Iranian Journal of Fuzzy Systems **Vol. 6, No. 4,** (2009) pp. 37-47

T-FUZZY CONGRUENCES AND T-FUZZY FILTERS OF A BL-ALGEBRA

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Abstract. In this note, we introduce the concept of a fuzzy filter of a BL-algebra, with respect to a t-norm briefly, T-fuzzy filters, and give some related results. In particular, we prove Representation Theorem in BL-algebras. Then we generalize the notion of a fuzzy congruence (in a BL-algebra) was defined by Lianzhen et al. to a new fuzzy congruence, specially with respect to a t-norm. We prove that there is a correspondence bijection between the set of all T-fuzzy filters of a BL-algebra and the set of all T-fuzzy congruences in that BL-algebra. Next, we show how T-fuzzy filters induce T-fuzzy congruences, and construct a new BL-algebras, called quotient BL-algebras, and give some homomorphism theorems.

1. Introduction

BL-algebras are the algebraic structure for Hájek basic logic [4]. Most familiar example of a BL-algebra is the unit interval [0,1] endowed with the structure induced by a continuous t-norm. In 1958, Chang [3] introduced the concept of an MV-algebra which is one of the most classes of BL-algebras. Turunen [10] introduced the notion of an implicative filter and a Boolean filter and proved that these notions are equivalent to BL-algebras. Boolean filters are an important class of filters, because the quotient BL-algebra induced by these filters are Boolean algebras. In [7] and [6], Lianzhen et al. introduced the concept of a fuzzy filter and fuzzy congruence in a BL-algebra and gave some related results. He proved that the fuzzy quotient algebras induced by fuzzy Boolean filters are Boolean algebras.

Now, in this paper, by considering [1], [6], [7], and the notion of a t-norm we define the notion of a fuzzy congruence with respect to a t-norm and give some related results.

2. Preliminaries

In this section, we give some fundamental definitions and results. For more details, we refer the readers to the references.

Definition 2.1. A BL-algebra is an algebra $(B, \vee, \wedge, *, \rightarrow, 0, 1)$ of type (2,2,2,2,0,0) such that
(BL1) $(B, \vee, \wedge, 0, 1)$ is a bounded lattice,
(BL2) $(B, *, 1)$ is a commutative monoid,

Key words and phrases: T-fuzzy filter, T-fuzzy congruence.

This research has been supported in part by Fuzzy Systems Research Center, University of Sistan and Bluchestan, Zahedan, Iran.

(BL3) $c \le a \to b$ if and only if $a * c \le b$, for all $a, b, c \in B$,
(BL4) $a \wedge b = a * (a \to b)$,
(BL5) $(a \to b) \vee (b \to a) = 1$.

Example 2.2. For $x, y \in [0,1]$ define $x * y = \min\{x, y\}$ and

$$x \to y = \begin{cases} 1 & x \le y, \\ y & x > y. \end{cases}$$

Then $([0,1], \vee, \wedge, *, \to, 0, 1)$ is a BL-algebra

In any BL-algebra the following hold:

(2.1) $x \le y$ if and only if $x \to y = 1$.

(2.2) $x * (x \to y) \le y$.

(2.3) $x \le y \to (x * y)$.

(2.4) $x * y \le x \wedge y$.

(2.5) $(x \to y) * (y \to z) \le x \to z$.

(2.6) $x \to x^- = x^{--} \to x^-$, where $x^- = x \to 0$.

(2.7) If $x \vee x^- = 1$, then $x \wedge x^- = 0$.

(2.8) $x \to (y \to z) = (x * y) \to z$.

(2.9) $x \to (y \to z) = y \to (x \to z)$.

(2.10) $y \le (y \to x) \to x$.

(2.11) $x \le y$ implies $y \to z \le x \to z$ and $z \to x \le z \to y$.

(2.12) $y \to x \le (z \to y) \to (z \to x)$.

(2.13) $x \to y \le (y \to z) \to (x \to z)$.

(2.14) $x \vee y = [(x \to y) \to y] \wedge [(y \to x) \to x]$.

(2.15) $(x \vee y) \to z = (x \to z) \wedge (y \to z)$.

Hájek [4] defined a *filter* of a BL-algebra $\mathcal{B}$ to be a nonempty subset F such that (i) for $a, b \in F$, $a * b \in F$, (ii) if $a \in F$ and $a \le b$ then $b \in F$.

Theorem 2.3. [4] *Let F be a filter of $\mathcal{B}$. Then the binary relation $\equiv_F$ which is defined by*

$$x \equiv_F y \quad \text{if and only if } x \to y \in F \quad \text{and } y \to x \in F$$

is a congruence relation on $\mathcal{B}$. Define $\bullet$, $\rightharpoonup$, $\sqcup$, $\sqcap$ on $\mathcal{B}/F$, the set of all congruence classes of $\mathcal{B}$, as follows:

$$[x] \bullet [y] = [x * y],\ [x] \rightharpoonup [y] = [x \to y],\ [x] \sqcup [y] = [x \vee y],\ [x] \sqcap [y] = [x \wedge y].$$

Then $(\mathcal{B}/F, \bullet, \rightharpoonup, \sqcup, \sqcap, \mathcal{B}, F)$ is a BL-algebra.

