

T-FUZZY CONGRUENCES AND T-FUZZY FILTERS OF A BL-ALGEBRA

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ABSTRACT. In this note, we introduce the concept of a fuzzy filter of a BL-algebra, with respect to a t-norm briefly, T-fuzzy filters, and give some related results. In particular, we prove Representation Theorem in BL-algebras. Then we generalize the notion of a fuzzy congruence (in a BL-algebra) was defined by Lianzhen et al. to a new fuzzy congruence, specially with respect to a t-norm. We prove that there is a correspondence bijection between the set of all T-fuzzy filters of a BL-algebra and the set of all T-fuzzy congruences in that BL-algebra. Next, we show how T-fuzzy filters induce T-fuzzy congruences, and construct a new BL-algebras, called quotient BL-algebras, and give some homomorphism theorems.

1. Introduction

BL-algebras are the algebraic structure for Hájek basic logic [4]. Most familiar example of a BL-algebra is the unit interval $[0,1]$ endowed with the structure induced by a continuous t-norm. In 1958, Chang [3] introduced the concept of an MV-algebra which is one of the most classes of BL-algebras. Turunen [10] introduced the notion of an implicative filter and a Boolean filter and proved that these notions are equivalent to BL-algebras. Boolean filters are an important class of filters, because the quotient BL-algebra induced by these filters are Boolean algebras. In [7] and [6], Lianzhen et al. introduced the concept of a fuzzy filter and fuzzy congruence in a BL-algebra and gave some related results. He proved that the fuzzy quotient algebras induced by fuzzy Boolean filters are Boolean algebras.

Now, in this paper, by considering [1], [6], [7], and the notion of a t-norm we define the notion of a fuzzy congruence with respect to a t-norm and give some related results.

2. Preliminaries

In this section, we give some fundamental definitions and results. For more details, we refer the readers to the references.

Definition 2.1. A BL-algebra is an algebra $(B, \vee, \wedge, *, \rightarrow, 0, 1)$ of type $(2,2,2,2,0,0)$ such that

(BL1) $(B, \vee, \wedge, 0, 1)$ is a bounded lattice,

(BL2) $(B, *, 1)$ is a commutative monoid,

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- (BL3) $c \leq a \rightarrow b$ if and only if $a * c \leq b$, for all $a, b, c \in B$,
 (BL4) $a \wedge b = a * (a \rightarrow b)$,
 (BL5) $(a \rightarrow b) \vee (b \rightarrow a) = 1$.

Example 2.2. For $x, y \in [0, 1]$ define $x * y = \min\{x, y\}$ and

$$x \rightarrow y = \begin{cases} 1 & x \leq y, \\ y & x > y. \end{cases}$$

Then $([0, 1], \vee, \wedge, *, \rightarrow, 0, 1)$ is a *BL*-algebra

In any *BL*-algebra the following hold:

- (2.1) $x \leq y$ if and only if $x \rightarrow y = 1$.
 (2.2) $x * (x \rightarrow y) \leq y$.
 (2.3) $x \leq y \rightarrow (x * y)$.
 (2.4) $x * y \leq x \wedge y$.
 (2.5) $(x \rightarrow y) * (y \rightarrow z) \leq x \rightarrow z$.
 (2.6) $x \rightarrow x^- = x^{-^-} \rightarrow x^-$, where $x^- = x \rightarrow 0$.
 (2.7) If $x \vee x^- = 1$, then $x \wedge x^- = 0$.
 (2.8) $x \rightarrow (y \rightarrow z) = (x * y) \rightarrow z$.
 (2.9) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$.
 (2.10) $y \leq (y \rightarrow x) \rightarrow x$.
 (2.11) $x \leq y$ implies $y \rightarrow z \leq x \rightarrow z$ and $z \rightarrow x \leq z \rightarrow y$.
 (2.12) $y \rightarrow x \leq (z \rightarrow y) \rightarrow (z \rightarrow x)$.
 (2.13) $x \rightarrow y \leq (y \rightarrow z) \rightarrow (x \rightarrow z)$.
 (2.14) $x \vee y = [(x \rightarrow y) \rightarrow y] \wedge [(y \rightarrow x) \rightarrow x]$.
 (2.15) $(x \vee y) \rightarrow z = (x \rightarrow z) \wedge (y \rightarrow z)$.

Hájek [4] defined a *filter* of a *BL*-algebra \mathcal{B} to be a nonempty subset F such that (i) for $a, b \in F$, $a * b \in F$, (ii) if $a \in F$ and $a \leq b$ then $b \in F$.

Theorem 2.3. [4] Let F be a filter of \mathcal{B} . Then the binary relation \equiv_F which is defined by

$$x \equiv_F y \text{ if and only if } x \rightarrow y \in F \text{ and } y \rightarrow x \in F$$

is a congruence relation on \mathcal{B} . Define $\bullet, \rightarrow, \sqcup, \sqcap$ on \mathcal{B}/F , the set of all congruence classes of \mathcal{B} , as follows:

$$[x] \bullet [y] = [x * y], [x] \rightarrow [y] = [x \rightarrow y], [x] \sqcup [y] = [x \vee y], [x] \sqcap [y] = [x \wedge y].$$

Then $(\mathcal{B}/F, \bullet, \rightarrow, \sqcup, \sqcap, \mathcal{B}, F)$ is a *BL*-algebra.

