (\mathcal{P}^*(H_1 \times H_2), \mathcal{T}_1^* \times \mathcal{T}_2^*)$,

defined by $h_1 \times h_2((x_1, y_1), (x_2, y_2)) = (h_1(x_1, y_1), h_2(x_2, y_2))$, is $\mathcal{T}_1^* \times \mathcal{T}_2^*$-fuzzy continuous, by Proposition 2.8. Similarly, by Proposition 4.7, the mapping α is $(\mathcal{T}_1 \times \mathcal{T}_1) \times (\mathcal{T}_2 \times \mathcal{T}_2)$-fuzzy continuous.

It is easy to see that $h = (h_1 \times h_2) \circ \alpha$ and therefore h is $\mathcal{T}_1^* \times \mathcal{T}_2^*$-fuzzy continuous, by Proposition 2.4, that is, h is fuzzy p-continuous. □

5. **Conclusions**

In this paper we have extended the results obtained by Ameri [1] and Hoskova [22] about topological hypergroupoids to the case of fuzzy topological hypergroupoids. This work could be continued in order to introduce the notion of fuzzy topological hypergroup as a generalization of a fuzzy topological group in the sense of Foster [20] or in the sense of Ma and Yu [27].

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