A function $T : [0,1] \times [0,1] \longrightarrow [0,1]$ satisfying the following properties is called a *t-norm*: for all $x, y, z \in [0,1]$,

(T1) $T(x, 1) = x$,
(T2) $T(x, y) \le T(x, z)$, if $y \le z$,
(T3) $T(x, y) = T(y, x)$,

(T4) $T(x, T(y, z)) = T(T(x, y), z)$.

Most familiar t-norms are:

- $T_D(x, y) = \begin{cases} x \wedge y & ; \quad x \vee y = 1, \\ 0 & ; \quad \text{otherwise.} \end{cases}$ (Drastic product)
- $T_L(x, y) = 0 \vee (x + y - 1)$. (Łukasiewicz)
- $T_P(x, y) = xy$. (Product)
- $T_M(x, y) = x \wedge y$. (Minimum)
- $T_{nM}(x, y) = \begin{cases} 0 & ; \quad x + y \leq 1, \\ x \wedge y & ; \quad \text{otherwise.} \end{cases}$ (Nilpotent minimum)

where $\wedge = \min$ and $\vee = \max$.

Every t-norm T satisfies the inequality $T_D(x, y) \leq T(x, y) \leq T_M(x, y)$.

For a t-norm T, $a \in [0, 1]$ is called an *idempotent* if $T(a, a) = a$.

It is obvious that 0 and 1 are idempotent elements of any t-norm which are called the *trivial* idempotent elements of T. For T_M, the idempotent elements are any element of $[0, 1]$, and for t-norm T_{nM}, the set $\{0\} \cup (0.5, 1]$ consists of all idempotent elements of T_{nM}.

Theorem 2.4. *(Representation Theorem) Let $\{B_\alpha : \alpha \in [0, 1]\}$ be a class of subsets of a nonempty set X. The necessary and sufficient condition for the existence of fuzzy subset A of X with $A_\alpha = B_\alpha$, for all $\alpha \in [0, 1]$, is that for every $M \subseteq [0, 1]$,*

$$B_{\bigvee\limits_{\alpha \in M} \alpha} = \bigcap_{\alpha \in M} B_\alpha. \tag{2.16}$$

Remark 2.5. If $\{B_\alpha : \alpha \in [0, 1]\}$ is a class of subsets of a nonempty set X for which (2.16) holds for every $M \subseteq [0, 1]$, then

(i) $B_0 = X$,
(ii) if $\alpha, \beta \in [0, 1]$ are such that $\alpha \leq \beta$, then $B_\beta \subseteq B_\alpha$.

Notation. From now on, in this paper, $\mathcal{B}$ denotes a BL-algebra.

3. T-fuzzy Filters

In [6], Lie et al. defined a fuzzy filter of $\mathcal{B}$ as a fuzzy subset f for which every level subset f_t, $t \in [0, 1]$, is empty or a filter of $\mathcal{B}$, and proved that a fuzzy subset f is a fuzzy filter if and only if

- $(\forall x \in \mathcal{B}) \quad f(1) \geq f(x)$,
- $(\forall x, y \in \mathcal{B}) \quad f(y) \geq \min\{f(x), f(x \to y)\}$.

Now, we generalize this concept.

Definition 3.1. A fuzzy subset $\mathfrak{F}$ of $\mathcal{B}$ is said to be a *T-fuzzy filter* if

- $(\forall x \in \mathcal{B}) \quad \mathfrak{F}(1) \geq \mathfrak{F}(x)$,
- $(\forall x, y \in \mathcal{B}) \quad \mathfrak{F}(y) \geq T(\mathfrak{F}(x), \mathfrak{F}(x \to y))$.

Note 3.2. Subsequently, in this paper, any T_M-fuzzy filter is called a fuzzy filter.

Example 3.3. [6] Let $\mathcal{B} = \{0, a, b, 1\}$, where $0 < a < b < 1$ with the operations $x \wedge y = \min\{x, y\}$, $x \vee y = \max\{x, y\}$ and $*$ and $\rightarrow$ as the following tables:

$*$	0	a	b	1
0	0	0	0	0
a	0	0	a	a
b	0	a	b	b
1	0	a	b	1

$\rightarrow$	0	a	b	1
0	1	1	1	1
a	a	1	1	1
b	0	a	1	1
1	0	a	b	1

Then $(\mathcal{B}, \vee, \wedge, *, \rightarrow, 0, 1)$ is a BL-algebra. If we define a fuzzy subset $\mathfrak{F}$ by $\mathfrak{F}(1) = t_1$, $\mathfrak{F}(b) = t_2$ and $\mathfrak{F}(0) = \mathfrak{F}(a) = t_3$ (with $0 \leq t_3 < t_2 < t_1 \leq 1$), then $\mathfrak{F}$ is a fuzzy filter of $\mathcal{B}$. Moreover, all nonempty level subsets, $\{1\}$ (for $t_2 < t \leq t_1$), $\{1, b\}$ (for $t_3 < t \leq t_2$), and $\mathcal{B}$ (for $t \leq t_3$) are filters of $\mathcal{B}$.

Example 3.4. Let $\mathcal{B} = [0, 1]$. Then $\mathcal{B}$ together with the ordinary maximum and minimum, $x * y = \min\{x, y\}$ and $x \rightarrow y = 1$ if $x \leq y$, otherwise $x \rightarrow y = y$ is a BL-algebra. It is easy to verify that $[n, 1]$ (with $n \in [0, 1]$) is a filter of $\mathcal{B}$. So, the fuzzy subset $\mathfrak{F}$ with $\mathfrak{F}(x) = x$, for all $x \in \mathcal{B}$, is a fuzzy filter of $\mathcal{B}$ for which $\mathfrak{F}_x = [x, 1]$, for all $x \in \mathcal{B}$.

