ON SOME STRUCTURES OF FUZZY NUMBERS

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Abstract. The operations in the set of fuzzy numbers are usually obtained by the Zadeh extension principle. But these definitions can have some disadvantages for the applications both by an algebraic point of view and by practical aspects. In fact the Zadeh multiplication is not distributive with respect to the addition, the shape of fuzzy numbers is not preserved by multiplication, the indeterminateness of the sum is too increasing. Then, for the applications in the Natural and Social Sciences it is important to individuate some suitable variants of the classical addition and multiplication of fuzzy numbers that have not the previous disadvantage. Here, some possible alternatives to the Zadeh operations are studied.

1. Introduction and Motivation

The operations in the set of fuzzy numbers are usually obtained by the Zadeh extension principle ([19], [21], [18]).

These definitions can have some disadvantages for the applications, e.g. to Social Sciences, both by an algebraic point of view and by logical and practical aspects.

In particular, the Zadeh multiplication is not distributive with respect to the addition, the shape of fuzzy numbers is not preserved by multiplication, the indeterminateness of the sum and product is often too increasing. Then, for the applications in the Natural and Social Sciences it is important to individuate some suitable variants of the Zadeh addition and multiplication of fuzzy numbers that have not the previous disadvantages. Some of these variants are considered in [1], [9], and [13].

In this paper we study three possible types of alternative operations:

(a) *Operations with bounded spreads*;

(b) *Operations up to an equivalence relation*;

(c) *Hyperoperations of fuzzy numbers.*

In the case (a) we introduce particular classes of "addition" and "multiplication" in such a way that the indeterminateness is not too enlarged. For some of these operations very interesting mathematical properties hold, in particular this way is the most suitable for the mathematical problems of convergence of sequences of fuzzy numbers. Moreover, among these operations, there are additions and multiplications that satisfy the distributive property.

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In the case (b) we introduce an equivalence relation in the set of fuzzy numbers compatible with given operations, e.g. the Zadeh operations or particular operations with bounded spreads. Moreover, we consider the induced operations on the equivalence classes. This way was proposed in [10] and [14] for triangular fuzzy numbers and attains the aim to obtain a vector space. This is very important, because the structure of linear space is the fundament of the theory of the coherent prevision and probability ([4], [13]) and so if we have a vector space of fuzzy numbers, then we can introduce an extension in a fuzzy ambit of many concepts and results on the coherent prevision and probability. For extensions of the probability in a fuzzy ambit see also [20], [11],[12]. An extension of the concept of coherence is in [15].

In the case (c) we introduce a very different point of view. This can also be seen as a generalization of the previous cases. We replace the operations with hyperoperations ([2], [3]) and so groups and vector spaces are replaced by hypergroups and vector hyperspaces. We obtain, in particular, a new and the most general way for the fuzzy extensions of the concepts of coherent prevision and probability.

2. Preliminaries

Definition 2.1. [8] Let $\mathcal{C}$ be the set of the compact intervals of $\mathbb{R}$. For every pair of intervals $[a, b]$ and $[c, d]$ in $\mathcal{C}$, we assume:

$$[a, b] + [c, d] = [a + c, b + d]; \tag{1}$$

$$[a, b] \cdot [c, d] = [\min\{ac, ad, bc, bd\}, \max\{ac, ad, bc, bd\}]. \tag{2}$$

The subtraction and the division are also defined on $\mathcal{C}$ by the formulae:

$$[a, b] - [c, d] = [a, b] + [-d, -c]; \tag{3}$$

$$\forall [c, d] : 0 \notin [c, d],\ [a, b]/[c, d] = [a, b] \cdot [1/d, 1/c]. \tag{4}$$

Remark 2.2. ([8], 103) The addition + defined by (1) is commutative, associative, having $0 = [0, 0]$ as neutral element. The multiplication $\cdot$ defined by (2) is commutative, associative, having $1 = [1, 1]$ as neutral element. Moreover, the following subdistributive property holds:

$$\forall [a, b], [c, d], [e, f] \in \mathcal{C},\ ([a, b] + [c, d]) \cdot [e, f] \subseteq [a, b] \cdot [e, f] + [c, d] \cdot [e, f]. \tag{5}$$

The equality holds if and only if $(a \geq 0, c \geq 0)$ or $(b \leq 0, d \leq 0)$ or $(e = f)$.