A function $T : [0, 1] \times [0, 1] \rightarrow [0, 1]$ satisfying the following properties is called a *t-norm*: for all $x, y, z \in [0, 1]$,

- (T1) $T(x, 1) = x$,
 (T2) $T(x, y) \leq T(x, z)$, if $y \leq z$,
 (T3) $T(x, y) = T(y, x)$,

$$(T4) \quad T(x, T(y, z)) = T(T(x, y), z).$$

Most familiar t-norms are:

- $T_D(x, y) = \begin{cases} x \wedge y & ; \quad x \vee y = 1, \\ 0 & ; \quad \text{otherwise.} \end{cases}$ (Drastic product)
- $T_L(x, y) = 0 \vee (x + y - 1).$ (Łukasiewicz)
- $T_P(x, y) = xy.$ (Product)
- $T_M(x, y) = x \wedge y.$ (Minimum)
- $T_{nM}(x, y) = \begin{cases} 0 & ; \quad x + y \leq 1, \\ x \wedge y & ; \quad \text{otherwise.} \end{cases}$ (Nilpotent minimum)

where $\wedge = \min$ and $\vee = \max$.

Every t-norm T satisfies the inequality $T_D(x, y) \leq T(x, y) \leq T_M(x, y)$.

For a t-norm T , $a \in [0, 1]$ is called an *idempotent* if $T(a, a) = a$.

It is obvious that 0 and 1 are idempotent elements of any t-norm which are called the *trivial* idempotent elements of T . For T_M , the idempotent elements are any element of $[0, 1]$, and for t-norm T_{nM} , the set $\{0\} \cup (0.5, 1]$ consists of all idempotent elements of T_{nM} .

Theorem 2.4. (*Representation Theorem*) Let $\{B_\alpha : \alpha \in [0, 1]\}$ be a class of subsets of a nonempty set X . The necessary and sufficient condition for the existence of fuzzy subset A of X with $A_\alpha = B_\alpha$, for all $\alpha \in [0, 1]$, is that for every $M \subseteq [0, 1]$,

$$(2.16) \quad B_{\bigvee_{\alpha \in M} \alpha} = \bigcap_{\alpha \in M} B_\alpha.$$

Remark 2.5. If $\{B_\alpha : \alpha \in [0, 1]\}$ is a class of subsets of a nonempty set X for which (2.16) holds for every $M \subseteq [0, 1]$, then

- (i) $B_0 = X$,
- (ii) if $\alpha, \beta \in [0, 1]$ are such that $\alpha \leq \beta$, then $B_\beta \subseteq B_\alpha$.

Notation. From now on, in this paper, \mathcal{B} denotes a BL-algebra.

3. T-fuzzy Filters

In [6], Lie et al. defined a fuzzy filter of \mathcal{B} as a fuzzy subset f for which every level subset f_t , $t \in [0, 1]$, is empty or a filter of \mathcal{B} , and proved that a fuzzy subset f is a fuzzy filter if and only if

- $(\forall x \in \mathcal{B}) \quad f(1) \geq f(x)$,
- $(\forall x, y \in \mathcal{B}) \quad f(y) \geq \min\{f(x), f(x \rightarrow y)\}$.

Now, we generalize this concept.

Definition 3.1. A fuzzy subset \mathfrak{F} of \mathcal{B} is said to be a *T-fuzzy filter* if

- $(\forall x \in \mathcal{B}) \quad \mathfrak{F}(1) \geq \mathfrak{F}(x)$,
- $(\forall x, y \in \mathcal{B}) \quad \mathfrak{F}(y) \geq T(\mathfrak{F}(x), \mathfrak{F}(x \rightarrow y))$.

Note 3.2. Subsequently, in this paper, any T_M -fuzzy filter is called a fuzzy filter.

Example 3.3. [6] Let $\mathcal{B} = \{0, a, b, 1\}$, where $0 < a < b < 1$ with the operations $x \wedge y = \min\{x, y\}$, $x \vee y = \max\{x, y\}$ and $*$ and \rightarrow as the following tables:

$*$	0	a	b	1	\rightarrow	0	a	b	1
0	0	0	0	0	0	1	1	1	1
a	0	0	a	a	a	a	1	1	1
b	0	a	b	b	b	0	a	1	1
1	0	a	b	1	1	0	a	b	1

Then $(\mathcal{B}, \vee, \wedge, *, \rightarrow, 0, 1)$ is a BL-algebra. If we define a fuzzy subset \mathfrak{F} by $\mathfrak{F}(1) = t_1$, $\mathfrak{F}(b) = t_2$ and $\mathfrak{F}(0) = \mathfrak{F}(a) = t_3$ (with $0 \leq t_3 < t_2 < t_1 \leq 1$), then \mathfrak{F} is a fuzzy filter of \mathcal{B} . Moreover, all nonempty level subsets, $\{1\}$ (for $t_2 < t \leq t_1$), $\{1, b\}$ (for $t_3 < t \leq t_2$), and \mathcal{B} (for $t \leq t_3$) are filters of \mathcal{B} .

Example 3.4. Let $\mathcal{B} = [0, 1]$. Then \mathcal{B} together with the ordinary maximum and minimum, $x * y = \min\{x, y\}$ and $x \rightarrow y = 1$ if $x \leq y$, otherwise $x \rightarrow y = y$ is a BL-algebra. It is easy to verify that $[n, 1]$ (with $n \in [0, 1]$) is a filter of \mathcal{B} . So, the fuzzy subset \mathfrak{F} with $\mathfrak{F}(x) = x$, for all $x \in \mathcal{B}$, is a fuzzy filter of \mathcal{B} for which $\mathfrak{F}_x = [x, 1]$, for all $x \in \mathcal{B}$.