Example 3.5. Let $\mathcal{B} = \{0, a, b, 1\}$ be the BL-algebra as in Example 3.3, and define a fuzzy subset $\mathfrak{F}$ of $\mathcal{B}$ by $\mathfrak{F}(0) = 2/3$, $\mathfrak{F}(a) = \mathfrak{F}(b) = 4/5$, and $\mathfrak{F}(1) = 5/6$. It is easy to verify that $\mathfrak{F}$ is a T_P-fuzzy filter of $\mathcal{B}$, but it is not a fuzzy filter, because $\mathfrak{F}(0) = 2/3 \not\geq 4/5 = \min\{\mathfrak{F}(a), \mathfrak{F}(a \rightarrow 0)\}$.

Definition 3.6. A T-fuzzy filter $\mathfrak{F}$ of $\mathcal{B}$ is called *imaginable* if every element of the image of $\mathfrak{F}$ is an idempotent element of T.

Obviously, any fuzzy filter is imaginable.

The following theorem demonstrats the connection between imaginable T-fuzzy filters and fuzzy filters of a BL-algebra.

Proposition 3.7. *A fuzzy subset of $\mathcal{B}$ is a fuzzy filter if and only if it is an imaginable T-fuzzy filter of $\mathcal{B}$.*

Proof. Let $\mathfrak{F}$ be an imaginable T-fuzzy filter of $\mathcal{B}$. It suffices to prove that $\mathfrak{F}(y) \geq \mathfrak{F}(x) \wedge \mathfrak{F}(x \rightarrow y)$. For this, let $x, y \in \mathcal{B}$. Then,

$$\begin{aligned} \mathfrak{F}(x) \wedge \mathfrak{F}(x \rightarrow y) &= T(\mathfrak{F}(x) \wedge \mathfrak{F}(x \rightarrow y), \mathfrak{F}(x) \wedge \mathfrak{F}(x \rightarrow y)) \\ &\leq T(\mathfrak{F}(x), \mathfrak{F}(x \rightarrow y)) \\ &\leq \mathfrak{F}(y) \end{aligned}$$

The converse is obvious. □

Theorem 3.8. *Let $\mathfrak{F}$ be a fuzzy subset of $\mathcal{B}$. Then, $\mathfrak{F}$ is an imaginable T-fuzzy filter if and only if it is imaginable and*

$$(\forall a, b, c \in \mathcal{B}) \quad a \rightarrow (b \rightarrow c) = 1 \;\Rightarrow\; \mathfrak{F}(c) \geq T(\mathfrak{F}(a), \mathfrak{F}(b)). \tag{3.1}$$

Proof. Assume that $\mathfrak{F}$ is an imaginable T-fuzzy filter of $\mathcal{B}$, and $a, b, c \in \mathcal{B}$ are such that $a \to (b \to c) = 1$. Then

$$\begin{aligned} \mathfrak{F}(c) &\geq T(\mathfrak{F}(b), \mathfrak{F}(b \to c)) \geq T(\mathfrak{F}(b), T(\mathfrak{F}(a), \mathfrak{F}(a \to (b \to c)))) \\ &= T(\mathfrak{F}(b), T(\mathfrak{F}(a), \mathfrak{F}(1))) \geq T(\mathfrak{F}(b), T(\mathfrak{F}(a), \mathfrak{F}(a))) = T(\mathfrak{F}(b), \mathfrak{F}(a)). \end{aligned}$$

Conversely, suppose that (3.1) holds, for $a, b, c \in \mathcal{B}$. Since, $a \to 1 = 1$, then $a \to (a \to 1) = 1$ and so by hypothesis, $\mathfrak{F}(1) \geq T(\mathfrak{F}(a), \mathfrak{F}(a)) = \mathfrak{F}(a)$. Now, let $a, b \in \mathcal{B}$. Since, $(a \to b) \to (a \to b) = 1$, $\mathfrak{F}(b) \geq T(\mathfrak{F}(a \to b), \mathfrak{F}(a))$, proving $\mathfrak{F}$ is a T-fuzzy filter. □

Corollary 3.9. *Let $\mathfrak{F}$ be a fuzzy subset of $\mathcal{B}$. Then, $\mathfrak{F}$ is an imaginable T-fuzzy filter if and only if it is imaginable and*

$$(\forall a, b, c \in \mathcal{B}) \quad a * b \leq c \;\Rightarrow\; \mathfrak{F}(c) \geq T(\mathfrak{F}(a), \mathfrak{F}(b)). \tag{3.2}$$

Theorem 3.10. *Let $\mathfrak{F}$ be a fuzzy subset of $\mathcal{B}$. Then, $\mathfrak{F}$ is an imaginable T-fuzzy filter if and only if it is imaginable and for all $x, y \in \mathcal{B}$,*

(i) *$x \leq y$ implies $\mathfrak{F}(x) \leq \mathfrak{F}(y)$,*
(ii) *$\mathfrak{F}(x * y) \geq T(\mathfrak{F}(x), \mathfrak{F}(y))$.*

Proof. (i) suppose that $\mathfrak{F}$ is an imaginable T-fuzzy filter of $\mathcal{B}$ and $x \leq y$, for $x, y \in \mathcal{B}$. Since, $x * x \leq x \leq y$, by Corollary 3.9, $\mathfrak{F}(y) \geq T(\mathfrak{F}(x), \mathfrak{F}(x)) = \mathfrak{F}(x)$. Now, because $x * y \leq x * y$, by (3.2), $\mathfrak{F}(x * y) \geq T(\mathfrak{F}(x), \mathfrak{F}(y))$.