Definition 2.3. For every pair of intervals $[a, b]$ and $[c, d]$ in $\mathcal{C}$, we assume:

$$[a, b] \leq [c, d] \Leftrightarrow a \leq c, b \leq d. \tag{6}$$

Remark 2.4. The relation (6) is a partial order relation on $\mathcal{C}$. Moreover it is compatible with the operations + and $\cdot$, i.e.

$$\forall [a, b], [c, d], [e, f] \in \mathcal{C},\ [a, b] \leq [c, d] \Rightarrow [a, b] + [e, f] \leq [c, d] + [e, f]; \tag{7}$$

$$\forall [a, b], [c, d] \in \mathcal{C},\ ([a, b] \geq 0, [c, d] \geq 0) \Rightarrow [a, b] \cdot [c, d] \geq 0. \tag{8}$$

Definition 2.5. [8] A fuzzy number is a real function with values in [0, 1] and having as domain the set of real numbers, $u : \mathbb{R} \to [0, 1]$, such that:

- **(FN1):** (**bounded support**) there are two real numbers a, b, with $a \leq b$, called *the endpoints* of u, such that:
  - $u(x) = 0$ for x not belonging to the closed interval $[a, b]$;
  - $u(x) > 0$ for x belonging to the open interval (a, b).
- **(FN2):** (**normality**) there are two real numbers c, d, with $a \leq c \leq d \leq b$ such that $u(x) = 1$ if and only if $x \in [c, d]$.
- **(FN3): (convexity)** $u(x)$ is a function increasing in the interval $[a, c]$ and decreasing in the interval $[d, b]$.
- **(FN4):** (**compactness**) for every $r \in (0, 1)$ the set $\{x \in \mathbb{R} : u(x) \geq r\}$ is a closed interval.

The set of real numbers x such that $u(x) > 0$ is said to be the *support* of u, denoted $S(u)$, and the interval $[c, d]$ is said to be the *core* of u, noted $C(u)$. The intervals $[a, c)$ and $(d, b]$ are the *left part* and the *right part* of u, respectively.

The fuzzy number u is said to be *simple* if $c = d$, i.e. $C(u)$ is a singleton. Moreover, u is said to be *degenerate* if $a = b$, i.e. $S(u) = \{c\}, c \in R$. In this case u is identified with the real number c.

The real numbers $L(u) = c - a$, $M(u) = d - c$, and $R(u) = b - d$ are the *left*, *middle*, and *right indeterminateness* of u, respectively. They are also called the *spreads* of u. Their sum $T(u) = b - a$ is the *total indeterminateness* of u.

For every r such that $0 \leq r \leq 1$ the set of the $x \in [a, b]$ such that $u(x) \geq r$ is noted $[u]^r$ and is said to be the *r-cut* of u. Let us denote with u^r_λ and u^r_ρ the left and right endpoints of $[u]^r$, respectively. In particular $[u]^0 = [u^0_\lambda = a, u^0_\rho = b]$ is the closure of the support of u and $[u]^1 = [u^1_\lambda = c, u^1_\rho = d]$ is the core of u.