Example 3.5. Let $\mathcal{B} = \{0, a, b, 1\}$ be the BL-algebra as in Example 3.3, and define a fuzzy subset \mathfrak{F} of \mathcal{B} by $\mathfrak{F}(0) = 2/3$, $\mathfrak{F}(a) = \mathfrak{F}(b) = 4/5$, and $\mathfrak{F}(1) = 5/6$. It is easy to verify that \mathfrak{F} is a T_P -fuzzy filter of \mathcal{B} , but it is not a fuzzy filter, because $\mathfrak{F}(0) = 2/3 \not\geq 4/5 = \min\{\mathfrak{F}(a), \mathfrak{F}(a \rightarrow 0)\}$.

Definition 3.6. A T -fuzzy filter \mathfrak{F} of \mathcal{B} is called *imaginable* if every element of the image of \mathfrak{F} is an idempotent element of T .

Obviously, any fuzzy filter is imaginable.

The following theorem demonstrates the connection between imaginable T -fuzzy filters and fuzzy filters of a BL-algebra.

Proposition 3.7. A fuzzy subset of \mathcal{B} is a fuzzy filter if and only if it is an imaginable T -fuzzy filter of \mathcal{B} .

Proof. Let \mathfrak{F} be an imaginable T -fuzzy filter of \mathcal{B} . It suffices to prove that $\mathfrak{F}(y) \geq \mathfrak{F}(x) \wedge \mathfrak{F}(x \rightarrow y)$. For this, let $x, y \in \mathcal{B}$. Then,

$$\begin{aligned} \mathfrak{F}(x) \wedge \mathfrak{F}(x \rightarrow y) &= T(\mathfrak{F}(x) \wedge \mathfrak{F}(x \rightarrow y), \mathfrak{F}(x) \wedge \mathfrak{F}(x \rightarrow y)) \\ &\leq T(\mathfrak{F}(x), \mathfrak{F}(x \rightarrow y)) \\ &\leq \mathfrak{F}(y) \end{aligned}$$

The converse is obvious. □

Theorem 3.8. Let \mathfrak{F} be a fuzzy subset of \mathcal{B} . Then, \mathfrak{F} is an imaginable T -fuzzy filter if and only if it is imaginable and

$$(3.1) \quad (\forall a, b, c \in \mathcal{B}) \quad a \rightarrow (b \rightarrow c) = 1 \Rightarrow \mathfrak{F}(c) \geq T(\mathfrak{F}(a), \mathfrak{F}(b)).$$

Proof. Assume that \mathfrak{F} is an imaginable T -fuzzy filter of \mathcal{B} , and $a, b, c \in \mathcal{B}$ are such that $a \rightarrow (b \rightarrow c) = 1$. Then

$$\begin{aligned} \mathfrak{F}(c) &\geq T(\mathfrak{F}(b), \mathfrak{F}(b \rightarrow c)) \geq T(\mathfrak{F}(b), T(\mathfrak{F}(a), \mathfrak{F}(a \rightarrow (b \rightarrow c)))) \\ &= T(\mathfrak{F}(b), T(\mathfrak{F}(a), \mathfrak{F}(1))) \geq T(\mathfrak{F}(b), T(\mathfrak{F}(a), \mathfrak{F}(a))) = T(\mathfrak{F}(b), \mathfrak{F}(a)). \end{aligned}$$

Conversely, suppose that (3.1) holds, for $a, b, c \in \mathcal{B}$. Since, $a \rightarrow 1 = 1$, then $a \rightarrow (a \rightarrow 1) = 1$ and so by hypothesis, $\mathfrak{F}(1) \geq T(\mathfrak{F}(a), \mathfrak{F}(a)) = \mathfrak{F}(a)$. Now, let $a, b \in \mathcal{B}$. Since, $(a \rightarrow b) \rightarrow (a \rightarrow b) = 1$, $\mathfrak{F}(b) \geq T(\mathfrak{F}(a \rightarrow b), \mathfrak{F}(a))$, proving \mathfrak{F} is a T -fuzzy filter. \square

Corollary 3.9. *Let \mathfrak{F} be a fuzzy subset of \mathcal{B} . Then, \mathfrak{F} is an imaginable T -fuzzy filter if and only if it is imaginable and*

$$(3.2) \quad (\forall a, b, c \in \mathcal{B}) \quad a * b \leq c \Rightarrow \mathfrak{F}(c) \geq T(\mathfrak{F}(a), \mathfrak{F}(b)).$$

Theorem 3.10. *Let \mathfrak{F} be a fuzzy subset of \mathcal{B} . Then, \mathfrak{F} is an imaginable T -fuzzy filter if and only if it is imaginable and for all $x, y \in \mathcal{B}$,*

- (i) $x \leq y$ implies $\mathfrak{F}(x) \leq \mathfrak{F}(y)$,
- (ii) $\mathfrak{F}(x * y) \geq T(\mathfrak{F}(x), \mathfrak{F}(y))$.