Conversely, suppose that $a * b \leq c$, for $a, b, c \in \mathcal{B}$. By (i) and (ii), $\mathfrak{F}(c) \geq \mathfrak{F}(a * b) \geq T(\mathfrak{F}(a), \mathfrak{F}(b))$. By Corollary 3.9, the proof is complete.

□

Proposition 3.11. *Any imaginable T-fuzzy filter $\mathfrak{F}$ of $\mathcal{B}$ satisfies the following properties.*

$$x \leq y \text{ implies } \mathfrak{F}(x) \leq \mathfrak{F}(y). \tag{3.3}$$

$$\mathfrak{F}(x * y) \geq T(\mathfrak{F}(x), \mathfrak{F}(y)). \tag{3.4}$$

$$\mathfrak{F}(x \wedge y) \geq T(\mathfrak{F}(x), \mathfrak{F}(y)). \tag{3.5}$$

$$\mathfrak{F}(x \vee y) \geq T(\mathfrak{F}(x), \mathfrak{F}(y)). \tag{3.6}$$

$$\mathfrak{F}(x \to y) = \mathfrak{F}(1) \text{ implies } \mathfrak{F}(x) \leq \mathfrak{F}(y). \tag{3.7}$$

$$\mathfrak{F}(x \to y) \leq \mathfrak{F}((y \to z) \to (x \to z)). \tag{3.8}$$

$$\mathfrak{F}(x \to y) \leq \mathfrak{F}((z \to x) \to (z \to y)). \tag{3.9}$$

$$\mathfrak{F}((x * z) \to (y * z)) \geq \mathfrak{F}(x \to y). \tag{3.10}$$

$$\mathfrak{F}(x \to z) \geq T(\mathfrak{F}(x \to y), \mathfrak{F}(y \to z)). \tag{3.11}$$

$$\mathfrak{F}(x^n) \geq \mathfrak{F}(x). \tag{3.12}$$

Lemma 3.12. [1] *Let μ be a fuzzy subset of a set X. Then:*

$$\mu_t = \bigcap_{s \in [0,t)} \mu_{s^>} \text{ and } \mu_{t^>} = \bigcup_{s \in (t,1]} \mu_s$$

for all $t \in [0, 1]$, where $\mu_t = \{x \in G : \mu(x) \geq t\}$ and $\mu_{t^>} = \{x \in G : \mu(x) > t\}$ are respectively the level subset and strong level subset of μ.

Theorem 3.13. *The intersection of any family of filters of a BL-algebra is again a filter.*

Proof. The proof is easy. □

Theorem 3.14. *For fuzzy subset $\mathfrak{F}$ of $\mathcal{B}$, the following statements are equivalent:*

(i) *$\mathfrak{F}$ is an imaginable T-fuzzy filter,*

(ii) *$\mathfrak{F}$ is imaginable and every nonempty strong level subset $\mathfrak{F}_{t^>}$, $t \in [0,1]$, is a filter of $\mathcal{B}$,*

(iii) *$\mathfrak{F}$ is imaginable and every nonempty level subset $\mathfrak{F}_t$, $t \in [0,1]$, is a filter of $\mathcal{B}$.*

Proof. (i) ⇒ (ii) Let $a, a \to b \in \mathfrak{F}_{t^>}$, for some $t \in [0,1]$, i.e, $\mathfrak{F}(a), \mathfrak{F}(a \to b) > t$. Then, $\mathfrak{F}(b) \geq T(\mathfrak{F}(a), \mathfrak{F}(a \to b)) > T(t,t) = t$, which implies that $b \in \mathfrak{F}_{t^>}$. Obviously, $1 \in \mathfrak{F}_{t^>}$. Hence, $\mathfrak{F}_{t^>}$ is a filter of $\mathcal{B}$.

(ii) ⇒ (iii) Let $a, a \to b \in \mathfrak{F}_t$, for some $t \in [0,1]$. So, $a, a \to b \in \mathfrak{F}_{s^>}$, for all $s \in [0,t)$, and hence, $b \in \mathfrak{F}_{s^>}$, for all $s \in [0,t)$. This implies that $b \in \mathfrak{F}_t$. Since, $1 \in \mathfrak{F}_{s^>}$, for all $s \in [0,t)$, by Lemma 3.12, we conclude that $1 \in \mathfrak{F}_t$.