Remark 2.6. We assume the following notations:

- (*endpoints notation*) $u \sim (a, c, d, b)$ stands u is a fuzzy number with endpoints a, b and core $[c, d]$; $u \sim (a, c, b)$ for $u \sim (a, c, c, b)$;
- (*spreads notation*) $u \sim [c, d, L, R]$ denotes that u is a fuzzy number with core $[c, d]$ and left and right spreads L and R, respectively; $u \sim [c, L, R]$ denotes $u \sim [c, c, L, R]$;
- (*r-cut spreads notation*) the numbers $L^r(u) = (c - u^r_\lambda)$ and $R^r(u) = (u^r_\rho - d)$ are called the *r-cut left spread* and the *r-cut right spread* of u, we write $[u]^r = [c, d, L^r(u), R^r(u)]$, and, if u is simple, we write also $[u]^r = [c, L^r(u), R^r(u)]$;
- (*sign*) the fuzzy number $u \sim (a, c, d, b)$ is said to be *positive*, *strictly positive*, *negative*, or *strictly negative*, if $a \geq 0$, $a > 0$, $b \leq 0$, or $b < 0$, respectively.

Definition 2.7. [8], [5] We say that the fuzzy number $u \sim (a, c, d, b)$ is a *trapezoidal fuzzy number*, denoted $u = (a, c, d, b)$, if:

$$x \in [a, c), a < c \Rightarrow u(x) = (x - a)/(c - a), \tag{9}$$

$$x \in (d, b], d < b \Rightarrow u(x) = (b - x)/(b - d). \tag{10}$$

A simple trapezoidal fuzzy number $u = (a, c, c, b)$ is said to be a *triangular fuzzy number*, denoted $u = (a, c, b)$. A trapezoidal fuzzy number $u = (c, c, d, d)$, with support equal to the core is said to be a *rectangular fuzzy number* and is identified with the compact interval $[c, d]$ of $\mathbb{R}$.

Remark 2.8. The necessary and sufficient conditions for $u \sim [c, d, L, R]$ be a trapezoidal fuzzy number, we write $u = [c, d, L, R]$, in terms of r-cut left and right spreads, are:

$$L^r(u) = (1 - r)(c - a) = (1 - r)L, \; R^r(u) = (1 - r)(b - d) = (1 - r)R. \tag{11}$$

3. Zadeh Operations and Order

Let Ψ and Φ be the set of all the fuzzy sets with universe the set $\mathbb{R}$ of real numbers and the set of the fuzzy numbers, respectively. If $\circ$ is an operation in $\mathbb{R}$, we can extend $\circ$ to Ψ with the extension principle of Zadeh as follows ([19], [21], [18], [8], [5]).

Let $u : \mathbb{R} \to [0, 1]$, $v : \mathbb{R} \to [0, 1]$ be two elements of Ψ. The extension principle of Zadeh assumes as "*product*" $u \circ v$ the fuzzy set, with universe $\mathbb{R}$, defined by the formula:

$$(u \circ v)(z) = sup\{min\{u(x), v(y)\} : x \circ y = z\}. \tag{12}$$

For a generic operation $\circ$, if u and v are fuzzy numbers, it may be that $u \circ v$ is not a fuzzy number, but if we consider the addition $+$, the substraction $-$, the multiplication $\cdot$, or, if $0 \notin [v]^0$, the division $/$, then $u \circ v$ is also a fuzzy number.

If $u \circ v$ is a fuzzy number then (12) can be expressed in terms of r-cuts with the formula:

$$\forall r \in [0, 1], [u \circ v]^r = [u]^r \circ [v]^r. \tag{13}$$

Then the Zadeh operations on Φ reduce to the ones on the set $\mathcal{C}$ of the compact intervals of $\mathbb{R}$. In an analogous way an order relation $\leq$ is introduced in Φ with the formula:

$$u \leq v \Leftrightarrow \forall r \in [0, 1], [u]^r \leq [v]^r. \tag{14}$$

The Zadeh definition of $u \circ v$ can have some drawbacks for the practical applications. For instance, they can arise by the following problems:

- **(S1):** The multiplication is not distributive with respect to the addition. In fact from (5) it follows that for every fuzzy numbers u, v, w, $(u + v) \cdot w \subseteq u \cdot w + v \cdot w$. The equality holds if u and v are both non negative or both non positive or w is a degenerate fuzzy number.
- **(S2):** If we wish to consider special classes of fuzzy numbers as the trapezoidal or triangular ones, they are not closed over the multiplication, because in general the product of two trapezoidal fuzzy numbers is not trapezoidal.
- **(S3):** The indeterminateness of the sum of two fuzzy numbers is the sum of the indeterminateness, then by aggregating as "sum" fuzzy numbers we can have an indeterminateness too increasing. This is in contradiction with the intuitive idea that many indeterminateness can compensate each other.

Specially for the applications in the Social Sciences it is important to individuate some suitable variants of the classical addition and multiplication of fuzzy numbers, called also, respectively, "addition" and "multiplication", that have the most part of the properties of the classical correspondent operations, but that have not the previous disadvantages. We present some proposals in the next sections.

4. **Operations with Bounded Spreads**

In order to control left and right spreads, we introduce a class of operations based on the concepts of t-conorm and of t-norm. As particular cases we have the Zadeh addition, and four operations, called *i-addition*, *i-multiplication*, *u-addition*, and *u-multiplication*, such that, in the set of simple fuzzy numbers, the i-multiplication is distributive with respect to the i-addition and the u-multiplication is distributive with respect to the u-addition.

We assume there exist two positive real numbers L_m and R_m that are the maximum left and right indeterminateness, respectively. Let S be the set of fuzzy numbers such that, for every $u \in S$, $L(u) \leq L_m$, and $R(u) \leq R_m$.

Let us denote with $\oplus$ a t-conorm, i.e. an operation

$$\oplus : (a, b) \in [0, 1] \times [0, 1] \to a \oplus b \in [0, 1]$$

associative, commutative, having 0 as neutral element and increasing with respect to every variable (see, e.g., [16], [15], [17], [8]). Moreover, let us denote with $\otimes$ a t-norm, i.e. an operation

$$\otimes : (a, b) \in [0, 1] \times [0, 1] \to a \otimes b \in [0, 1]$$

associative, commutative, having 1 as neutral element and increasing with respect to every variable (see, e.g., [16], [15], [17], [8]).

For every $\diamond \in \{\oplus, \otimes\}$ let us introduce an addition $+_\diamond$ and a multiplication $\cdot_\diamond$ on S, called $\diamond$-*addition* and $\diamond$-*multiplication*, respectively, in such a way that the left and right indeterminateness of the sum $u +_\diamond v$ and the product $u \cdot_\diamond v$ are not greater than L_m and R_m, respectively.

Definition 4.1. We define the $\diamond$-addition on S by formulae:

$$C(u +_\diamond v) = C(u) + C(v); \tag{15}$$

$$\forall r \in [0, 1), L^r(u +_\diamond v) = [(L^r(u)/L_m) \diamond (L^r(v)/L_m)]L_m; \tag{16}$$

$$\forall r \in [0, 1), R^r(u +_\diamond v) = [(R^r(u)/R_m) \diamond (R^r(v)/R_m)]R_m. \tag{17}$$

Analogous formulae are introduced for the $\diamond$-multiplication:

$$C(u \cdot_\diamond v) = C(u) \cdot C(v); \tag{18}$$

$$\forall r \in [0, 1), L^r(u \cdot_\diamond v) = [(L^r(u)/L_m) \diamond (L^r(v)/L_m)]L_m; \tag{19}$$

$$\forall r \in [0, 1), R^r(u \cdot_\diamond v) = [(R^r(u)/R_m) \diamond (R^r(v)/R_m)]R_m. \tag{20}$$

By previous definitions it follows:

Proposition 4.2. *Every $\diamond$-addition is associative, commutative, with neutral element 0; moreover every $\diamond$-multiplication is associative, commutative, with neutral element 1.*

Remark 4.3. If $\diamond$ is the *bounded sum*, i.e.

$$\forall a, b \in [0,1], a \diamond b = \min\{a + b, 1\}$$

then we have:

$$L^r(u +_\diamond v) = \min\{L^r(u) + L^r(v), L_m\};$$
$$R^r(u +_\diamond v) = \min\{R^r(u) + R^r(v), R_m\}.$$

Then, if we work with fuzzy numbers having left and right spreads very smaller than L_m and R_m, respectively, then the $\diamond$-addition reduces to the Zadeh addition.