Proof. (i) suppose that \mathfrak{F} is an imaginable T -fuzzy filter of \mathcal{B} and $x \leq y$, for $x, y \in \mathcal{B}$. Since, $x * x \leq x \leq y$, by Corollary 3.9, $\mathfrak{F}(y) \geq T(\mathfrak{F}(x), \mathfrak{F}(x)) = \mathfrak{F}(x)$. Now, because $x * y \leq x * y$, by (3.2), $\mathfrak{F}(x * y) \geq T(\mathfrak{F}(x), \mathfrak{F}(y))$.

Conversely, suppose that $a * b \leq c$, for $a, b, c \in \mathcal{B}$. By (i) and (ii), $\mathfrak{F}(c) \geq \mathfrak{F}(a * b) \geq T(\mathfrak{F}(a), \mathfrak{F}(b))$. By Corollary 3.9, the proof is complete. \square

Proposition 3.11. *Any imaginable T -fuzzy filter \mathfrak{F} of \mathcal{B} satisfies the following properties.*

- (3.3) $x \leq y$ implies $\mathfrak{F}(x) \leq \mathfrak{F}(y)$.
- (3.4) $\mathfrak{F}(x * y) \geq T(\mathfrak{F}(x), \mathfrak{F}(y))$.
- (3.5) $\mathfrak{F}(x \wedge y) \geq T(\mathfrak{F}(x), \mathfrak{F}(y))$.
- (3.6) $\mathfrak{F}(x \vee y) \geq T(\mathfrak{F}(x), \mathfrak{F}(y))$.
- (3.7) $\mathfrak{F}(x \rightarrow y) = \mathfrak{F}(1)$ implies $\mathfrak{F}(x) \leq \mathfrak{F}(y)$.
- (3.8) $\mathfrak{F}(x \rightarrow y) \leq \mathfrak{F}((y \rightarrow z) \rightarrow (x \rightarrow z))$.
- (3.9) $\mathfrak{F}(x \rightarrow y) \leq \mathfrak{F}((z \rightarrow x) \rightarrow (z \rightarrow y))$.
- (3.10) $\mathfrak{F}((x * z) \rightarrow (y * z)) \geq \mathfrak{F}(x \rightarrow y)$.
- (3.11) $\mathfrak{F}(x \rightarrow z) \geq T(\mathfrak{F}(x \rightarrow y), \mathfrak{F}(y \rightarrow z))$.
- (3.12) $\mathfrak{F}(x^n) \geq \mathfrak{F}(x)$.

Lemma 3.12. [1] *Let μ be a fuzzy subset of a set X . Then:*

$$\mu_t = \bigcap_{s \in [0, t)} \mu_{s>} \text{ and } \mu_{t>} = \bigcup_{s \in (t, 1]} \mu_s$$

for all $t \in [0, 1]$, where $\mu_t = \{x \in G : \mu(x) \geq t\}$ and $\mu_{t>} = \{x \in G : \mu(x) > t\}$ are respectively the level subset and strong level subset of μ .

Theorem 3.13. *The intersection of any family of filters of a BL-algebra is again a filter.*

Proof. The proof is easy. \square

Theorem 3.14. *For fuzzy subset \mathfrak{F} of \mathcal{B} , the following statements are equivalent:*

- (i) \mathfrak{F} is an imaginable T -fuzzy filter,
- (ii) \mathfrak{F} is imaginable and every nonempty strong level subset $\mathfrak{F}_{t>}$, $t \in [0, 1]$, is a filter of \mathcal{B} ,
- (iii) \mathfrak{F} is imaginable and every nonempty level subset \mathfrak{F}_t , $t \in [0, 1]$, is a filter of \mathcal{B} .

Proof. (i) \Rightarrow (ii) Let $a, a \rightarrow b \in \mathfrak{F}_{t>}$, for some $t \in [0, 1]$, i.e. $\mathfrak{F}(a), \mathfrak{F}(a \rightarrow b) > t$. Then, $\mathfrak{F}(b) \geq T(\mathfrak{F}(a), \mathfrak{F}(a \rightarrow b)) > T(t, t) = t$, which implies that $b \in \mathfrak{F}_{t>}$. Obviously, $1 \in \mathfrak{F}_{t>}$. Hence, $\mathfrak{F}_{t>}$ is a filter of \mathcal{B} .

(ii) \Rightarrow (iii) Let $a, a \rightarrow b \in \mathfrak{F}_t$, for some $t \in [0, 1]$. So, $a, a \rightarrow b \in \mathfrak{F}_{s>}$, for all $s \in [0, t)$, and hence, $b \in \mathfrak{F}_{s>}$, for all $s \in [0, t)$. This implies that $b \in \mathfrak{F}_t$. Since, $1 \in \mathfrak{F}_{s>}$, for all $s \in [0, t)$, by Lemma 3.12, we conclude that $1 \in \mathfrak{F}_t$.