(iii) ⇒ (i) The proof is easy. □

Theorem 3.15. *(Representation Theorem) Let $\{F_\alpha : \alpha \in [0,1]\}$ be a class of filters of $\mathcal{B}$. The necessary and sufficient condition the existence of a T-fuzzy filter $\mathfrak{F}$ of $\mathcal{B}$ such that for all $\alpha \in [0,1]$, $\mathfrak{F}_\alpha = F_\alpha$ is that for all $M \subseteq [0,1]$,*

$$F_{\bigvee_{\alpha \in M} \alpha} = \bigcap_{\alpha \in M} F_\alpha. \tag{3.13}$$

Proof. Suppose that (3.13) holds, and define a fuzzy subset $\mathfrak{F}$ of $\mathcal{B}$ by

$$\mathfrak{F}(x) = \bigvee_{\alpha \in [0,1]} \alpha \chi_{F_\alpha}(x), \ \forall x \in \mathcal{B}.$$

We shall prove that $\mathfrak{F}$ is a T-fuzzy filter. Let $x \in \mathcal{B}$. Since every nonempty filter contains 1, $\{\alpha \in [0,1] : x \in F_\alpha\} \subseteq \{\alpha \in [0,1] : 1 \in F_\alpha\}$. This implies that $\mathfrak{F}(x) = \vee_{\alpha \in [0,1]} \alpha \chi_{F_\alpha}(x) \leq \bigvee_{\alpha \in [0,1]} \alpha \chi_{F_\alpha}(1) = \mathfrak{F}(1)$. Now, let $x, y \in \mathcal{B}$ and

$$T\left(\bigvee_{\beta \in [0,1]} \beta \chi_{F_\beta}(x), \bigvee_{\gamma \in [0,1]} \gamma \chi_{F_\gamma}(x \to y)\right) = T(\mathfrak{F}(x), \mathfrak{F}(x \to y)) = t.$$

Then $\vee_{\beta \in [0,1]} \beta \chi_{F_\beta}(x) \geq t$ and $\bigvee_{\gamma \in [0,1]} \gamma \chi_{F_\gamma}(x \to y) \geq t$. Now, let $M = \{\alpha \in [0,1] : x \in F_\alpha\}$. Then,

$$x \in \bigcap_{\beta \in M} F_\beta = F_{\bigvee_{\beta \in M} \beta} \subseteq F_t. \tag{3.14}$$

Similarly, we can deduce that $x \to y \in F_t$. Combining (3.14) we get $y \in F_t$ and so

$$\mathfrak{F}(y) = \bigvee_{\alpha \in [0,1]} \alpha \chi_{F_\alpha}(y) \geq t = T(\mathfrak{F}(x), \mathfrak{F}(x \to y)).$$

Now, we show that $\mathfrak{F}_\alpha = F_\alpha$, for all $\alpha \in [0,1]$. Let $x \in F_t$, for some $t \in [0,1]$. Then, $\mathfrak{F}(x) = \vee_{\alpha \in [0,1]} \alpha \chi_{F_\alpha}(x) \geq t$, that is, $x \in \mathfrak{F}_t$ and so $F_t \subseteq \mathfrak{F}_t$. Now, if $x \in \mathfrak{F}_t$, then $\vee_{\alpha\in[0,1]}\alpha\chi_{F_\alpha}(x) = \mathfrak{F}(x) \geq t$ and so for $M = \{\alpha \in [0,1] : x \in F_\alpha\}$ we have $x \in \cap_{\alpha\in M} F_\alpha = F_{\underset{\alpha\in M}{\vee}\alpha} \subseteq F_t$, that is, $\mathfrak{F}_t \subseteq F_t$. Thus, we proved that $\mathfrak{F}_t = F_t$, for all $t \in [0,1]$.

Conversely, suppose that there exists a T-fuzzy filter $\mathfrak{F}$ of $\mathcal{B}$ such that for all $t \in [0,1]$, $\mathfrak{F}_t = F_t$. We prove the validity of (3.13). For this, let $M \subseteq [0,1]$ and $x \in F_\beta$, where $\beta = \vee_{\alpha\in M}\alpha$. Then, $x \in \mathfrak{F}_\beta$ and so $\mathfrak{F}(x) \geq \alpha$, for all $\alpha \in M$. This implies that $x \in \mathfrak{F}_\alpha = F_\alpha$, for all $\alpha \in M$, and so $x \in \bigcap_{\alpha\in M} F_\alpha$. Hence,

$$F_{\underset{\alpha\in M}{\vee}\alpha} \subseteq \bigcap_{\alpha\in M} F_\alpha.$$

The converse inequality is proved similarly. □

4. **T-fuzzy Congruences**

Definition 4.1. A fuzzy relation Θ on $\mathcal{B}$ is said to be a *T-fuzzy equivalence* if

- $(\forall x \in \mathcal{B}) \quad \Theta(x,x) = \sup_{(y,z)\in\mathcal{B}^2} \Theta(y,z)$,
- $(\forall x, y \in \mathcal{B}) \quad \Theta(x,y) = \Theta(y,x)$,
- $(\forall x, y, z \in \mathcal{B}) \quad T(\Theta(x,y), \Theta(y,z)) \leq \Theta(x,z)$.

Θ is said to be *compatible* if for all $x, y, z \in \mathcal{B}$,

- $\Theta(x,y) \leq \Theta(x * z, y * z)$,
- $\Theta(x,y) \leq T(\Theta(x \to z, y \to z), \Theta(z \to x, z \to y))$.

Any T-fuzzy compatible equivalence on $\mathcal{B}$ is called a *T-fuzzy congruence*. The fuzzy subset $\Theta^x : \mathcal{B} \to [0,1]$, which is defined by $\Theta^x(y) = \Theta(x,y)$, is called the *fuzzy congruence class* containing x. Let $\mathcal{B}/\Theta$ be the set of all T-fuzzy congruence classes Θ^x.