Remark 4.4. If $\diamond = \oplus_u$ is the *fuzzy union*, i.e.

$$\forall a, b \in [0,1], a \diamond b = a \oplus_u b = \max\{a, b\},$$

then we have:

$$L^r(u +_\diamond v) = \max\{L^r(u), L^r(v)\};$$
$$R^r(u +_\diamond v) = \max\{R^r(u), R^r(v)\}.$$

In this case let us call *u-addition* and *u-multiplication* the correspondent $\oplus_u$-addition and $\oplus_u$-multiplication, respectively.

Remark 4.5. If $\diamond = \oplus_d$ is the *drastic union* ([8], 78), i.e.

$$\forall a, b \in [0,1], a \diamond b = \begin{cases} a, & \text{when } b = 0; \\ b, & \text{when } a = 0; \\ 1, & \text{otherwise;} \end{cases}$$

then we have:

$$L^r(u +_\diamond v) = \begin{cases} L^r(u), & \text{when } L^r(v) = 0; \\ L^r(v), & \text{when } L^r(u) = 0; \\ L_m, & \text{otherwise.} \end{cases}$$

$$R^r(u +_\diamond v) = \begin{cases} R^r(u), & \text{when } R^r(v) = 0; \\ R^r(v), & \text{when } R^r(u) = 0; \\ R_m, & \text{otherwise.} \end{cases}$$

Remark 4.6. For every t-conorm $\oplus$ and for every $a, b \in [0,1]$ it is ([8], 78) $\max\{a, b\} \le a \oplus b \le a \oplus_d b$. It follows that, for every $u, v \in S$, it is:

$$L^r(u +_{\oplus_u} v) \le L^r(u +_\oplus v) \le L^r(u +_{\oplus_d} v), \tag{21}$$

$$R^r(u +_{\oplus_u} v) \le R^r(u +_\diamond v) \le R^r(u +_{\oplus_d} v). \tag{22}$$

Remark 4.7. If $\diamond$ is the *bounded difference*, i.e.

$$\forall a, b \in [0,1], a \diamond b = \max\{a + b - 1, 0\},$$

then we have:

$$L^r(u +_\diamond v) = \max\{L^r(u) + L^r(v) - L_m, 0\};$$
$$R^r(u +_\diamond v) = \max\{R^r(u) + R^r(v) - R_m, 0\}.$$

Remark 4.8. If $\diamond = \otimes_i$ is the *fuzzy intersection*, i.e.

$$\forall a, b \in [0,1], a \diamond b = a \otimes_i b = \min\{a, b\}$$

then we have:

$$L^r(u +_\diamond v) = \min\{L^r(u), L^r(v)\};$$
$$R^r(u +_\diamond v) = \min\{R^r(u), R^r(v)\}.$$

In this case let us call *i-addition* and *i-multiplication* the correspondent $\otimes_i$-addition and $\otimes_i$-multiplication, respectively.