(iii) \Rightarrow (i) The proof is easy. \square

Theorem 3.15. (*Representation Theorem*) *Let $\{F_\alpha : \alpha \in [0, 1]\}$ be a class of filters of \mathcal{B} . The necessary and sufficient condition the existence of a T -fuzzy filter \mathfrak{F} of \mathcal{B} such that for all $\alpha \in [0, 1]$, $\mathfrak{F}_\alpha = F_\alpha$ is that for all $M \subseteq [0, 1]$,*

$$(3.13) \quad F_{\bigvee_{\alpha \in M} \alpha} = \bigcap_{\alpha \in M} F_\alpha.$$

Proof. Suppose that (3.13) holds, and define a fuzzy subset \mathfrak{F} of \mathcal{B} by

$$\mathfrak{F}(x) = \bigvee_{\alpha \in [0, 1]} \alpha \chi_{F_\alpha}(x), \quad \forall x \in \mathcal{B}.$$

We shall prove that \mathfrak{F} is a T -fuzzy filter. Let $x \in \mathcal{B}$. Since every nonempty filter contains 1, $\{\alpha \in [0, 1] : x \in F_\alpha\} \subseteq \{\alpha \in [0, 1] : 1 \in F_\alpha\}$. This implies that $\mathfrak{F}(x) = \bigvee_{\alpha \in [0, 1]} \alpha \chi_{F_\alpha}(x) \leq \bigvee_{\alpha \in [0, 1]} \alpha \chi_{F_\alpha}(1) = \mathfrak{F}(1)$. Now, let $x, y \in \mathcal{B}$ and

$$T \left(\bigvee_{\beta \in [0, 1]} \beta \chi_{F_\beta}(x), \bigvee_{\gamma \in [0, 1]} \gamma \chi_{F_\gamma}(x \rightarrow y) \right) = T(\mathfrak{F}(x), \mathfrak{F}(x \rightarrow y)) = t.$$

Then $\bigvee_{\beta \in [0, 1]} \beta \chi_{F_\beta}(x) \geq t$ and $\bigvee_{\gamma \in [0, 1]} \gamma \chi_{F_\gamma}(x \rightarrow y) \geq t$. Now, let $M = \{\alpha \in [0, 1] : x \in F_\alpha\}$. Then,

$$(3.14) \quad x \in \bigcap_{\beta \in M} F_\beta = F_{\bigvee_{\beta \in M} \beta} \subseteq F_t.$$

Similarly, we can deduce that $x \rightarrow y \in F_t$. Combining (3.14) we get $y \in F_t$ and so

$$\mathfrak{F}(y) = \bigvee_{\alpha \in [0, 1]} \alpha \chi_{F_\alpha}(y) \geq t = T(\mathfrak{F}(x), \mathfrak{F}(x \rightarrow y)).$$

Now, we show that $\mathfrak{F}_\alpha = F_\alpha$, for all $\alpha \in [0, 1]$. Let $x \in F_t$, for some $t \in [0, 1]$. Then, $\mathfrak{F}(x) = \bigvee_{\alpha \in [0, 1]} \alpha \chi_{F_\alpha}(x) \geq t$, that is, $x \in \mathfrak{F}_t$ and so $F_t \subseteq \mathfrak{F}_t$. Now, if $x \in \mathfrak{F}_t$,

then $\bigvee_{\alpha \in [0,1]} \alpha \chi_{F_\alpha}(x) = \mathfrak{F}(x) \geq t$ and so for $M = \{\alpha \in [0,1] : x \in F_\alpha\}$ we have $x \in \bigcap_{\alpha \in M} F_\alpha = F_{\bigvee_{\alpha \in M} \alpha} \subseteq F_t$, that is, $\mathfrak{F}_t \subseteq F_t$. Thus, we proved that $\mathfrak{F}_t = F_t$, for all $t \in [0,1]$.

Conversely, suppose that there exists a T-fuzzy filter \mathfrak{F} of \mathcal{B} such that for all $t \in [0,1]$, $\mathfrak{F}_t = F_t$. We prove the validity of (3.13). For this, let $M \subseteq [0,1]$ and $x \in F_\beta$, where $\beta = \bigvee_{\alpha \in M} \alpha$. Then, $x \in \mathfrak{F}_\beta$ and so $\mathfrak{F}(x) \geq \alpha$, for all $\alpha \in M$. This implies that $x \in \mathfrak{F}_\alpha = F_\alpha$, for all $\alpha \in M$, and so $x \in \bigcap_{\alpha \in M} F_\alpha$. Hence,

$$F_{\bigvee_{\alpha \in M} \alpha} \subseteq \bigcap_{\alpha \in M} F_\alpha.$$

The converse inequality is proved similarly. □

4. T-fuzzy Congruences

Definition 4.1. A fuzzy relation Θ on \mathcal{B} is said to be a *T-fuzzy equivalence* if

- $(\forall x \in \mathcal{B}) \quad \Theta(x, x) = \sup_{(y,z) \in \mathcal{B}^2} \Theta(y, z),$
- $(\forall x, y \in \mathcal{B}) \quad \Theta(x, y) = \Theta(y, x),$
- $(\forall x, y, z \in \mathcal{B}) \quad T(\Theta(x, y), \Theta(y, z)) \leq \Theta(x, z).$

Θ is said to be *compatible* if for all $x, y, z \in \mathcal{B}$,

- $\Theta(x, y) \leq \Theta(x * z, y * z),$
- $\Theta(x, y) \leq T(\Theta(x \rightarrow z, y \rightarrow z), \Theta(z \rightarrow x, z \rightarrow y)).$

Any T-fuzzy compatible equivalence on \mathcal{B} is called a *T-fuzzy congruence*. The fuzzy subset $\Theta^x : \mathcal{B} \rightarrow [0,1]$, which is defined by $\Theta^x(y) = \Theta(x, y)$, is called the *fuzzy congruence class* containing x . Let \mathcal{B}/Θ be the set of all T-fuzzy congruence classes Θ^x .

Proposition 4.2. For (imaginable) T-fuzzy congruence Θ in \mathcal{B} , Θ^1 is an (imaginable) T-fuzzy filter of \mathcal{B} .