Proposition 4.2. *For (imaginable) T-fuzzy congruence Θ in $\mathcal{B}$, Θ^1 is an (imaginable) T-fuzzy filter of $\mathcal{B}$.*

Proof. Let $x \in \mathcal{B}$. Then, $\Theta^1(1) = \Theta(1,1) = \sup_{(y,z)\in\mathcal{B}^2} \Theta(y,z) \geq \Theta(1,x) = \Theta^1(x)$. Now, let $x, y \in \mathcal{B}$. Then

$$\begin{aligned}
\Theta^1(y) &= \Theta(1,y) \geq T(\Theta(1, x \to y), \Theta(x \to y, y)) \quad \text{by transitivity} \\
&= T(\Theta^1(x \to y), \Theta(x \to y, 1 \to y)) \\
&\geq T(\Theta^1(x \to y), \Theta(x,1)) \quad \text{by compatibility} \\
&= T(\Theta^1(x), \Theta^1(x \to y)).
\end{aligned}$$

This completes the proof. □

Theorem 4.3. *For imaginable T-fuzzy filter $\mathfrak{F}$ of $\mathcal{B}$ there is an imaginable T-fuzzy congruence Θ such that $\Theta^1 = \mathfrak{F}$.*

Proof. Let $\mathfrak{F}$ be an imaginable T-fuzzy filter of $\mathcal{B}$. Define a fuzzy relation Θ in $\mathcal{B}$ by

$$\Theta(x,y) = T(\mathfrak{F}(x \to y), \mathfrak{F}(y \to x)).$$

Then $\Theta(1,1) = T(\mathfrak{F}(1), \mathfrak{F}(1)) \geq T(\mathfrak{F}(x \to y), \mathfrak{F}(y \to x)) = \Theta(x, y)$, for all $x, y \in \mathcal{B}$. So, $\Theta(1,1) \geq \sup_{(x,y)\in\mathcal{B}^2} \Theta(x, y)$ means that $\Theta(1,1) = \sup_{(x,y)\in\mathcal{B}^2} \Theta(x, y)$. Symmetry is clear. For transitivity, let $x, y, z \in \mathcal{B}$. Then,

$$\begin{array}{rcl}\Theta(x,y) & = & T(\mathfrak{F}(x \to y), \mathfrak{F}(y \to x)) \\ & \geq & T(T(\mathfrak{F}(x \to z), \mathfrak{F}(z \to y)), T(\mathfrak{F}(y \to z), \mathfrak{F}(z \to x))) \quad \text{by } (3.11) \\ & = & T(T(\mathfrak{F}(x \to z), \mathfrak{F}(z \to x)), T(\mathfrak{F}(z \to y), \mathfrak{F}(y \to z))) \\ & = & T(\Theta(x, z), \Theta(z, y)).\end{array}$$

The compatibility of Θ is easily followed from (3.8), (3.9) and (3.10).

Obviously, if $a, b \in [0, 1]$ are idempotent, then $T(a, b)$ is also an idempotent element of T. Also, it is clear that $\Theta^1 = \mathfrak{F}$. This completes the proof. □

Remark. The T-fuzzy congruence Θ in the above theorem is called the T-fuzzy congruence induced by T-fuzzy filter $\mathfrak{F}$ and denoted by $\Theta_{\mathfrak{F}}$.

Proposition 4.4. *Let Θ be an imaginable T-fuzzy congruence in $\mathcal{B}$. Then $\Theta_{\Theta^1} = \Theta$.*

Proof. Let $x, y \in \mathcal{B}$. Then,

$$\begin{array}{rcl}\Theta_{\Theta^1}(x,y) & = & T(\Theta^1(x \to y), \Theta^1(y \to x)) \\ & = & T(\Theta(1, x \to y), \Theta(1, y \to x)) \\ & = & T(\Theta(x \to y, 1), \Theta(1, y \to x)) \\ & \leq & \Theta(x \to y, y \to x).\end{array}$$

On the other hand, by compatibility and (2.2) we get

$$\Theta(x \to y, y \to x) \leq \Theta(x * (x \to y), x * (y \to x)) \leq \Theta(y, 1)$$

and

$$\Theta(x \to y, y \to x) \leq \Theta(y * (x \to y), y * (y \to x)) \leq \Theta(1, x).$$

Hence,

$$\begin{array}{rcl}\Theta(x \to y, y \to x) & = & T(\Theta(x \to y, y \to x), \Theta(x \to y, y \to x)) \leq T(\Theta(y, 1), \Theta(1, x)) \\ & \leq & \Theta(y, x) = \Theta(x, y)\end{array}$$

and so $\Theta_{\Theta^1}(x, y) \leq \Theta(x, y)$. For the converse inequality, we have

$$\begin{array}{rcl}\Theta(x,y) & \leq & T(\Theta(x \to x, y \to x), \Theta(x \to x, x \to y)) \quad \text{by compatibility} \\ & = & T(\Theta(1, y \to x), \Theta(1, x \to y)) \\ & = & T(\Theta^1(y \to x), \Theta^1(x \to y)) \\ & = & \Theta_{\Theta^1}(x, y).\end{array}$$

Thus, $\Theta = \Theta_{\Theta^1}$. □

Proposition 4.5. *For imaginable T-fuzzy filter $\mathfrak{F}$ of $\mathcal{B}$, $\Theta^1_{\mathfrak{F}} = \mathfrak{F}$.*