Remark 4.9. If $\diamond = \otimes_d$ is the *drastic intersection* ([8], 63), i.e.

$$\forall a, b \in [0,1], a \diamond b = \begin{cases} a, & \text{when } b = 1; \\ b, & \text{when } a = 1; \\ 0, & \text{otherwise}; \end{cases}$$

then we have:

$$L^r(u +_\diamond v) = \begin{cases} L^r(u), & \text{when } L^r(v) = 1; \\ L^r(v), & \text{when } L^r(u) = 1; \\ 0, & \text{otherwise.} \end{cases}$$

$$R^r(u +_\diamond v) = \begin{cases} R^r(u), & \text{when } R^r(v) = 1; \\ R^r(v), & \text{when } R^r(u) = 1; \\ 0, & \text{otherwise.} \end{cases}$$

Remark 4.10. For every t-norm $\otimes$ and for every $a, b \in [0,1]$ it is ([8], 65) $a \otimes_d b \leq a \otimes b \leq \min\{a, b\}$. It follows that, for every $u, v \in S$, it is:

$$L^r(u +_{\otimes_d} v) \leq L^r(u +_\otimes v) \leq L^r(u +_{\otimes_i} v), \tag{23}$$

$$R^r(u +_{\otimes_d} v) \leq R^r(u +_\otimes v) \leq R^r(u +_{\otimes_i} v). \tag{24}$$

The following theorem holds:

Theorem 4.11. *Let Φ^s be the set of simple fuzzy numbers. Then the u-addition, the u-multiplication, the i-addition and the i-multiplication are defined on Φ^s. Moreover the u-multiplication is distributive with respect to the u-addition and the i-multiplication is distributive with respect to the i-addition.*

Remark 4.12. The set of triangular fuzzy numbers is a subset of Φ^s and is closed with respect to the u-addition, u-multiplication, i-addition, and i-multiplication. Since triangular fuzzy numbers are very simple to handle, from previous theorem we have a further justification of their wide utilization.

The $\diamond$-additions introduced in this section seem to be reasonable, and each of these can be advantageous for particular situations; on the contrary, the $\diamond$-multiplications have some drawbacks, in particular they do not satisfy the intuitive idea that the spreads of the product must depend not only on the correspondent spreads of the factors, but also on the sizes of their cores. Nevertheless the u-multiplication and the i-multiplication have some interest, because of the distributive property.

5. Operations up to an Equivalence Relation

In the book by de Finetti [4] and in [10] and [14] it is proved that the concepts of coherent prevision of a random number, and in particular the concept of coherent probability, is based on the properties of the random numbers to be a group and a vector space over the real numbers. It follows that, in order to extend the concepts of coherent prevision and probability in a fuzzy ambit, it is necessary to obtain a structure of vector space based on fuzzy numbers.

But, whatever definition of sum is utilized, the sum $u + v$ of two fuzzy numbers has left and right spreads greater than the correspondent spreads of u and v. Then we cannot have the additive inverse of a non degenerate fuzzy number and fuzzy numbers are neither a group nor a vector space.

In this section we prove that we can overcome this obstacle by introducing a suitable equivalence relation $\sim$ on the set Φ of fuzzy numbers and by considering the quotient set $\Phi/\sim$ and the induced structures. In fact, in this case we obtain a vector space.

Let S be a set of fuzzy numbers, and let $\sim$ be an equivalence relation on S. For every $u \in S$, let $[u]$ denote the equivalence class of u. Let $+$ be an operation defined on S. The relation $\sim$ is compatible with the operation $+$ if, for every $u, u', v, v' \in S$, the following implication holds:

$$([u] = [u'], [v] = [v']) \Rightarrow [u + v] = [u' + v'].$$

Let $*$ be a scalar multiplication on S, i.e. a function $* : \mathbb{R} \times S \to S$. The relation $\sim$ is compatible with $*$ if,

$$\forall \alpha \in \mathbb{R}, u, u' \in S, ([u] = [u']) \Rightarrow ([\alpha * u] = [\alpha * u']).$$

If $\sim$ is compatible with $+$ and $*$, then we can consider the induced operations $+$ and $*$ on $S/\sim$, defined as:

$$\forall [u], [v] \in S/\sim, [u] + [v] = [u + v]; \forall \alpha \in \mathbb{R}, [u] \in S/\sim, \alpha * [u] = [\alpha * u].$$