Proof. Let $x \in \mathcal{B}$. Then, $\Theta^1(1) = \Theta(1, 1) = \sup_{(y,z) \in \mathcal{B}^2} \Theta(y, z) \geq \Theta(1, x) = \Theta^1(x)$. Now, let $x, y \in \mathcal{B}$. Then

$$\begin{aligned} \Theta^1(y) &= \Theta(1, y) \geq T(\Theta(1, x \rightarrow y), \Theta(x \rightarrow y, y)) \quad \text{by transitivity} \\ &= T(\Theta^1(x \rightarrow y), \Theta(x \rightarrow y, 1 \rightarrow y)) \\ &\geq T(\Theta^1(x \rightarrow y), \Theta(x, 1)) \quad \text{by compatibility} \\ &= T(\Theta^1(x), \Theta^1(x \rightarrow y)). \end{aligned}$$

This completes the proof. □

Theorem 4.3. For imaginable T-fuzzy filter \mathfrak{F} of \mathcal{B} there is an imaginable T-fuzzy congruence Θ such that $\Theta^1 = \mathfrak{F}$.

Proof. Let \mathfrak{F} be an imaginable T-fuzzy filter of \mathcal{B} . Define a fuzzy relation Θ in \mathcal{B} by

$$\Theta(x, y) = T(\mathfrak{F}(x \rightarrow y), \mathfrak{F}(y \rightarrow x)).$$

Then $\Theta(1, 1) = T(\mathfrak{F}(1), \mathfrak{F}(1)) \geq T(\mathfrak{F}(x \rightarrow y), \mathfrak{F}(y \rightarrow x)) = \Theta(x, y)$, for all $x, y \in \mathcal{B}$. So, $\Theta(1, 1) \geq \sup_{(x,y) \in \mathcal{B}^2} \Theta(x, y)$ means that $\Theta(1, 1) = \sup_{(x,y) \in \mathcal{B}^2} \Theta(x, y)$. Symmetry is clear. For transitivity, let $x, y, z \in \mathcal{B}$. Then,

$$\begin{aligned} \Theta(x, y) &= T(\mathfrak{F}(x \rightarrow y), \mathfrak{F}(y \rightarrow x)) \\ &\geq T(T(\mathfrak{F}(x \rightarrow z), \mathfrak{F}(z \rightarrow y)), T(\mathfrak{F}(y \rightarrow z), \mathfrak{F}(z \rightarrow x))) \quad \text{by (3.11)} \\ &= T(T(\mathfrak{F}(x \rightarrow z), \mathfrak{F}(z \rightarrow x)), T(\mathfrak{F}(z \rightarrow y), \mathfrak{F}(y \rightarrow z))) \\ &= T(\Theta(x, z), \Theta(z, y)). \end{aligned}$$

The compatibility of Θ is easily followed from (3.8), (3.9) and (3.10).

Obviously, if $a, b \in [0, 1]$ are idempotent, then $T(a, b)$ is also an idempotent element of T . Also, it is clear that $\Theta^1 = \mathfrak{F}$. This completes the proof. \square

Remark. The T -fuzzy congruence Θ in the above theorem is called the T -fuzzy congruence induced by T -fuzzy filter \mathfrak{F} and denoted by $\Theta_{\mathfrak{F}}$.

Proposition 4.4. *Let Θ be an imaginable T -fuzzy congruence in \mathcal{B} . Then $\Theta_{\Theta^1} = \Theta$.*

Proof. Let $x, y \in \mathcal{B}$. Then,

$$\begin{aligned} \Theta_{\Theta^1}(x, y) &= T(\Theta^1(x \rightarrow y), \Theta^1(y \rightarrow x)) \\ &= T(\Theta(1, x \rightarrow y), \Theta(1, y \rightarrow x)) \\ &= T(\Theta(x \rightarrow y, 1), \Theta(1, y \rightarrow x)) \\ &\leq \Theta(x \rightarrow y, y \rightarrow x). \end{aligned}$$

On the other hand, by compatibility and (2.2) we get

$$\Theta(x \rightarrow y, y \rightarrow x) \leq \Theta(x * (x \rightarrow y), x * (y \rightarrow x)) \leq \Theta(y, 1)$$

and

$$\Theta(x \rightarrow y, y \rightarrow x) \leq \Theta(y * (x \rightarrow y), y * (y \rightarrow x)) \leq \Theta(1, x).$$

Hence,

$$\begin{aligned} \Theta(x \rightarrow y, y \rightarrow x) &= T(\Theta(x \rightarrow y, y \rightarrow x), \Theta(x \rightarrow y, y \rightarrow x)) \leq T(\Theta(y, 1), \Theta(1, x)) \\ &\leq \Theta(y, x) = \Theta(x, y) \end{aligned}$$

and so $\Theta_{\Theta^1}(x, y) \leq \Theta(x, y)$. For the converse inequality, we have

$$\begin{aligned} \Theta(x, y) &\leq T(\Theta(x \rightarrow x, y \rightarrow x), \Theta(x \rightarrow x, x \rightarrow y)) \quad \text{by compatibility} \\ &= T(\Theta(1, y \rightarrow x), \Theta(1, x \rightarrow y)) \\ &= T(\Theta^1(y \rightarrow x), \Theta^1(x \rightarrow y)) \\ &= \Theta_{\Theta^1}(x, y). \end{aligned}$$