Proof. The proof is easy. □

Theorem 4.6. *(Correspondence Theorem) There is a correspondence between $\mathcal{C}$, the set of all imaginable T-fuzzy congruences in $\mathcal{B}$, and $\mathcal{F}$, the set of all imaginable T-fuzzy filters of $\mathcal{B}$.*

Proof. Define the maps $\Phi : \mathcal{F} \longrightarrow \mathcal{C}$ by $\mathfrak{F} \mapsto \Theta_{\mathfrak{F}}$, and $\Psi : \mathcal{C} \longrightarrow \mathcal{F}$ by $\Psi(\Theta) = \Theta^1$. By Propositions 4.4 and 4.5, it is easy to check that $\Phi\Psi = 1_{\mathcal{C}}$ and $\Psi\Phi = 1_{\mathcal{F}}$, proving that Φ is a bijection. $\square$

Corollary 4.7. *Every imaginable T-fuzzy congruence in $\mathcal{B}$ is determined by an imaginable T-fuzzy filter.*

Lemma 4.8. $\Theta^x_{\mathfrak{F}} = \Theta^y_{\mathfrak{F}}$ *if and only if* $\mathfrak{F}(x \to y) = \mathfrak{F}(y \to x) = \mathfrak{F}(1)$.

Proof. Let $\Theta^x_{\mathfrak{F}} = \Theta^y_{\mathfrak{F}}$, for $x, y \in \mathcal{B}$. Then $\Theta_{\mathfrak{F}}(x, x) = \Theta_{\mathfrak{F}}(y, x)$ and hence

$$\begin{aligned} \mathfrak{F}(1) &= \mathfrak{F}(x \to x) = T(\mathfrak{F}(x \to x), \mathfrak{F}(x \to x)) = T(\mathfrak{F}(x \to y), \mathfrak{F}(y \to x)) \\ &\leq \mathfrak{F}(x \to y), \mathfrak{F}(y \to x) \end{aligned}$$

which implies that $\mathfrak{F}(x \to y) = \mathfrak{F}(y \to x) = \mathfrak{F}(1)$.

Conversely, suppose that $\mathfrak{F}(x \to y) = \mathfrak{F}(y \to x) = \mathfrak{F}(1)$ and let $z \in \mathcal{B}$. By (3.11),

$$\mathfrak{F}(z \to x) \geq T(\mathfrak{F}(z \to y), \mathfrak{F}(y \to x)) = \mathfrak{F}(z \to y)$$

and

$$\mathfrak{F}(z \to y) \geq T(\mathfrak{F}(z \to x), \mathfrak{F}(x \to y)) = \mathfrak{F}(z \to x).$$

This implies that $\mathfrak{F}(z \to y) = \mathfrak{F}(z \to x)$ and similarly we can deduce that $\mathfrak{F}(x \to z) = \mathfrak{F}(y \to z)$. Consequently,

$$\Theta^x_{\mathfrak{F}}(z) = T(\mathfrak{F}(x \to z), \mathfrak{F}(z \to x)) = T(\mathfrak{F}(y \to z), \mathfrak{F}(z \to y)) = \Theta^y_{\mathfrak{F}}(z).$$

$\square$

For a T-fuzzy filter $\mathfrak{F}$ of $\mathcal{B}$ let $\mathfrak{F}_* = \{x \in \mathcal{B} : \mathfrak{F}(x) = \mathfrak{F}(1)\}$. Obviously, $\mathfrak{F}_*$ is a filter of $\mathcal{B}$.

Corollary 4.9. *Let $\mathfrak{F}$ be an imaginable T-fuzzy filter of $\mathcal{B}$. Then $\Theta^x_{\mathfrak{F}} = \Theta^y_{\mathfrak{F}}$ if and only if $x \equiv_{\mathfrak{F}_*} y$.*

Let $\mathfrak{F}$ be a T-fuzzy filter of $\mathcal{B}$. For $x \in \mathcal{B}$, the fuzzy subset $\mathfrak{F}_x$ is called a *fuzzy coset* of $\mathfrak{F}$ such that $\mathfrak{F}_x(y) = T(\mathfrak{F}(x \to y), \mathfrak{F}(y \to x))$. Let $\mathcal{B}/\mathfrak{F} = \{\mathfrak{F}_x : x \in \mathcal{B}\}$. Obviously, $\mathfrak{F}_x = \Theta^x_{\mathfrak{F}}$ and so $\mathcal{B}/\mathfrak{F} = \mathcal{B}/\Theta_{\mathfrak{F}}$. Now, on $\mathcal{B}/\mathfrak{F}$, define the following operations

$$\mathfrak{F}_x \sqcup \mathfrak{F}_y = \mathfrak{F}_{x \vee y},\ \mathfrak{F}_x \sqcap \mathfrak{F}_y = \mathfrak{F}_{x \wedge y},\ \mathfrak{F}_x \bullet \mathfrak{F}_y = \mathfrak{F}_{x * y},\ \mathfrak{F}_x \rightharpoonup \mathfrak{F}_y = \mathfrak{F}_{x \to y}.$$

It should be noted that the union operation induces a partial ordering '$\preceq$' on $\mathcal{B}/\mathfrak{F}$ by '$\mathfrak{F}_x \preceq \mathfrak{F}_y$ if and only if $\mathfrak{F}_x \sqcup \mathfrak{F}_y = \mathfrak{F}_y$'.