For every fuzzy number u and for every $r \in [0, 1]$, let us put

$$D^r(u) = R^r(u) - L^r(u).$$

A particular important case is the equivalence $\sim_{sc}$ in the set Φ of all the fuzzy numbers, we call *spread compensation*, defined by:

$$u \sim_{sc} v \Leftrightarrow (C(u) = C(v) \quad \texttt{and} \quad \forall r \in [0, 1), D^r(u) = D^r(v)).$$

The following propositions hold:

Proposition 5.1. *The spread compensation $\sim_{sc}$ is compatible with Zadeh addition and scalar multiplication of a fuzzy number by a real number.*

Proposition 5.2. *The addition $+$ induced by Zadeh addition in the quotient set $\Phi/\sim_{sc}$ is associative, commutative, and has $[0]$ as neutral element. Moreover, an element $[u]$ of $\Phi/\sim_{sc}$ has additive inverse $[v] = -[u]$ if and only if u is simple.*

Propositions 5.1 and 5.2 imply the following result of structural importance, as we will stress:

Proposition 5.3. *Let Φ^s be the set of simple fuzzy numbers. Then $\Phi^s/\sim_{sc}$, with respect to the addition + induced by Zadeh addition is a group. If $*$ is the operation induced by the scalar multiplication of a fuzzy number by a real number, then $(\Phi^s/\sim_{sc}, +, *)$ is a vector space.*

The structure of vector space obtained in the set of simple fuzzy numbers, in particular triangular numbers, induced by spread compensation equivalence, allows us to extend the probabilistic concept of coherent prevision and probability, and some of the results given in [4], to the class of random fuzzy numbers (see [14], [10]).

6. Hyperstructures of Fuzzy Numbers

A totally different and general point of view is to replacing the Zadeh or the alternative operations with hyperoperations (see [2], [3]).

This way is also a generalization of the theory of the previous section by at least two points of view:

- **(EQ):** every operation $\star$ between two equivalence classes $[u], [v] \in \Phi/\sim$ can be considered as an hyperoperation on Φ that associates to the pair (u, v) the set $[u \star v]$;
- **(HY):** we construct hypergroups and vector hyperspaces, that generalize the structures of group and vector space, respectively; then a generalization of the concepts of coherent prevision and probability follows.

We introduce two hyperoperations σ and δ in the set Φ of fuzzy numbers with the following properties:

- **(H1):** the Zadeh addition and all the alternative $\oplus$-additions with spreads not greater than the ones of the Zadeh addition are restrictions of σ;
- **(H2):** (Φ, σ) is a commutative semihypergroup;
- **(H3):** the sets T and Δ of trapezoidal and triangular fuzzy numbers, respectively, are subsemihypergroups of (Φ, σ);
- **(H4):** (Φ, δ) is a commutative semihypergroup, extension of (Φ, σ), and its restriction to the set Φ^s of the simple fuzzy numbers is a hypergroup.

Definition 6.1. We define hyperaddition σ on Φ the hyperoperation such that to every pair (u, v) of elements of Φ associates the set of all the fuzzy numbers w such that:

$$C(w) = C(u) + C(v); \tag{25}$$

$$\forall r \in [0,1], L^r(w) \in [max\{L^r(u), L^r(v)\}; L^r(u) + L^r(v)]; \tag{26}$$

$$\forall r \in [0,1], R^r(w) \in [max\{R^r(u), R^r(v)\}; R^r(u) + R^r(v)]. \tag{27}$$

The main properties of the hypergroupoid (Φ, σ) are given in the following theorems. In particular (H1), (H2), and (H3) hold.