Thus, $\Theta = \Theta_{\Theta^1}$. \square

Proposition 4.5. *For imaginable T -fuzzy filter \mathfrak{F} of \mathcal{B} , $\Theta_{\mathfrak{F}}^1 = \mathfrak{F}$.*

Proof. The proof is easy. \square

Theorem 4.6. (*Correspondence Theorem*) *There is a correspondence between \mathcal{C} , the set of all imaginable T -fuzzy congruences in \mathcal{B} , and \mathcal{F} , the set of all imaginable T -fuzzy filters of \mathcal{B} .*

Proof. Define the maps $\Phi : \mathcal{F} \longrightarrow \mathcal{C}$ by $\mathfrak{F} \mapsto \Theta_{\mathfrak{F}}$, and $\Psi : \mathcal{C} \longrightarrow \mathcal{F}$ by $\Psi(\Theta) = \Theta^1$. By Propositions 4.4 and 4.5, it is easy to check that $\Phi\Psi = 1_{\mathcal{C}}$ and $\Psi\Phi = 1_{\mathcal{F}}$, proving that Φ is a bijection. \square

Corollary 4.7. *Every imaginable T -fuzzy congruence in \mathcal{B} is determined by an imaginable T -fuzzy filter.*

Lemma 4.8. $\Theta_{\mathfrak{F}}^x = \Theta_{\mathfrak{F}}^y$ if and only if $\mathfrak{F}(x \rightarrow y) = \mathfrak{F}(y \rightarrow x) = \mathfrak{F}(1)$.

Proof. Let $\Theta_{\mathfrak{F}}^x = \Theta_{\mathfrak{F}}^y$, for $x, y \in \mathcal{B}$. Then $\Theta_{\mathfrak{F}}(x, x) = \Theta_{\mathfrak{F}}(y, x)$ and hence

$$\begin{aligned} \mathfrak{F}(1) &= \mathfrak{F}(x \rightarrow x) = T(\mathfrak{F}(x \rightarrow x), \mathfrak{F}(x \rightarrow x)) = T(\mathfrak{F}(x \rightarrow y), \mathfrak{F}(y \rightarrow x)) \\ &\leq \mathfrak{F}(x \rightarrow y), \mathfrak{F}(y \rightarrow x) \end{aligned}$$

which implies that $\mathfrak{F}(x \rightarrow y) = \mathfrak{F}(y \rightarrow x) = \mathfrak{F}(1)$.

Conversely, suppose that $\mathfrak{F}(x \rightarrow y) = \mathfrak{F}(y \rightarrow x) = \mathfrak{F}(1)$ and let $z \in \mathcal{B}$. By (3.11),

$$\mathfrak{F}(z \rightarrow x) \geq T(\mathfrak{F}(z \rightarrow y), \mathfrak{F}(y \rightarrow x)) = \mathfrak{F}(z \rightarrow y)$$

and

$$\mathfrak{F}(z \rightarrow y) \geq T(\mathfrak{F}(z \rightarrow x), \mathfrak{F}(x \rightarrow y)) = \mathfrak{F}(z \rightarrow x).$$

This implies that $\mathfrak{F}(z \rightarrow y) = \mathfrak{F}(z \rightarrow x)$ and similarly we can deduce that $\mathfrak{F}(x \rightarrow z) = \mathfrak{F}(y \rightarrow z)$. Consequently,

$$\Theta_{\mathfrak{F}}^x(z) = T(\mathfrak{F}(x \rightarrow z), \mathfrak{F}(z \rightarrow x)) = T(\mathfrak{F}(y \rightarrow z), \mathfrak{F}(z \rightarrow y)) = \Theta_{\mathfrak{F}}^y(z).$$

\square

For a T -fuzzy filter \mathfrak{F} of \mathcal{B} let $\mathfrak{F}_* = \{x \in \mathcal{B} : \mathfrak{F}(x) = \mathfrak{F}(1)\}$. Obviously, \mathfrak{F}_* is a filter of \mathcal{B} .

Corollary 4.9. *Let \mathfrak{F} be an imaginable T -fuzzy filter of \mathcal{B} . Then $\Theta_{\mathfrak{F}}^x = \Theta_{\mathfrak{F}}^y$ if and only if $x \equiv_{\mathfrak{F}_*} y$.*

Let \mathfrak{F} be a T -fuzzy filter of \mathcal{B} . For $x \in \mathcal{B}$, the fuzzy subset \mathfrak{F}_x is called a *fuzzy coset* of \mathfrak{F} such that $\mathfrak{F}_x(y) = T(\mathfrak{F}(x \rightarrow y), \mathfrak{F}(y \rightarrow x))$. Let $\mathcal{B}/\mathfrak{F} = \{\mathfrak{F}_x : x \in \mathcal{B}\}$. Obviously, $\mathfrak{F}_x = \Theta_{\mathfrak{F}}^x$ and so $\mathcal{B}/\mathfrak{F} = \mathcal{B}/\Theta_{\mathfrak{F}}$. Now, on \mathcal{B}/\mathfrak{F} , define the following operations

$$\mathfrak{F}_x \sqcup \mathfrak{F}_y = \mathfrak{F}_{x \vee y}, \quad \mathfrak{F}_x \sqcap \mathfrak{F}_y = \mathfrak{F}_{x \wedge y}, \quad \mathfrak{F}_x \bullet \mathfrak{F}_y = \mathfrak{F}_{x * y}, \quad \mathfrak{F}_x \dashv \mathfrak{F}_y = \mathfrak{F}_{x \rightarrow y}.$$

It should be noted that the union operation induces a partial ordering ' \preceq ' on \mathcal{B}/\mathfrak{F} by ' $\mathfrak{F}_x \preceq \mathfrak{F}_y$ if and only if $\mathfrak{F}_x \sqcup \mathfrak{F}_y = \mathfrak{F}_y$ '.