In virtue of the above observations we give the following easy result.

Theorem 4.10. *Let $\mathfrak{F}$ be an imaginable T-fuzzy filter of $\mathcal{B}$. Then $(\mathcal{B}/\mathfrak{F}, \sqcup, \sqcap, \bullet, \rightharpoonup, \mathfrak{F}_0, \mathfrak{F}_1)$ is a BL-algebra.*

Proof. The conditions (BL1), (BL2), (BL4) and (BL5) are easily proved. For (BL3), let $\mathfrak{F}_x, \mathfrak{F}_y, \mathfrak{F}_z \in \mathcal{B}/\mathfrak{F}$. Then

$$\begin{aligned}
\mathfrak{F}_x \bullet \mathfrak{F}_y \leq \mathfrak{F}_z \quad &\Leftrightarrow \quad \mathfrak{F}_{x*y} \leq \mathfrak{F}_z \Leftrightarrow \mathfrak{F}_{x*y} \sqcup \mathfrak{F}_z = \mathfrak{F}_z \Leftrightarrow \mathfrak{F}_{(x*y)\vee z} = \mathfrak{F}_z \\
&\Leftrightarrow \quad \mathfrak{F}((x*y) \rightarrow z) = \mathfrak{F}(1) \quad \text{by Lemma 4.8 and (2.15)} \\
&\Leftrightarrow \quad \mathfrak{F}(x \rightarrow (y \rightarrow z)) = \mathfrak{F}(1) \quad \text{by (2.8)} \\
&\Leftrightarrow \quad \mathfrak{F}_x = \mathfrak{F}_y \rightharpoonup \mathfrak{F}_z.
\end{aligned}$$

□

Theorem 4.11. *Let $\mathfrak{F}$ be an imaginable T-fuzzy filter of $\mathcal{B}$. Then $\mathcal{B}/\mathfrak{F} \simeq \mathcal{B}/\mathfrak{F}_*$.*

Proof. It is easy to verify that the mapping $\Phi : \mathcal{B}/\mathfrak{F} \longrightarrow \mathcal{B}/\mathfrak{F}_*$ by $\Phi(\mathfrak{F}_x) = [x]_{\mathfrak{F}_*}$ is an isomorphism. □

Theorem 4.12. *(Homomorphism Theorem) Let $f : \mathcal{B} \longrightarrow \mathcal{C}$ be a homomorphism of BL-algebras and $\mathfrak{F}$ an imaginable T-fuzzy filter of $\mathcal{B}$ such that $\mathfrak{F}_* = kerf$. Then $\mathcal{B}/\mathfrak{F} \simeq f(\mathcal{B})$.*

Proof. Let $\mathfrak{F}$ be an imaginable T-fuzzy filter of $\mathcal{B}$. Then $\mathfrak{F}_*$ is a filter of $\mathcal{B}$ and so $\mathcal{B}/\mathfrak{F}_* \simeq \mathfrak{F}(\mathcal{B})$. Thus, Theorem 4.11 completes the proof. □

Let $\mathfrak{F}$ and $\mathfrak{G}$ be two T-fuzzy filters of $\mathcal{B}$. We say that $\mathfrak{G}$ is a T-fuzzy filter of $\mathfrak{F}$ if $\mathfrak{G} \subseteq \mathfrak{F}$. In this case, we define a fuzzy subset $\mathfrak{F}/\mathfrak{G}$ of $\mathcal{B}/\mathfrak{G}$ by $(\mathfrak{F}/\mathfrak{G})(\mathfrak{G}^x) = \mathfrak{F}(x)$.

Lemma 4.13. *For two T-fuzzy filters $\mathfrak{F}$ and $\mathfrak{G}$ of $\mathcal{B}$, $\mathfrak{F}/\mathfrak{G}$ is a T-fuzzy filter of $\mathcal{B}/\mathfrak{G}$.*

Proof. The proof is easy. □

Theorem 4.14. *(Second Isomorphism Theorem) Let $\mathfrak{F}$ and $\mathfrak{G}$ be two T-fuzzy filters of $\mathcal{B}$ such that $\mathfrak{G} \subseteq \mathfrak{F}$. Then $(\mathcal{B}/\mathfrak{G})/(\mathfrak{F}/\mathfrak{G}) \simeq \mathcal{B}/\mathfrak{F}$.*

Proof. Define $\Psi : \mathcal{B}/\mathfrak{G} \longrightarrow \mathcal{B}/\mathfrak{F}$ with $\mathfrak{G}^x \mapsto \mathfrak{F}^x$. It is easy to verify that Ψ is an epimorphism. Now,

$$\begin{aligned}
ker\Psi \quad &= \quad \{\mathfrak{G}^x : \mathfrak{F}^x = \Psi(\mathfrak{G}^x) = \mathfrak{F}^1\} = \{\mathfrak{G}^x : \mathfrak{F}(x) = \mathfrak{F}(1)\} \\
&= \quad \{g^x : (\mathfrak{F}/\mathfrak{G})(\mathfrak{G}^x) = (\mathfrak{F}/\mathfrak{G})(g^1)\} = (\mathfrak{F}/\mathfrak{G})_*.
\end{aligned}$$

By Homomorphism Theorem the proof is complete. □

Acknowledgements. This paper is partially supported by the Fuzzy Systems and Its Applications Center of Excellence, Shahid Bahonar University of Kerman, Iran.

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