Theorem 6.2. *The pair (Φ, σ) is a commutative semihypergroup having 0 as scalar neutral element. It is an extension of all the $\oplus$-additions having spreads not greater than the correspondent spreads of the Zadeh addition. The pairs (T, σ) and (Δ, σ) are subsemihypergroups.*

Theorem 6.3. *For every $u, v \in \Phi$, there exists a $x \in \Phi$ such that $v \in u\sigma x$ if and only if every spread of* u *is less than or equal to the correspondent spread of* v*, i.e. $M(u) \leq M(v)$ and for every $r \in [0, 1)$, $L^r(u) \leq L^r(v)$ and $R^r(u) \leq R^r(v)$.*

Definition 6.4. We define hyperaddition δ on Φ the hyperoperation such that to every pair (u, v) of elements of Φ associates the set of all the fuzzy numbers w such that:

$$C(w) = C(u) + C(v); \tag{28}$$

$$\forall r \in [0,1], L^r(w) \in [min\{L^r(u), L^r(v)\}; L^r(u) + L^r(v)]; \tag{29}$$

$$\forall r \in [0,1], R^r(w) \in [min\{R^r(u), R^r(v)\}; R^r(u) + R^r(v)]. \tag{30}$$

It follows from (28), (29), and (30) that there exists a $x \in \Phi$ such that $v \in u\sigma x$ if and only if $M(u) \leq M(v)$. In particular this happens if $C(u)$ is a singleton. Then we have the following theorem.

Theorem 6.5. *The pair (Φ, δ) is a commutative semihypergroup and its restriction to the set Φ^s of the simple fuzzy numbers is a hypergroup. The pairs (Δ, δ) is a subhypergroup.*

Remark 6.6. It follows from Theorem 6.3 that the intervals of the left and right spreads defined by hyperoperation δ are the minimal intervals containing the correspondent spreads of the Zadeh additions and such that $(\Phi \cdot \delta)$ is a hypergroup.

7. Conclusions

A natural application of the fuzzy operations introduced in Sec.4 and Sec.5 is the aggregation of the scores in multicriteria or multiobjective decision making, in particular in the ambit of Social Sciences.

Let $A = \{a_1, a_2, ..., a_m\}$ be the set of *alternatives* and $K = \{c_1, c_2, ..., c_n\}$ be the set of *criteria*. For instance A can be a set of individuals that ask for a particular job and K a set of criteria to verify their competence. Let us suppose that a decision maker D assigns to every pair (a_i, c_j) of alternative-criterion a value of a linguistic variable, represented by a fuzzy number $b_{i,j}$ contained in the interval $[0, 1]$, that measures the degree in which the alternative a_i satisfies the criterion c_j.

The decision maker D must aggregate the scores $b_{i,j}$ of the alternative a_i in order to obtain a global score $g(a_i)$. At this aim he can utilize an appropriate "fuzzy addition" $\oplus$, chosen among the ones introduced in Sec.4 or in Sec.5, and satisfying some suitable properties, e.g. preserving shapes, having moderate spreads, and so on. The global score $g(a_i)$ is obtained by the formula.

$$g(a_i) = b_{i,1} \oplus b_{i,2} ... \oplus b_{i,n}. \tag{31}$$

The hyperoperations introduced in Sec.6 can open a wide field of researches on the procedures to aggregate scores in multicriteria or multiobjective decision making. Let $\oplus$ be one of the hyperadditions considered in Sec.6. For every $u, v \in \Phi$, the sets of spreads of $u \oplus v$ are intervals defined by (29), (30), (26), and (27). If I is one of these intervals, the decision maker D can introduce a membership function μ on I, where, for every $x \in I$, $\mu(x)$ is the "degree of compatibility" of x with the opinions of D on the aggregation of the spreads of u and v. Then D replaces the set I with a fuzzy subset, that we assume be a fuzzy number. The operations introduced in Sec.4 and Sec.5 are particular cases in which μ is a degenerate fuzzy number.

One of the future focuses and aims of our research is to investigate the applications in the Social Sciences of such aggregation procedures and their logical and mathematical properties.

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