In virtue of the above observations we give the following easy result.

Theorem 4.10. *Let \mathfrak{F} be an imaginable T -fuzzy filter of \mathcal{B} . Then $(\mathcal{B}/\mathfrak{F}, \sqcup, \sqcap, \bullet, \dashv, \mathfrak{F}_0, \mathfrak{F}_1)$ is a BL -algebra.*

Proof. The conditions (BL1), (BL2), (BL4) and (BL5) are easily proved. For (BL3), let $\mathfrak{F}_x, \mathfrak{F}_y, \mathfrak{F}_z \in \mathcal{B}/\mathfrak{F}$. Then

$$\begin{aligned} \mathfrak{F}_x \bullet \mathfrak{F}_y \leq \mathfrak{F}_z &\Leftrightarrow \mathfrak{F}_{x*y} \leq \mathfrak{F}_z \Leftrightarrow \mathfrak{F}_{x*y} \sqcup \mathfrak{F}_z = \mathfrak{F}_z \Leftrightarrow \mathfrak{F}_{(x*y) \vee z} = \mathfrak{F}_z \\ &\Leftrightarrow \mathfrak{F}((x * y) \rightarrow z) = \mathfrak{F}(1) \quad \text{by Lemma 4.8 and (2.15)} \\ &\Leftrightarrow \mathfrak{F}(x \rightarrow (y \rightarrow z)) = \mathfrak{F}(1) \quad \text{by (2.8)} \\ &\Leftrightarrow \mathfrak{F}_x = \mathfrak{F}_y \rightarrow \mathfrak{F}_z. \end{aligned}$$

□

Theorem 4.11. *Let \mathfrak{F} be an imaginable T-fuzzy filter of \mathcal{B} . Then $\mathcal{B}/\mathfrak{F} \simeq \mathcal{B}/\mathfrak{F}_*$.*

Proof. It is easy to verify that the mapping $\Phi : \mathcal{B}/\mathfrak{F} \longrightarrow \mathcal{B}/\mathfrak{F}_*$ by $\Phi(\mathfrak{F}_x) = [x]_{\mathfrak{F}_*}$ is an isomorphism. □

Theorem 4.12. *(Homomorphism Theorem) Let $f : \mathcal{B} \longrightarrow \mathcal{C}$ be a homomorphism of BL-algebras and \mathfrak{F} an imaginable T-fuzzy filter of \mathcal{B} such that $\mathfrak{F}_* = \ker f$. Then $\mathcal{B}/\mathfrak{F} \simeq f(\mathcal{B})$.*

Proof. Let \mathfrak{F} be an imaginable T-fuzzy filter of \mathcal{B} . Then \mathfrak{F}_* is a filter of \mathcal{B} and so $\mathcal{B}/\mathfrak{F}_* \simeq \mathfrak{F}(\mathcal{B})$. Thus, Theorem 4.11 completes the proof. □

Let \mathfrak{F} and \mathfrak{G} be two T-fuzzy filters of \mathcal{B} . We say that \mathfrak{G} is a T-fuzzy filter of \mathfrak{F} if $\mathfrak{G} \subseteq \mathfrak{F}$. In this case, we define a fuzzy subset $\mathfrak{F}/\mathfrak{G}$ of \mathcal{B}/\mathfrak{G} by $(\mathfrak{F}/\mathfrak{G})(\mathfrak{G}^x) = \mathfrak{F}(x)$.

Lemma 4.13. *For two T-fuzzy filters \mathfrak{F} and \mathfrak{G} of \mathcal{B} , $\mathfrak{F}/\mathfrak{G}$ is a T-fuzzy filter of \mathcal{B}/\mathfrak{G} .*

Proof. The proof is easy. □

Theorem 4.14. *(Second Isomorphism Theorem) Let \mathfrak{F} and \mathfrak{G} be two T-fuzzy filters of \mathcal{B} such that $\mathfrak{G} \subseteq \mathfrak{F}$. Then $(\mathcal{B}/\mathfrak{G})/(\mathfrak{F}/\mathfrak{G}) \simeq \mathcal{B}/\mathfrak{F}$.*

Proof. Define $\Psi : \mathcal{B}/\mathfrak{G} \longrightarrow \mathcal{B}/\mathfrak{F}$ with $\mathfrak{G}^x \mapsto \mathfrak{F}^x$. It is easy to verify that Ψ is an epimorphism. Now,

$$\begin{aligned} \ker \Psi &= \{\mathfrak{G}^x : \mathfrak{F}^x = \Psi(\mathfrak{G}^x) = \mathfrak{F}^1\} = \{\mathfrak{G}^x : \mathfrak{F}(x) = \mathfrak{F}(1)\} \\ &= \{g^x : (\mathfrak{F}/\mathfrak{G})(\mathfrak{G}^x) = (\mathfrak{F}/\mathfrak{G})(g^1)\} = (\mathfrak{F}/\mathfrak{G})_*. \end{aligned}$$

By Homomorphism Theorem the proof is complete. □

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